NASA
TT
F-554
c.1

V. D. Andreev

THEORY OF INERTIAL NAVIGATION

Aided Systems

LOAN COPY: RET
AFWL (WLOL
KIRTLAND AFB,

TECH LIBRARY KAFB, NM
0068995

10 NOV
Laboratory

Translated from Russian

Published for the National Aeronautics and Space Administration
and the National Science Foundation, Washington, D.C.
by the Israel Program for Scientific Translations

TECH LIBRARY KAFB, NM
0068995

V. D. Andreev

THEORY OF INERTIAL NAVIGATION

Aided Systems

(Teoriya inertsial'noi navigatsii: korrektiruemye sistemy)

Izdatel'stva "Nauka"
Moskva 1967

Translated from Russian

Israel Program for Scientific Translations
Jerusalem 1969

NASA TT F-564
TT 69-55027

Published Pursuant to an Agreement with
THE NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
and
THE NATIONAL SCIENCE FOUNDATION, WASHINGTON, D.C.

Copyright © 1969
Israel Program for Scientific Translations Ltd.
IPST Cat. No. 5444

Translated by Z. Lerman

Printed in Jerusalem by IPST Press
Binding: Wiener Bindery Ltd., Jerusalem

Available from the
U.S. DEPARTMENT OF COMMERCE
Clearinghouse for Federal Scientific and Technical Information
Springfield, Va. 22151

VII/16/3

PREFACE

Modern aircraft, rockets, and sea vessels require high-precision self-contained navigation and guidance systems, which are further expected to be noise-proof, can be set up at a short notice, and will function unattended.

The last years have been marked by considerable advances in the field of navigation and guidance systems for moving objects. Alongside with the development of new principles, considerable efforts have been devoted to inertial navigation systems which determine the current position of the object by straightforward integration of the accelerations measured on board the moving object.

Inertial systems are currently the most promising and universal means of self-contained, autonomous navigation. However, the dynamic properties of completely autonomous inertial systems are such that the errors of their elements lead to an inevitable buildup of position errors with time.

When a self-contained navigation system is allowed to function continuously for some time, the measurement errors in the navigation parameters may eventually exceed the permissible limit value. The inertial system is therefore never made completely autonomous: it is generally aided by external sources of guidance information, such as astronomical (stellar) correction, altimeter correction, Doppler correction, and radio-navigation aiding.

Introduction of external aiding information into the inertial system alters the unperturbed operation equations of the system (its algorithm). The dynamics of its perturbed operation, i.e., the equations of errors, are also altered. Different external sources of guidance information may differently affect the error equations, and in the final account we thus end up with different dependences of the navigation errors on instrumental errors and on the errors in the external guidance information. The number of possible alternatives here is fairly large.

Some particular problems related to the theory of aided inertial systems are treated in previously published books.* These, however, mainly cover isolated, individual topics, and the analysis is generally far from being rigorous or amenable to far-reaching generalizations. The aim of the present book is conversely to present a systematic and rigorous treatment of the fundamental problems in the theory of aided inertial system, proceeding from a single point of view. The main emphasis is on the analysis of the various means of introducing the correction signals into the system and the investigation of the error equations. The ultimate goal of this study is a comparative analysis of the dependence of navigation errors on instrumental errors for various external sources of guidance information.

The book is largely based on my original researches, carried out and partly published in recent years. It is a direct continuation of my "Theory

* See, e.g., McClure, C.L. Theory of Inertial Guidance. — Prentice Hall. 1960.

of Inertial Navigation: Autonomous Systems," published in Moscow in 1966 by Nauka Publishing House, and naturally draws upon the results and techniques of the previous volume.

The book comprises seven chapters. There are also three Appendices, covering a number of topics which are not an organic part of the theory.

Chapter 1 is introductory in nature: it reiterates the fundamental results of the theory of autonomous inertial systems.

Chapter 2 is concerned with the general aspects of the application of external guidance information and with the theory of inertial systems where an altimeter provides additional information on the distance of the moving object from the Earth's surface. Systems with three arbitrarily oriented accelerometers and systems with two horizontal accelerometers are considered. The ideal (unperturbed) operation equations are derived, as well as the corresponding error equations. The stability of the system is investigated and the solution of the error equations is derived.

Chapter 3 discusses Schuler-tuned gyropendulum systems, particular examples of which are the common Schuler pendulum and various classical gyroscopic instruments, such as the two-gyro vertical and the Geckeler—Anschütz three-degrees-of-freedom gyrocompass horizon. Particular stress is laid on the dynamic analogy between the Schuler-tuned gyropendulum systems and the two-accelerometer (or two-axis) inertial guidance systems. Ishlinskii* was the first to call attention to the possible analogy between these two categories of guidance systems, and I proved the existence of complete dynamic analogy between them. Consequently, a number of important results from the theory of Schuler-tuned systems can be readily generalized to two-accelerometer inertial systems.

Chapter 4 deals with various simplified forms of the ideal operation equations of inertial systems, including the simplifications associated with the nearly spherical figure of the Earth and the nearly central field of its gravitation. Simplifications resulting from particular constraints imposed on the object trajectory are also considered. These include close trajectories of objects moving near the Earth's surface, trajectories which are close to some orthodromy on the Earth's surface, and cases of constrained velocity. By examining the simplifications in the ideal operation equations (i.e., in the algorithm of the navigation system), we can learn to simplify the design of the inertial system by omitting or replacing some of its elements.

Chapter 5 presents the theory of inertial systems where altimeter correction is reinforced by additional information from a Doppler velocity meter. Introduction of damping in the perturbed operation equations of inertial systems inevitably leads to velocity errors. Suitable guidance information supplied by a Doppler velocity meter will eliminate these errors. The residual errors will then depend only on the instrumental inaccuracy of the system elements and the inevitable errors in the Doppler information. The corresponding relations are expressed by the error equations, which are derived and investigated for different Doppler correction techniques.

Chapter 6 is concerned with astronomical (stellar) correction procedures, whereby telescopes are used to fix the directions to certain stars. Combined astro-Doppler correction is also considered in this chapter. It is shown that, unlike Doppler and altimeter correction routines, astrocorrection

* Ishlinskii, A.Yu. The equations of the problem of determination of the position of a moving object using gyroscopes and accelerometers. — Prikladnaya Matematika i Mekhanika, Vol. 20, No. 6. 1957.

affects only the second group of error equations. The resulting changes in the error equations are studied for linear and relay-type correction routines. A phase astrocorrection system using star trackers is dealt with in a separate section.

Chapter 7 deals with the dynamics of autonomous and aided inertial systems with random errors. The analysis is carried out within the framework of the correlation theory of random processes. The main object of this chapter is to study the dispersion of the navigation errors as a function of the statistical characteristics of the instrumental errors for various correction techniques.

Appendix I at the end of the book describes the effect of the scattered solar radiation in the atmosphere on the normal operation of a phase astrocorrection system. Appendix II is concerned with close navigation using a directional gyro and a Doppler velocity meter. Appendix III investigates the problem of direct accelerometer guidance, with or without external guidance information. The material presented in these Appendices has been previously published in different periodical journals (the third appendix, in cooperation with I. V. Novozhilov).

One more point should be borne in mind. The present volume is a direct continuation of my "Theory of Inertial Guidance: Autonomous Systems" and largely draws upon the results of the previous book. However, Chapter 1 summarizes in a compact form all the fundamental results of the theory of autonomous inertial systems and the book can therefore be read as an independent, self-contained monograph.

This volume appears to be the first attempt at a systematic discussion of the theory of aided inertial systems. As such, it is not free from errors, and all remarks and criticism will be appreciated.

In conclusion I would like to acknowledge the great help of A.Yu. Ishlinskii, whose assistance in formulating a number of problems analyzed in this book was invaluable. I am also grateful to N.P. Bukanov, G.I. Vasil'ev-Lyulin, E.A. Devyanin, A.P. Dem'yanovskii, I.M. Lisovich, I.V. Novozhilov, N.A. Parusnikov, P.V. Tarasov, and V.V. Shed'ko for their part in the discussion of various sections of the book.

V. Andreev

Table of Contents

Preface iii

Chapter 1. ELEMENTS OF THE THEORY OF AUTONOMOUS INERTIAL NAVIGATION SYSTEMS 1

§1.1. The ideal operation equations of inertial navigation systems 1

1.1.1. The fundamental equation of inertial navigation and its integration in fixed-orientation reference axes (1). 1.1.2. Fundamental coordinate systems fixed to the Earth (3). 1.1.3. The gravitational field of the Earth (7). 1.1.4. Ideal operation equations of inertial systems measuring the Cartesian coordinates of the object (10). 1.1.5. Determination of curvilinear coordinates (13). 1.1.6. The ideal operation equations in orthodromic, geocentric, and geographical coordinates (17).

§1.2. The equations of errors for inertial navigation systems. . . . 22

1.2.1. Perturbed inertial systems. Main instrumental errors (22). 1.2.2. Position error equations (23). 1.2.3. Alternative forms of the position error equations (26). 1.2.4. Orientation error equations for the sensitive axes and fixed directions in space (29).

§1.3. Some results of the analysis of error equations of autonomous inertial systems 31

1.3.1. The second group of error equations (31). 1.3.2. Stability of the first group of equations for latitudinal motion (33). 1.3.3. Solution of the first group of error equations for a stationary object and for an object moving along the arc of a fixed-attitude great circle (35). 1.3.4. Integration of the first group of equations for Keplerian motion (39). 1.3.5. Position and orientation errors as a function of instrumental and initial setting errors (44).

Chapter 2. AIDED INERTIAL SYSTEMS: ALTIMETER CORRECTION 48

§2.1. General considerations pertaining to aided inertial systems 48

§2.2. Ideal operation equations and error equations for inertial systems with altimeter correction 52

2.2.1. Three-accelerometer system (52). 2.2.2. Two-accelerometer system (54).

§2.3. Stability analysis and integration of the first group of error equations for two-accelerometer systems 60

2.3.1. *Stability analysis in cases leading to equations with constant coefficients* (60). 2.3.2. Solution of the first group of error equations in cases which lead to equations with constant coefficients (65). 2.3.3. Integration of the first group of error equations for Keplerian motion (70). 2.3.4. Investigation of the first group of equations for arbitrary motion near the Earth (74).

§2.4. Stability analysis and integration of the first group of error equations for a three-accelerometer system with altimeter correction 81
2.4.1. Stability analysis (81). 2.4.2. Internal dissipation forces (83). 2.4.3. Stability of the first group of equations for a close object (85). 2.4.4. Solution of the first group of equations for arbitrary motion at a constant distance from the Earth's center (86). 2.4.5. Integration of the first group of error equations for Keplerian motion (89).

§2.5. Dependence of navigation errors on instrumental and setting inaccuracies. Comparison with autonomous inertial systems 94
2.5.1. Dependence of position and orientation errors on setting errors (94). 2.5.2. Navigation errors due to accelerometer and gyroscope errors (98). 2.5.3. Altimeter errors (100).

Chapter 3. SCHULER-TUNED GYROPENDULUM SYSTEMS. ANALOGY WITH TWO-ACCELEROMETER INERTIAL SYSTEMS . . 103

§3.1. Schuler pendulum 103
3.1.1. The conditions of a relative equilibrium with the pendulum axis caged to the geocentric vertical (103). 3.1.2. Equations of small oscillations of the Schuler pendulum about the relative equilibrium (110). 3.1.3. Equations of free oscillations of the Schuler pendulum about a position of relative equilibrium: the case of finite angles (115).

§3.2. The two-gyro vertical and the Geckeler—Anschütz gyrocompass horizon 117
3.2.1. The two-gyro vertical (117). 3.2.2. The Geckeler—Anschütz gyrocompass horizon (122).

§3.3. A general Schuler-tuned system 126
3.3.1. Conditions of existence of relative equilibrium (126). 3.3.2. Perturbed motion about the relative equilibrium (128).

§3.4. Analogy between Schuler-tuned systems and two-accelerometer inertial systems 131
3.4.1. Comparison of the first group of error equations with the equations of oscillations of a gyropendulum system about relative equilibrium (131). 3.4.2. Sufficient stability conditions for two-accelerometer inertial systems (135). 3.4.3. Damping of oscillations in a Schuler-tuned system (137).

Chapter 4. SIMPLIFIED OPERATION EQUATIONS OF IDEAL AND PERTURBED TWO-ACCELEROMETER INERTIAL SYSTEMS . . . 140

§4.1. General considerations. 140

§4.2. Simplifications in an inertial system measuring the orthodromic coordinates 141
4.2.1. Exact ideal operation equations. Error equations (141). 4.2.2. Some particular cases of azimuthal orientation of accelerometers (151). 4.2.3. Simplifications which follow from the nearly spherical figure of the Earth (153). 4.2.4. Simplifications associated with the closeness of the trajectory to the Earth's surface and the orthodromy plane (158). 4.2.5. Different methods of accounting for the horizontal component of the Earth's gravitation (165). 4.2.6. Other simplifications (171).

§4.3. Simplifications in an inertial system which measures the geographical coordinates. 174
4.3.1. Exact ideal operation equations. Error equations (174). 4.3.2. Simplifications of the equations for geographical coordinates (178).

§4.4. Determination of orthodromic coordinates with accelerometers oriented in the plane of the geographical horizon 180

Chapter 5. INERTIAL SYSTEMS WITH DOPPLER CORRECTION. 188

§5.1. Asymptotic stability of inertial systems. Velocity errors . 188

§5.2. Basic information on Doppler velocity meters 193

§5.3. Doppler damping of an inertial system 196
5.3.1. Unperturbed (ideal) operation equations. Error equations (196). 5.3.2. Ideal operation equations and error equations for an orthodromic system (200). 5.3.3. Stability analysis (203). 5.3.4. Solution of the error equations. Effect of instrumental errors (214).

§5.4. Changing the natural frequencies of a Doppler-aided system 219
5.4.1. Ideal operation equations. Error equations (219). 5.4.2. Analysis of error equations (222).

§5.5. Damping with simultaneous frequency change 225
5.5.1. Ideal operation equations. Error equations (225). 5.5.2. Orthodromic coordinates (231). 5.5.3. Stability analysis (235). 5.5.4. The solution of error equations. The effect of instrumental errors. Choice of correction coefficients (245).

§5.6. Some additional remarks on Doppler correction 252
5.6.1. Correction techniques using a gyroscopically stabilized platform (252). 5.6.2. Doppler correction increasing the order of the error equations. Linear correction with variable coefficients. Nonlinear correction (259).

Chapter 6. ASTRONOMICAL CORRECTION OF INERTIAL NAVIGATION SYSTEMS. COMBINED ASTRO-DOPPLER CORRECTION 264

§6.1. Introductory remarks 264

§6.2. Earth's motion. Determination of directions to celestial bodies in axes fixed to the Earth 265
6.2.1. Simplified description of the Earth's motion (265). 6.2.2. Determination of directions to celestial bodies (267). 6.2.3. Earth's motion and measurement of time (272). 6.2.4. Fixing the position on the Earth from astronomical observations (275).

§6.3. Astrocorrection using two stars 276
6.3.1. Information obtained from telescopic observations. Astrocorrection techniques (276). 6.3.2. Simultaneous linear correction using two stars (280). 6.3.3. Simultaneous "relay-type" correction using two stars (286). 6.3.4. Alternate two-star correction (288). 6.3.5. The correcting torques (290).

§6.4. Astrocorrection with phase indication of stars 291
6.4.1. The principle of phase indication (291). 6.4.2. The dynamics of a closed phase astrocorrection system near the state of equilibrium. Stability analysis (294).

§6.5. Combined astro-Doppler correction 300
6.5.1. System with simultaneous delivery of Doppler and astrocorrection signals to gyro torquers (300). 6.5.2. One-star correction (307).

Chapter 7. DYNAMICS OF INERTIAL SYSTEMS WITH RANDOM INSTRUMENTAL ERRORS 309

§ 7.1. Fundamental relations of the correlation theory of random processes. 309

§ 7.2. The effect of random instrumental errors on autonomous inertial systems 317
7.2.1. The case of a stationary object (317). 7.2.2. The case of a satellite in a circular orbit (326).

§ 7.3. The effect of random instrumental errors on aided systems. 335
7.3.1. Altimeter correction (335). 7.3.2. Combined altimeter and Doppler correction (345). 7.3.3. Simultaneous altimeter and astro-Doppler correction (354).

§ 7.4. Dynamics of a phase astrocorrection system with random errors 362

Appendix I. SCATTERED SOLAR RADIATION IN THE ATMOSPHERE AS AN EXAMPLE OF DETERMINATE ERRORS IN PHASE ASTRO-CORRECTION SYSTEMS 367

Appendix II. NAVIGATION USING A DIRECTIONAL GYROSCOPE AND A DOPPLER VELOCITY METER 389

Appendix III. ACCELEROMETER GUIDANCE 400

References 409

Chapter 1

ELEMENTS OF THE THEORY OF AUTONOMOUS INERTIAL NAVIGATION SYSTEMS

§1.1. THE IDEAL OPERATION EQUATIONS OF INERTIAL NAVIGATION SYSTEMS

1.1.1. The fundamental equation of inertial navigation and its integration in fixed-orientation reference axes

The main object of any navigation system is to determine the exact position of a body, i.e., the coordinates of some point, such as the center of mass, in a given frame of reference. The rate of change of these coordinates, the parameters characterizing the orientation of the moving object in a given frame of reference, and the rate of change of these attitude parameters are also of interest in navigation.

A distinctive feature of inertial navigation is that the body coordinates are essentially obtained through integration of Newton's equations of motion of the center of mass of the body. The components of the resultant force vector applied to the body, which are required for the integration of these equations, are obtained from the readings of special instruments — accelerometers, which measure the projections of the force on their sensitive axis. The attitude of the accelerometer sensitive axis is fixed with gyroscopes or simply determined from accelerometer readings.

The inertial (Galilean) coordinate system, where Newton's laws are valid by definition, is the main frame of reference used in inertial navigation.

A spatial accelerometer (Figure 1.1) can be schematically described as a point mass m held in a weightless elastic suspension with three degrees of freedom in the body of the instrument. The equation of motion of the accelerometer sensitive mass in some inertial frame of reference is

$$m\frac{d^2r}{dt^2} = mF(r) + f. \tag{1.1}$$

In this equation r is the radius-vector of the point O where the accelerometer sensitive mass is situated in the inertial frame; $F(r)$ is the geometrical sum of the Newtonian attraction on unit sensitive mass by all the celestial bodies, including the nearby masses in the object where the accelerometer is mounted; f is the elastic deformation force of the suspension.

For an observer in the fixed frame of the accelerometer case, the only force on the sensitive mass is the elastic force of the suspension; the

parameters characterizing this force are the suspension deformations, and the accelerometer readings actually give the elastic deformation of the suspension. Taking the force f proportional to the vector deformation of the suspension n ($f = kn$) and putting for simplicity $m/k = 1$, we obtain from (1.1)

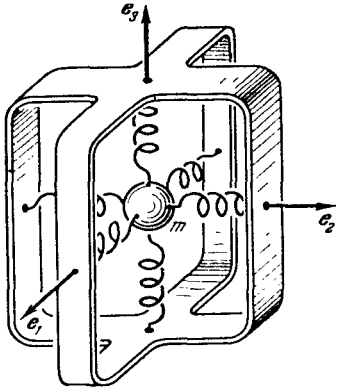

FIGURE 1.1.

$$n = \frac{d^2r}{dt^2} - F(r). \tag{1.2}$$

For an object moving near the Earth, Eq. (1.2) can be written with fair accuracy in the form*

$$n = \frac{d^2r}{dt^2} - g(r). \tag{1.3}$$

Here r is the radius-vector of the point O (where the sensitive mass is) in the coordinate system $O_1\xi_*\eta_*\zeta_*$, whose origin is at the center of the Earth and the orientation of its axes is fixed relative to directions from the Earth's center to distant (fixed) stars. $g(r)$ stands in (1.3) for the gravitational attraction of the Earth. The differentiation in (1.3) is done in the $O_1\xi_*\eta_*\zeta_*$ system. The main error involved in the substitution of Eq. (1.3) for Eq. (1.2) is associated with the difference in the gravitational attraction of the Sun and the Moon at O_1 (Earth's center) and O (the position of the object). Near the Earth's surface this error is equivalent to the instrumental error of the accelerometer, which is at most 10^{-4} cm/sec^2.

Actual accelerometers are usually one-component devices. The sensitive mass of a one-component (linear) accelerometer has only one degree of freedom relative to the accelerometer body (Figure 1.2). It can move only along a straight line, the sensitive axis of the accelerometer. The reaction force of the suspension spring (whose deformation is measured) is applied along this axis. The reading of a one-component accelerometer gives the projection

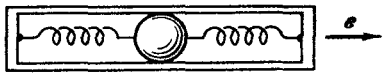

FIGURE 1.2.

$$n_e = n \cdot e, \tag{1.4}$$

where e is the unit vector in the direction of the sensitive axis.

Three one-component accelerometers with non-coplanar sensitive axes are clearly equivalent to a single spatial accelerometer.

We have so far regarded the elastic suspension in the accelerometer as made of an elastic spring. In practice, accelerometers utilize elastic forces of a different origin, mostly electromagnetic. This point, however, is of no significance for the derivation of Eqs. (1.3), (1.4).

Eq. (1.3) is the fundamental equation of inertial navigation. Integration of this equation is the main task of an autonomous inertial-navigation system.

* See Andreev, V.D. Theory of Inertial Navigation: Autonomous Systems. — Nauka. 1966.

Formally, the most natural integration technique calls for the integration of Eq. (1.3) in axes of fixed orientation. Let three linear accelerometers be aligned along the axes of a gyroscopically-stabilized platform, which are parallel to the axes ξ_*, η_*, ζ_*. The accelerometer readings are then the projections of the vector $\boldsymbol{n}$ on the axes ξ_*, η_*, ζ_*:

$$\left.\begin{aligned} n_{\xi_*} &= \ddot{\xi}_* - g_{\xi_*}, \quad n_{\eta_*} = \ddot{\eta}_* - g_{\eta_*}, \\ n_{\zeta_*} &= \ddot{\zeta}_* - g_{\zeta_*}. \end{aligned}\right| \tag{1.5}$$

where superior dot denotes differentiation with respect to time. From these relations we get

$$\left.\begin{aligned} \xi_* &= \int_0^t \int_0^t (n_{\xi_*} - g_{\xi_*})\, dt\, dt - \dot{\xi}_*^0 t - \xi_*^0, \\ \eta &= \int_0^t \int_0^t (n_{\eta_*} + g_{\eta_*})\, dt\, dt + \dot{\eta}_*^0 t + \eta_*^0, \\ \zeta_* &= \int_0^t \int_0^t (n_{\zeta_*} + g_{\zeta_*})\, dt\, dt + \dot{\zeta}_*^0 t + \zeta_*^0. \end{aligned}\right| \tag{1.6}$$

where $\xi_*^0, \eta_*^0, \zeta_*^0, \dot{\xi}_*^0, \dot{\eta}_*^0, \dot{\zeta}_*^0$ are the corresponding initial values.

If the projections $g_{\xi_*}, g_{\eta_*}, g_{\zeta_*}$ are known functions of the coordinates ξ_*, η_*, ζ_*, as is the case, say, in a spherical field of gravity, Eqs. (1.6) can be applied to determine the current position ξ_*, η_*, ζ_* of the object from the readings of the three accelerometers and the corresponding initial conditions.

In principle, Eqs. (1.6) provide the complete solution of the problem of inertial navigation. Indeed, as the motion of the Earth in the $O_1\xi_*\eta_*\zeta_*$ frame is known, we can easily transform from the Cartesian coordinates ξ_*, η_*, ζ_* to any other system of coordinates fixed to the Earth. The rate of change of any set of coordinates is also readily obtained. Since the orientation of the object in the $O_1\xi\eta\zeta$ frame is determined by its attitude relative to the stable platform (e.g., the rotation angles of the platform gimbals), we can find the parameters characterizing the orientation of the moving object relative to any frame fixed to the Earth.

Note that Eqs. (1.6) specify the coordinates of the point O where the sensitive masses of the accelerometers are. It is these coordinates that are henceforth regarded as the coordinates of the object.

1.1.2. Fundamental coordinate systems fixed to the Earth

Consider a right orthogonal trihedron $O_1\xi\eta\zeta$ with its origin at the center of the Earth. The axis $O_1\zeta$ points along the absolute spin vector $\boldsymbol{u}$ of the Earth. With fair accuracy (see footnote on p. 2) the vector $\boldsymbol{u}$ can be taken to point in a fixed direction relative to directions to the distant star; it is thus constantly directed along the axis of the Earth's figure and its magnitude u is constant. The axis $O_1\xi$ of the $O_1\xi\eta\zeta$ frame is directed along the intersection line of the equatorial plane and the plane of the Greenwich meridian. Without loss of generality, we can take the trihedron $O_1\xi\eta\zeta$ to coincide with the trihedron $O_1\xi_*\eta_*\zeta_*$ at the initial time. Then, at any other

time the relative orientation of these axes is specified by the following matrix of direction cosines:

$$\begin{array}{c|ccc} & \xi & \eta & \zeta \\ \xi_1 & \cos ut & -\sin ut & 0 \\ \eta_1 & \sin ut & \cos ut & 0 \\ \zeta_1 & 0 & 0 & 1. \end{array} \tag{1.7}$$

The Earth's level surface (geoid) is adequately approximated to by the Clairaut ellipsoid of revolution:

$$\frac{\xi^2+\eta^2}{a^2}+\frac{\zeta^2}{b^2}=1, \tag{1.8}$$

where a and b are the Earth's semimajor and semiminor axis. For the Krasovskii's ellipsoid, in current use in the USSR, we have $a = 6{,}378{,}245$ m, $b = 6{,}356{,}863$ m.*

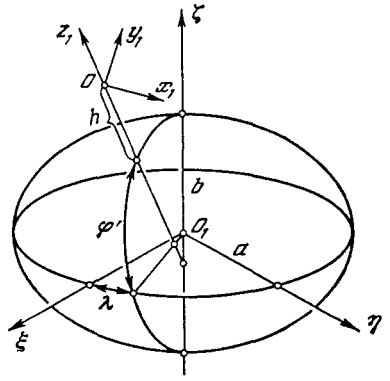

FIGURE 1.3.

The position of any point O in the $O_1\xi\eta\zeta$ frame can be determined (Figure 1.3) in terms of the geographical coordinates h, φ', λ, where h is the length of the normal dropped from the point O to the Clairaut ellipsoid (height above ocean level), φ' is the geographical latitude (the angle between the normal to the ellipsoid and the equatorial plane), λ is the longitude (the angle between the plane of the Greenwich meridian and the plane of the meridian through O).

The coordinates ξ, η, ζ are related to the coordinates h, φ', λ by the equalities

$$\left.\begin{aligned} \xi &= \left[\frac{a}{(1-e^2\sin^2\varphi')^{1/2}}+h\right]\cos\varphi'\cos\lambda, \\ \eta &= \left[\frac{a}{(1-e^2\sin^2\varphi')^{1/2}}+h\right]\cos\varphi'\sin\lambda, \\ \zeta &= \left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{1/2}}+h\right]\sin\varphi', \end{aligned}\right\}$$

* The parameters of ellipsoids used in other countries can be found, say, in Graur, A.V. Matematicheskaya kartografiya (Mathematical Cartography). — Izd. LGU. Leningrad. 1956.

where e^2 is the square of the first eccentricity of the Clairaut ellipsoid,

$$e^2 = \frac{a^2 - b^2}{a^2}. \tag{1.9}$$

With the point O we associate a right orthogonal trihedron $Ox_1y_1z_1$, the co-moving axes of the geographical coordinate grid (see Figure 1.3). The axis z_1 of this trihedron is directed along the positive normal to the Clairaut ellipsoid, the axis y_1 lies in the meridional plane through O pointing northward. The relative attitude of the two trihedra $\xi\eta\zeta$ and $x_1y_1z_1$ is characterized by the following matrix of direction cosines:

$$\begin{array}{c|ccc} & x_1 & y_1 & z_1 \\ \hline \xi & -\sin\lambda & -\sin\varphi'\cos\lambda & \cos\varphi'\cos\lambda \\ \eta & \cos\lambda & -\sin\varphi'\sin\lambda & \cos\varphi'\sin\lambda \\ \zeta & 0 & \cos\varphi' & \sin\varphi' \end{array} \tag{1.10}$$

The planes x_1z_1 and y_1z_1 are the planes of the principal normal sections of the surface $h = \text{const}$. The corresponding radii of curvature are given by

$$\left.\begin{aligned} r_2 &= \frac{a}{(1 - e^2\sin^2\varphi')^{1/2}} + h, \\ r_3 &= \frac{a(1 - e^2)}{(1 - e^2\sin^2\varphi')^{3/2}} + h. \end{aligned}\right\} \tag{1.11}$$

On a par with geographical coordinates, we can use the geocentric coordinates to define the position of the point O; the geocentric coordinates

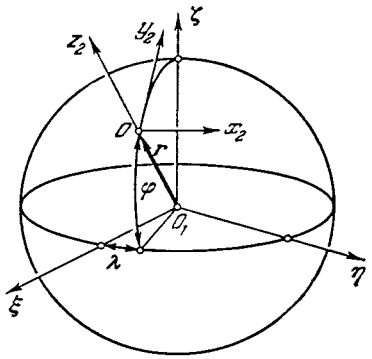

FIGURE 1.4.

include the distance r of the point O from the Earth's center O_1, the geocentric latitude φ, and the longitude λ. In this case (Figure 1.4)

$$\left.\begin{aligned} \xi &= r\cos\varphi\cos\lambda, \quad \eta = r\cos\varphi\sin\lambda, \\ \zeta &= r\sin\varphi. \end{aligned}\right\} \tag{1.12}$$

The geocentric latitude coincides with the geographical latitude. The relation between the geographical coordinates h, φ' and the geocentric coordinates r and φ is obtained by comparing (1.8a) and (1.12) (see footnote on p. 2):

$$\left.\begin{aligned} \operatorname{tg}\varphi &= \left[1 - \frac{ae^2}{a + h\,(1 - e^2\sin^2\varphi')^{1/2}}\right]\operatorname{tg}\varphi', \\ r &= \left[\frac{a\,(1-e^2)}{(1 - e^2\sin^2\varphi')^{1/2}} + h\right]\frac{\sin\varphi'}{\sin\varphi}. \end{aligned}\right\} \tag{1.13}$$

We introduce the co-moving trihedron $Ox_2y_2z_2$ associated with the geocentric coordinate grid (Figure 1.4); its axis z_2 is directed along the radius-vector of the point r, and the axis y_2 lies in the meridional plane through O pointing northward. The matrix of the direction cosines which characterizes the orientation of the trihedron $x_2y_2z_2$ relative to the trihedron $\xi\eta\zeta$ is clearly obtained from matrix (1.10) by writing φ for φ'. The relative orientation of the geographical and the geocentric co-moving axes is determined by the following direction cosines:

	x_2	y_2	z_2
x_1	1	0	0
y_1	0	$\cos(\varphi' - \varphi)$	$-\sin(\varphi' - \varphi)$
z_1	0	$\sin(\varphi' - \varphi)$	$\cos(\varphi' - \varphi)$,

(1.14)

and, by the first equality in (1.13),

$$\operatorname{tg}(\varphi' - \varphi) = \frac{ae^2\sin\varphi'\cos\varphi'}{(1 - e^2\sin^2\varphi')^{1/2}\,[a\,(1 - e^2\sin^2\varphi')^{1/2} + h]}. \tag{1.15}$$

Apart from the geocentric coordinates r, φ, λ, one often uses the so-called orthodromic coordinates (also spherical). They can be defined in the following way. Consider a trihedron $O_1\xi'\eta'\zeta'$, rigidly fixed to the Earth, so that its orientation relative to the trihedron $O_1\xi\eta\zeta$ does not change. Let the relative orientation of these axes be expressed by the matrix of direction cosines

	ξ'	η'	ζ'
ξ	β_{11}	β_{12}	β_{13}
η	β_{21}	β_{22}	β_{23}
ζ	β_{31}	β_{32}	β_{33},

(1.16)

where all the elements are constant.

The position of the point O relative to $O_1\xi'\eta'\zeta'$ axes can be determined using the coordinates r, z, S, which are analogous to the geocentric coordinates r, φ, λ, so that (Figure 1.5)

$$\left.\begin{aligned} \xi' &= r\cos z\cos S, \quad \eta' = r\cos z\sin S, \\ \zeta' &= r\sin z. \end{aligned}\right\} \tag{1.17}$$

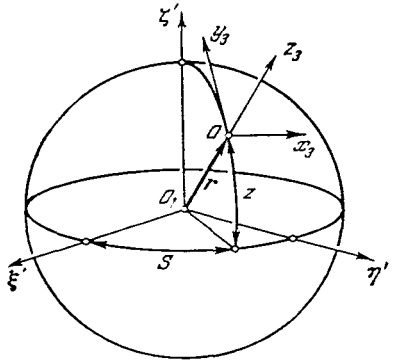

FIGURE 1.5.

The relation between φ, λ and S, z follows from the equalities

$$\left.\begin{aligned}\cos\varphi\cos\lambda &= \beta_{11}\cos z\cos S + \beta_{12}\cos z\sin S + \beta_{13}\sin z,\\ \cos\varphi\sin\lambda &= \beta_{21}\cos z\cos S + \beta_{22}\cos z\sin S + \beta_{23}\sin z,\\ \sin\varphi &= \beta_{31}\cos z\cos S + \beta_{32}\cos z\sin S + \beta_{33}\sin z.\end{aligned}\right\} \qquad (1.18)$$

In orthodromic coordinates the equivalent of the equatorial plane is the plane $O_1\xi'\eta'$ (the plane of principal orthodromy). When the trihedra $\xi'\eta'\zeta'$ and $\xi\eta\zeta$ coincide, orthodromic coordinates reduce to geocentric coordinates.

We now introduce the co-moving trihedron $Ox_3y_3z_3$ of the geocentric coordinate grid. Its axis z_3 coincides with r and the axis y_3 lies in the plane through r and the axis ζ', pointing in the direction of the ζ' axis. The relative orientation of the trihedra $x_2y_2z_2$ and $x_3y_3z_3$ is then given by the following matrix of direction cosines:

	x_3	y_3	z_3
x_2	$\cos\psi$	$\sin\psi$	0
y_2	$-\sin\psi$	$\cos\psi$	0
z_2	0	0	1,

(1.19)

where

$$\left.\begin{aligned}\sin\psi &= \frac{1}{\cos\varphi}(\beta_{31}\sin S - \beta_{32}\cos S),\\ \cos\psi &= \frac{1}{\cos\varphi}(-\beta_{31}\sin z\cos S - \beta_{32}\sin z\sin S + \beta_{33}\cos z).\end{aligned}\right\} \qquad (1.20)$$

1.1.3. The gravitational field of the Earth

The projections of the vector $g(r)$ — the regularized gravitational acceleration of the Earth — on the axes ξ, η, ζ are obtained from the solution of the Stokes problem for a spheroidal level surface. They have the form*

* Mikhailov, A.A. Kurs gravimetrii i teorii figury Zemli (A Course in Gravimetry and the Theory of the Earth's Figure). — Redbyuro GUGK, SNK SSSR. 1939; Grushinskii, P.P. Teoriya figury Zemli (Theory of the Earth's Figure). — Fizmatgiz. 1963.

$$\left.\begin{aligned} g_\xi &= -P\xi + C\frac{\partial K}{\partial \xi}, \quad g_\eta = -P\eta + C\frac{\partial K}{\partial \eta}, \\ g_\zeta &= -Q\zeta + C\frac{\partial K}{\partial \zeta}. \end{aligned}\right\} \tag{1.21}$$

Here

$$\left.\begin{aligned} P &= 2\pi D\mu \frac{a^2 b}{(a^2-b^2)^{3/2}}\left(\operatorname{arctg} l' - \frac{l'}{1+l'^2}\right), \\ Q &= 4\pi D\mu \frac{a^2 b}{(a^2-b^2)^{3/2}}(l' - \operatorname{arctg} l'), \\ K &= 2\pi D\mu \frac{a^2 b}{(a^2-b^2)^{1/2}} \operatorname{arctg} l', \end{aligned}\right\} \tag{1.22}$$

where l' is the second eccentricity of the ellipsoid through point O which is confocal with the Clairaut ellipsoid and is described by the equation

$$\frac{\xi^2+\eta^2}{a^2+\nu} + \frac{\zeta^2}{b^2+\nu} = 1. \tag{1.23}$$

Thus

$$l' = \sqrt{\frac{a^2-b^2}{b^2+\nu}} \tag{1.24}$$

The constants $D\mu$ and C entering Eqs. (1.21), (1.22) are obtained from the conditions

$$\left.\begin{aligned} \frac{u^2}{2\pi D\mu} &= \frac{3+l^2}{l^2}\operatorname{arctg} l - \frac{3}{l^2}, \\ g_e &= 2C\pi D\mu a + P_0 a - u^2 a, \end{aligned}\right\} \tag{1.25}$$

where $l^2 = (a^2-b^2)/b^2$ is the second eccentricity of the Clairaut ellipsoid, g_e is the gravitational acceleration on the equator, P_0 is the value of P on the surface of the Clairaut ellipsoid.

From Eqs. (1.21), (1.22), (1.12) and matrix (1.10) we obtain the following expressions for the projections of the gravitational acceleration on the comoving geocentric axes:

$$\left.\begin{aligned} g_{x_2} &= 0, \\ g_{y_2} &= 2\pi D\mu \sin\varphi\cos\varphi\left[\frac{a^2 br}{(a^2-b^2)^{3/2}}\left(3\operatorname{arctg} l' - \frac{l'}{1+l'^2} - 2l'\right) - \right. \\ &\quad \left. - Ca^3 be^2 \frac{a}{r}\frac{(b^2+\nu)^{1/2}}{(b^2+\nu)^2\cos^2\varphi + (a^2+\nu)^2\sin^2\varphi}\right], \\ g_{z_2} &= -2\pi D\mu\left\{\frac{a^2 br}{(a^2-b^2)^{3/2}}\left[2(l' - \operatorname{arctg} l') + \right.\right. \\ &\quad + \cos^2\varphi\left(3\operatorname{arctg} l' - \frac{l'}{1+l'^2} - 2l'\right)\Big] + \\ &\quad \left. + Cab\frac{a}{r}\frac{[(b^2+\nu)\cos^2\varphi + (a^2+\nu)\sin^2\varphi](b^2+\nu)^{1/2}}{(b^2+\nu)^2\cos^2\varphi + (a^2+\nu)^2\sin^2\varphi}\right\}. \end{aligned}\right\} \tag{1.26}$$

Using (1.23)—(1.25), we expand the right-hand sides of Eqs. (1.26) in rapidly converging series* in powers of the first eccentricity of the

* Andreev, V.D. A solution of the Stokes problem for a geoid level surface.—Prikladnaya Matematika i Mekhanika, Vol. 30, No. 2. 1966.

Clairaut ellipsoid $e = \sqrt{a^2 - b^2}/a$:

$$\left.\begin{aligned}
g_{y_1} &= \frac{g_e}{2}(q - e^2)\left(\frac{a}{r}\right)^2 \sin 2\varphi \left[1 + \right. \\
&\quad \left. + e^2 \frac{7e^2 - 30q}{14(q - e^2)}\right]\left\{1 + \left[\frac{30q - 21e^2}{14(q - e^2)} + \right.\right. \\
&\quad \left.\left. + \sin^2\varphi \frac{7e^2 - 10q}{2(q - e^2)}\right] e^2 \left(\frac{a}{r}\right)^2 + \ldots\right\}, \\
g_{z_1} &= -g_e\left(\frac{a}{r}\right)^2\left\{1 - \frac{1}{2}e^2 - \frac{1}{8}e^4 + q\left(\frac{3}{2} - \frac{15}{28}e^2\right) + \right. \\
&\quad + \left[\frac{1}{2}e^2 - \frac{1}{4}e^4 + q\left(-\frac{1}{2} + \frac{15}{14}e^2\right) - \right. \\
&\quad \left.\left. - \sin^2\varphi\left(\frac{3}{2}e^2 - \frac{3}{4}e^4 + q\left(-\frac{3}{2} + \frac{45}{14}e^2\right)\right)\right]\left(\frac{a}{r}\right)^2 + \right. \\
&\quad + \left[\frac{3}{8}e^2 - \frac{15}{28}q + \sin^2\varphi\left(\frac{5}{8}e^2 - \frac{25}{28}q\right) - \right. \\
&\quad \left.\left. - \sin^2 2\varphi\left(\frac{35}{32}e^2 - \frac{25}{16}q\right)\right] e^2\left(\frac{a}{r}\right)^4 + \ldots\right\},
\end{aligned}\right\} \tag{1.27}$$

where q is the ratio of the centrifugal acceleration to the gravitational acceleration at the equator, $q = \omega^2 a\, g_e$.

The numerical values of the coefficients in Eqs. (1.27) show that, to an accuracy of about 0.02 cm/sec^2, these expressions can be replaced by the simpler formulas

$$\left.\begin{aligned}
g_{y_1} &= \frac{g_e}{2}(q - e^2)\left(\frac{a}{r}\right)^2 \sin 2\varphi, \\
g_{z_1} &= -g_e\left(\frac{a}{r}\right)^2\left[1 - \frac{1}{2}e^2 + \frac{3}{2}q + \frac{q - e^2}{2}(-1 + 3\sin^2\varphi)\left(\frac{a}{r}\right)^2\right].
\end{aligned}\right\} \tag{1.28}$$

Taking $g_e = 978.049$ cm/sec^2, $\omega = 7.292116\ \text{sec}^{-1}$, we obtain using Krasovskii's ellipsoid parameters ($e^2 = 0.0066934$)

$$\left.\begin{aligned}
\frac{g_e}{2}(q - e^2) &= -1.58, \quad \frac{q - e^2}{2} = 0.0016, \\
1 - \frac{e^2}{2} - \frac{3q}{2} &= 0.0018.
\end{aligned}\right\} \tag{1.29}$$

Eqs. (1.27), (1.28) give the projections of $g(r)$ on the co-moving geocentric axes as a function of the geocentric latitude φ and the distance r from the Earth's center. From these relations we can find the projections g_{y_1}, g_{z_1} on the geographical axes as a function of the geographical latitude φ' and the distance h from the surface of the Clairaut ellipsoid (height above ocean level).

In Eqs. (1.27) we now change over from r and φ to h and φ'. We will consider only the case of small h, when the ratio h/a is of the order e^2, i.e., when $h \approx 100$ km. Using (1.13), we find

$$\left.\begin{aligned}
g_{y_1} &= \frac{g_e(q - e^2)}{2}\sin 2\varphi'\left(1 - \frac{4h}{a} - \frac{qe^2}{q - e^2} - \frac{e^2}{2}\frac{e^2 + 2q}{q - e^2}\sin^2\varphi'\right), \\
g_{z_1} &= -g_e\left[1 - \frac{e^2}{2}\sin^2\varphi' + q\left(1 + \frac{1}{3}\sin^2\varphi'\right) + \right. \\
&\quad + e^4\left(-\frac{1}{8}\sin^2\varphi' - \frac{3}{32}\sin^2 2\varphi'\right) + \\
&\quad + e^2 q\left(-\frac{17}{28}\sin^2\varphi' + \frac{1}{16}\sin^2 2\varphi'\right) + \\
&\quad \left. + \frac{h}{a}e^2(3\sin^2\varphi' - 1) - \frac{hq}{a}(1 + 6\sin^2\varphi') - \frac{2h}{a} + \frac{3h^2}{a^2}\right].
\end{aligned}\right\} \tag{1.30}$$

Now, using Eq. (1.15) and matrix (1.14), we find

$$\left.\begin{aligned}
g_{y_1} &= g_e \sin 2\varphi' \left[\frac{q}{2}\left(1+\frac{1}{2}e^2\sin^2\varphi'\right)+\frac{h}{a}\left(\frac{1}{2}e^2-2q\right)\right],\\
g_{z_1} &= -g_e\left[1-\frac{1}{2}e^2\sin^2\varphi'+q\left(1+\frac{3}{2}\sin^2\varphi'\right)+\right.\\
&+e^4\left(-\frac{1}{8}\sin^2\varphi'+\frac{1}{32}\sin^2 2\varphi'\right)+\\
&+e^2q\left(-\frac{17}{28}\sin^2\varphi'-\frac{3}{16}\sin^2 2\varphi'\right)+\\
&\left.+\frac{h}{a}e^2(3\sin^2\varphi'-1)-\frac{hq}{a}(1+6\sin^2\varphi')-\frac{2h}{a}+\frac{3h^2}{a^2}\right].
\end{aligned}\right\} \qquad (1.31)$$

For $h=0$, Eqs. (1.31) give the projections g_{y_1}, g_{z_1} on the Earth's surface (on the surface of the level ellipsoid). If we now add the y_1 and z_1 projections of the centrifugal acceleration due to the Earth's spin the first sum vanishes, and the second leads to the Helmert—Cassinis expression for the normal force of gravity:*

$$g = g_e(1+\beta\sin^2\varphi'+\beta_1\sin^2 2\varphi'), \qquad (1.32)$$

where the coefficients β and β_1 are expressed in the standard Clairaut form with second order terms:

$$\beta = \frac{5}{2}q-\alpha-\frac{17}{14}q\alpha,$$
$$\beta_1 = \frac{\alpha^2}{8}-\frac{5}{8}q\alpha, \quad \alpha = \frac{a-b}{a}.$$

1.1.4. Ideal operation equations of inertial systems measuring the Cartesian coordinates of the object

Consider an inertial navigation system comprising three accelerometers which are mounted on the platform of a three-degree absolute angular velocity meter.** Their sensitive axes point along the axes of a right orthogonal system $Oxyz$ fixed to the platform (Figure 1.6).

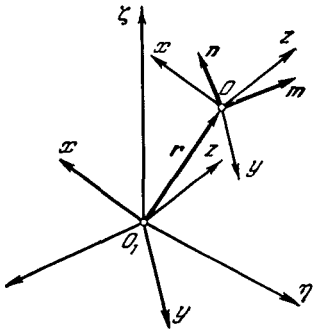

FIGURE 1.6.

* See footnote on p. 7.

** Andreev, V.D. Theory of Inertial Navigation: Autonomous Systems. — Nauka. 1966.

Let the inertial system measure the Cartesian coordinates ξ, η, ζ in a frame $O_1\xi\eta\zeta$ fixed to the Earth, the rates of change of these coordinates $\dot{\xi}, \dot{\eta}, \dot{\zeta}$, and the orientation parameters of the object relative to the axes ξ, η, ζ. Let n_x, n_y, n_z be the accelerometer readings, m_x, m_y, m_z the readings of the absolute angular velocity meters. We introduce the vectors

$$\left.\begin{aligned} \boldsymbol{n} &= n_x\boldsymbol{x} + n_y\boldsymbol{y} + n_z\boldsymbol{z}, \\ \boldsymbol{m} &= \boldsymbol{\omega} = m_x\boldsymbol{x} + m_y\boldsymbol{y} + m_z\boldsymbol{z}, \end{aligned}\right\} \tag{1.33}$$

where $\boldsymbol{\omega}$ is the absolute angular velocity of the platform, i.e., of the axes $Oxyz$, and $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ are the unit vectors along these axes.

We introduce a coordinate system O_1xyz with its origin at the Earth's center O_1, and axes parallel to the corresponding axes of the $Oxyz$ system. Clearly,

$$\boldsymbol{r} = \xi\boldsymbol{\xi} + \eta\boldsymbol{\eta} + \zeta\boldsymbol{\zeta} = x\boldsymbol{x} + y\boldsymbol{y} + z\boldsymbol{z}. \tag{1.34}$$

Writing $\boldsymbol{v} = d\boldsymbol{r}/dt$, we find

$$\left.\begin{aligned} \boldsymbol{v} &= \dot{\boldsymbol{r}} + \boldsymbol{\omega} \times \boldsymbol{r}, \\ \frac{d\boldsymbol{r}}{dt} &= \frac{d\boldsymbol{v}}{dt} = \dot{\boldsymbol{v}} + \boldsymbol{\omega} \times \boldsymbol{v}, \end{aligned}\right\} \tag{1.35}$$

where dot denotes differentiation in the coordinate system O_1xyz.

Integration in the O_1xyz frame gives from Eqs. (1.3), (1.33), (1.35)

$$\left.\begin{aligned} \boldsymbol{v} &= \int_0^t (\boldsymbol{n} - \boldsymbol{m} \times \boldsymbol{v} + \boldsymbol{g})\, dt + \boldsymbol{v}^0, \\ \boldsymbol{r} &= \int_0^t (\boldsymbol{v} - \boldsymbol{m} \times \boldsymbol{r})\, dt + \boldsymbol{r}^0. \end{aligned}\right\} \tag{1.36}$$

Eqs. (1.36) specify the coordinates of the object in the O_1xyz frame, if the projections g_x, g_y, g_z are known (e.g., if the Earth's gravitational field is spherical).

To find the coordinates ξ, η, ζ of the object, note that the trihedron O_1xyz rotates relative to the trihedron $O_1\xi\eta\zeta$ with angular velocity

$$\boldsymbol{\omega} - \boldsymbol{u} = \boldsymbol{m} - \boldsymbol{u}, \tag{1.37}$$

where $\boldsymbol{u}$ is the vector absolute angular velocity of the Earth's rotation.

If the initial relative orientation of the trihedra O_1xyz and $O_1\xi\eta\zeta$ is known, their relative orientation at any time (for given vector angular velocity (1.37)) reduces to a solution of standard Poisson's equations

$$\left.\begin{aligned} \boldsymbol{\xi} &= \int_0^t \boldsymbol{\xi} \times (\boldsymbol{m} - \boldsymbol{u})\, dt + \boldsymbol{\xi}^0, \\ \boldsymbol{\eta} &= \int_0^t \boldsymbol{\eta} \times (\boldsymbol{m} - \boldsymbol{u})\, dt + \boldsymbol{\eta}^0, \\ \boldsymbol{\zeta} &= \int_0^t \boldsymbol{\zeta} \times (\boldsymbol{m} - \boldsymbol{u})\, dt + \boldsymbol{\zeta}^0. \end{aligned}\right\} \tag{1.38}$$

Now

$$\xi = \xi \cdot r, \quad \eta = \eta \cdot r, \quad \zeta = \zeta \cdot r, \tag{1.39}$$

and the problem of determination of the coordinates ξ, η, ζ has been solved. The velocities $\dot{\xi}, \dot{\eta}, \dot{\zeta}$ are found in an obvious manner.

The orientation of an object in space is specified in terms of its orientation relative to the $Oxyz$ axes, so that one either has to determine the attitude of the trihedron $Oxyz$ relative to the object or measure the angular velocity of rotation of $Oxyz$ relative to the object and then solve a set of equations analogous to Eqs. (1.38). If the angular velocimeter platform is rigidly fixed to the object, the orientation of the object is found directly from Eqs. (1.38). In what follows the orientation of the object is considered known if the spatial orientation of the sensitive elements of the inertial navigation system has been determined.

In a spherical gravitational field

$$g(r) = -\frac{r}{r^3} g(r)$$

Eqs. (1.36) and (1.38) are solved independently. In fact, however,

$$g = \operatorname{grad} V, \quad V = \frac{\mu}{r} + \varepsilon(\xi, \eta, \zeta), \tag{1.40}$$

where $\varepsilon(\xi, \eta, \zeta)$ is the force function of the small nonspherical component of the gravitational field. Therefore, before g_x, g_y, g_z can be determined we should solve Eqs. (1.38) and then find the simultaneous solution of Eqs. (1.36), (1.39) and (1.40).

Note that in the derivation of Eqs. (1.36), (1.38)—(1.40) we allowed an arbitrary orientation of the angular velocimeter platform (the $Oxyz$ axes) both relative to the Earth and relative to the object. Various particular cases should be considered at this point.

The $Oxyz$ axes may have a fixed attitude in the inertial space (e.g., coinciding with the $O\xi_*\eta_*\zeta_*$ axes). This can be achieved using a gyroscopically-stabilized platform or by hooking up the accelerometers to free gyroscopes. Eqs. (1.36) then reduce to Eqs. (1.6), and Eqs. (1.38) drop out.

Alternatively, the angular velocimeter platform may be rigidly fixed to the object. Since the orientation of the object is quite arbitrary, Eqs. (1.36), (1.38)—(1.40) retain their form.

There is also an intermediate case, when the orientation of the $Oxyz$ frame, with the accelerometers aligned along its axes, is a known function of time and coordinates being measured. This is achieved either by a controlled gyroframe (or a controlled gyroscopically-stabilized platform) or by a special kinematic arrangement mounted on a stabilized platform or connected to free gyroscopes.

The orientation of the $Oxyz$ axes is a known function of time if their attitude relative to the Earth is fixed (e.g., the axes x, y, z and ξ, η, ζ coincide). When a gyroscopically-stabilized platform is used, the accelerometer axes should rotate relative to the platform in accordance with the matrix of direction cosines (1.7). If, on the other hand, a controlled gyroframe is used, the control torques M_y^2, M_x^1, M_x^3 are defined by the equalities

$$M_y^2 = 0, \quad M_x^1 = 0, \quad M_x^3 = -H_3 u, \tag{1.41}$$

which follow from the precession equations of the gyroframe:

$$M_x^1 = -H_2\omega_x, \quad M_y^1 = H_1\omega_y, \quad M_z^1 = -H_1\omega_z. \tag{1.42}$$

Eqs. (1.38), (1.39) drop out, since the coordinates ξ, η, ζ immediately follow from (1.36) in this case.

As an example in which the orientation of the $Oxyz$ axes is a function of the coordinates being measured by the inertial system, consider the case when the axis z points constantly along the radius-vector $\boldsymbol{r}$. Then

$$\left.\begin{aligned} &x = 0, \quad y = 0, \quad z = r, \\ &\boldsymbol{v} = \dot{r}\boldsymbol{z} + (\omega_y x - \omega_x y)\times\boldsymbol{r} \end{aligned}\right| \tag{1.43}$$

and hence

$$v_x = r\omega_y, \quad v_y = -r\omega_x, \quad v_z = \dot{r}. \tag{1.44}$$

The control torques are thus

$$M_x^1 = H_2\frac{v_y}{r}, \quad M_y^1 = H_1\frac{v_x}{r}, \quad M_z^1 = -H_1 m_z. \tag{1.45}$$

In Eqs. (1.36), (1.38) we should take $x = y = 0$, $z = r$, $m_x = -v_y/r$, $m_y = v_x/r$. m_z and M_z^1 are quite arbitrary. This torque can therefore be chosen so as to simplify the equations, e.g.,

$$M_z^1 = 0, \quad m_z = \omega_z = 0. \tag{1.46}$$

1.1.5. Determination of curvilinear coordinates

A useful technique is to replace the coordinates $O_1\xi,\eta,\zeta$ and $O_1\xi\eta\zeta$ with homogeneous coordirates $O_1\xi^1\xi^2\xi^3$ and $O_1\eta^1\eta^2\eta^3$. Let the position of the object O in the coordinate system $O_1\xi^1\xi^2\xi^3$ be specified by some curvilinear, in general non-orthogonal and non-stationary coordinates x^1, x^2, x^3, so that

$$\xi^s = \xi^s(x^1, x^2, x^3, t), \quad \boldsymbol{r} = \xi^s\boldsymbol{\xi}_s, \tag{1.47}$$

where $\boldsymbol{\xi}_s$ are the unit vectors along the axes ξ^s. Here and in what follows summation from 1 to 3 over repeating indices is implied. The non-repeating indices take on the values 1, 2, 3.

Let the Jacobian

$$J = \frac{D(\xi^1, \xi^2, \xi^3)}{D(x^1, x^2, x^3)} \neq 0. \tag{1.48}$$

The inertial system measuring the coordinates x^1, x^2, x^3 is schematically described as follows. It comprises a gyrostabilized platform whose axes (without loss of generality) can be taken to point along the ξ^s axes of the system $O_1\xi^1\xi^2\xi^3$ ($O_1\xi_s\eta_s\zeta_s$). Three accelerometers are mounted on the stabilized platform; the unit vectors of their sensitive axes are $\boldsymbol{e}_s$ and the accelerometer readings are w_{e_s}. The kinematic system is such that it ensures a given dependence of the orientations of the unit vectors $\boldsymbol{e}_s$ on the time t and

the coordinates $\varkappa^s$ measured by the inertial system:

$$\boldsymbol{e}_s = \boldsymbol{e}_s(\varkappa^1, \varkappa^2, \varkappa^3, t). \tag{1.49}$$

The vectors $\boldsymbol{e}_s$ are not coplanar.

Let us express n_{e_s} in terms of the coordinates $\varkappa^k$. We introduce the covariant (fundamental) base vectors*

$$\boldsymbol{r}_s = \frac{\partial \boldsymbol{r}}{\partial \varkappa^s}, \quad \boldsymbol{r} = \xi^k \boldsymbol{\xi}_k. \tag{1.50}$$

Because of (1.48), the vectors $\boldsymbol{r}_s$ are not coplanar. We also introduce the contravariant base vectors $\boldsymbol{r}^s$ and a metric tensor $\mathbf{A}$ in the curvilinear coordinate space $\varkappa^s$. The covariant, contravariant, and mixed components of this tensor will be designated a_{sk}, a^{sk}, and a^k_s, respectively.

From (1.3), (1.4), and (1.40) we now have

$$n_{e_s} = (\ddot{\varkappa}^k + \Gamma^k_{mn}\dot{\varkappa}^m\dot{\varkappa}^n + 2\Gamma^k_{0n}\dot{\varkappa}^n + \Gamma^k_{00} - \operatorname{grad}^l V \eta^k_l)\, e_{sk}. \tag{1.51}$$

Here Γ^k_{mn} are the Christoffel symbols of the second kind, expressed in terms of the components of the metric tensor $\mathbf{A}$; Γ^k_{0n} and Γ^k_{00} are three-index symbols defined by

$$\left.\begin{aligned} \Gamma^s_{00} &= a^{sn}\Gamma_{00,n}, \quad \Gamma_{00,n} = \frac{\partial^2 \boldsymbol{r}}{\partial t^2}\cdot \boldsymbol{r}_n, \\ \Gamma^s_{0k} &= a^{sn}\Gamma_{0k,n}, \quad \Gamma_{0k,n} = \frac{\partial^2 \boldsymbol{r}}{\partial t\, \partial \varkappa^k}\cdot \boldsymbol{r}_n; \end{aligned}\right\} \tag{1.52}$$

$\operatorname{grad}^l V$ and η^k_l are the contravariant components of the vectors $\boldsymbol{g} = \operatorname{grad} V$ and $\boldsymbol{\eta}_l$ in the fundamental base, and e_{sk} are the covariant components of the vectors $\boldsymbol{e}_s$.

We will restrict the discussion to the case when $\boldsymbol{e}_s$ are along the contravariant base vectors $\boldsymbol{r}^s$.** Then

$$\boldsymbol{e}_s = \frac{\boldsymbol{r}^s}{\sqrt{a^{ss}}}; \quad e_{sk} = 0 \text{ for } s \neq k; \quad e_{ss} = \frac{1}{\sqrt{a^{ss}}}. \tag{1.53}$$

Using Eqs. (1.51) and (1.53), we write the first group of ideal operation equations of an inertial system in the form** (no summation over s!):

$$\left.\begin{aligned} \frac{\dot{\varkappa}^s}{\sqrt{a^{ss}}} &= \int_0^t \left\{ n_{e_s} - \frac{1}{\sqrt{a^{ss}}}\left[\dot{\varkappa}^s \frac{d}{dt}\ln\sqrt{a^{ss}} + \Gamma^s_{mn}\dot{\varkappa}^m\dot{\varkappa}^n + \right.\right. \\ &\quad \left.\left. + 2\Gamma^s_{0n}\dot{\varkappa}^n + \Gamma^s_{00} - \operatorname{grad}^l V \eta^s_l\right]\right\} dt + \frac{\dot{\varkappa}^s(0)}{\sqrt{a^{ss}(0)}}, \\ \varkappa^s &= \int_0^t \left(\frac{\dot{\varkappa}^s}{\sqrt{a^{ss}}}\right)\sqrt{a^{ss}}\, dt + \varkappa^s(0). \end{aligned}\right\} \tag{1.54}$$

Eqs. (1.54) and (1.53) should be supplemented by equations for η^k_l and expressions specifying η^l and $\varkappa^s$ as a function of time.

We have

$$\eta^k = \alpha^k_s \xi^s, \quad \eta_l = \alpha^s_l \xi_s. \tag{1.55}$$

* See, e.g., Lur'e, A.I. Analiticheskaya mekhanika (Analytical Mechanics). — Fizmatgiz. 1961.

** The general case is considered in "Theory of Inertial Navigation: Autonomous Systems," Nauka. 1966.

where α_s^k are the direction cosines between the axes ξ^k and η^s (matrix (1.7)). From the second equality in (1.55) we get

$$\eta_l^k = \alpha_{ls} a^{km} \frac{\partial \xi^s}{\partial \varkappa^m}, \quad \alpha_{ls} = \alpha_s^l \tag{1.56}$$

Using the definition of η_l^k and the obvious equality $r = \eta^k \eta_k$, we obtain

$$\eta^k = \frac{1}{2} \eta_k^s \frac{\partial r^2}{\partial \varkappa^s}, \quad r^2 = r \cdot r. \tag{1.57}$$

The direction cosines of the vectors e_s in relation to the axes of the stable platform are

$$e_s \cdot \xi_k = \frac{r^s \cdot \xi_k}{\sqrt{a^{ss}}}. \tag{1.58}$$

Consider a particular case of Eqs. (1.54)—(1.58), when $\varkappa^1$, $\varkappa^2$, $\varkappa^3$ are orthogonal coordinates. In this case, the fundamental base vectors are perpendicular to one another. The contravariant base vectors point along the fundamental base vectors. The metric tensor is thus diagonal, and its diagonal elements are expressed in terms of the Lamé coefficients h_s:

$$a_{ss} = \frac{1}{a^{ss}} = h_s^2. \tag{1.59}$$

For orthogonal coordinates, the only nonvanishing Christoffel symbols Γ_{nm} are either those with $k = m$ or those with $n = m$. Moreover,

$$\left.\begin{aligned} \frac{d}{dt} \ln \sqrt{a^{ss}} &= -\Gamma_{sk}^s \dot{\varkappa}^k - \Gamma_{0s}^s, \\ \Gamma_{0k,\,s} &= -\Gamma_{0s,\,k}. \end{aligned}\right\} \tag{1.60}$$

Eqs. (1.54) are thus rewritten in the form (no summation over s!)

$$\left.\begin{aligned} h_s \dot{\varkappa}^s &= \int_0^t \{ n_{e_s} - h_s [\Gamma_{0s}^s \dot{\varkappa}^s + \Gamma_{ks}^s \dot{\varkappa}^k \dot{\varkappa}^s + \Gamma_{kk}^s (\dot{\varkappa}^k)^2 + \\ &\quad + 2\Gamma_{0k}^s \dot{\varkappa}^k + \Gamma_{00}^s - \operatorname{grad}{}^l V \eta_l^s] \}\, dt + h_s(0)\, \dot{\varkappa}^s(0), \\ \varkappa^s &= \int_0^t (h_s \dot{\varkappa}^s) \frac{1}{h_s}\, dt + \varkappa^s(0), \end{aligned}\right\} \tag{1.61}$$

where the sum is over all $k \neq s$.

Eqs. (1.61) replace Eqs. (1.54) in the case of orthogonal coordinates. Eqs. (1.55), (1.57) remain valid, and Eqs. (1.58), (1.56) are replaced by

$$e_s \cdot \xi_k = \frac{1}{h_s} \frac{\partial \xi_0^k}{\partial \varkappa^s}, \quad \eta_l^k = \alpha_{ls} \frac{\partial \xi_0^s}{\partial \varkappa^k} \frac{1}{h_k^2}. \tag{1.62}$$

In the case of orthogonal curvilinear coordinates $\varkappa^s$, the kinematic system may clearly employ a controlled gyroframe, since the directions r^s (or equivalently r_s) form a rigid trihedron which, in particular, may be made to coincide with the gyroframe platform axes.

To form the control torques (1.42), we require the projections $\omega_{(s)}$ of the absolute angular velocity ω of the trihedron $r_1r_2r_3$ on the directions of the base vectors. The corresponding expressions are*

$$\omega_{(s)} = \sqrt{a^{ss}}\,\epsilon^{snk}\,(\Gamma_{nm,\,k}\dot{\varkappa}^m + \Gamma_{0n,\,k}), \tag{1.63}$$

where $\Gamma_{nm,\,k}$ are Christoffel symbols of the first kind.

The indices s and k in (1.63) are different. The sum is only over n. As we have noted before, the only nonvanishing Christoffel symbols in the case of orthogonal coordinates are those with $k=m$ or $n=m$. The Levi-Civita symbols are $\epsilon^{snk} = \pm 1/h_s h_n h_k$, where the sign is determined by the order of the indices s, n, k. Using Eqs. (1.59), we obtain from (1.63) (taking $snk = 123, 231, 312$)

$$\left.\begin{aligned}
\omega_{(1)} &= \frac{1}{h_2h_3}\,(\Gamma_{2m,\,3}\dot{\varkappa}^m + \Gamma_{02,\,3}),\\
\omega_{(2)} &= \frac{1}{h_1h_3}\,(\Gamma_{3m,\,1}\dot{\varkappa}^m + \Gamma_{03,\,1}),\\
\omega_{(3)} &= \frac{1}{h_1h_2}\,(\Gamma_{1m,\,2}\dot{\varkappa}^m + \Gamma_{01,\,2}).
\end{aligned}\right\} \tag{1.64}$$

If the coordinates $\varkappa^s$ determine the position of the object in the $O_1\eta^1\eta^2\eta^3$ axes fixed to the Earth, they are stationary relative to the Earth, so that

$$\eta^s = \eta^s(\varkappa^1, \varkappa^2, \varkappa^3) \tag{1.65}$$

and by matrix (1.7)

$$\left.\begin{aligned}
\xi^1 &= \eta^1\cos ut - \eta^2\sin ut,\\
\xi^2 &= \eta^1\sin ut + \eta^2\cos ut,\\
\xi^3 &= \eta^3.
\end{aligned}\right\} \tag{1.66}$$

In this case, when calculating the metric tensor elements, the Lamé coefficients, and the Christoffel symbols, we may take $t=0$, since the coordinate grid $\varkappa^1\varkappa^2\varkappa^3$ moves (rotates with angular velocity ω) as one whole and the properties of the space described by the curvilinear coordinates $\varkappa^1, \varkappa^2, \varkappa^3$ are independent of time. For $\Gamma_{0k,\,s}$ and $\Gamma_{00,\,s}$, using Eqs. (1.52), (1.65), and (1.66), we also obtain time-independent expressions in this case:

$$\left.\begin{aligned}
\Gamma_{0k,\,s} &= u\left(\frac{\partial\eta^1}{\partial\varkappa^k}\frac{\partial\eta^2}{\partial\varkappa^s} - \frac{\partial\eta^2}{\partial\varkappa^k}\frac{\partial\eta^1}{\partial\varkappa^s}\right),\\
\Gamma_{00,\,s} &= -\frac{u^2}{2}\frac{\partial}{\partial\varkappa^s}[(\eta^1)^2 + (\eta^2)^2].
\end{aligned}\right\} \tag{1.67}$$

One further point deserves special mention. In the case of curvilinear orthogonal coordinates, the first three equations of (1.61) can be replaced by the first vector equation in (1.36). Projecting this equation on the directions r_2, r_3, r_1, we get

$$v_{(s)} = \int_0^t \left[n_{(s)} - \omega_{(l)}v_{(k)}\,\epsilon_{lks}\,\frac{1}{h_lh_kh_s} + g_{(s)}\right]dt + v_{(s)}(0). \tag{1.67a}$$

* Andreev, V.D. Unperturbed operation equations of an inertial system measuring curvilinear coordinates. — Prikladnaya Matematika i Mekhanika, Vol. 29, No. 5. 1965.

Now (no summation over s!)

$$v_{(s)} = h_s \left(\dot{\varkappa}^s + \frac{\partial \boldsymbol{r}}{\partial t} \cdot \boldsymbol{r}^s \right), \tag{1.68}$$

whence

$$\varkappa^s = \int_0^t \left[\frac{v_{(s)}}{h_s} - \frac{\partial \boldsymbol{r}}{\partial t} \cdot \boldsymbol{r}^s \right] dt + \varkappa^s(0). \tag{1.69}$$

These equations can replace the last three equations in (1.61).

Using the definition of the contravariant vectors and Eqs. (1.65), (1.66), we get

$$\frac{\partial \boldsymbol{r}}{\partial t} \cdot \boldsymbol{r}^s = \frac{u}{h_s^2} \left(\eta^1 \frac{\partial \eta^2}{\partial \varkappa^s} - \eta^2 \frac{\partial \eta^1}{\partial \varkappa^s} \right). \tag{1.70}$$

Note that if Eqs. (1.63), (1.68) are inserted for $\omega_{(s)}$, $v_{(s)}$ in (1.67a), simple manipulations lead to the first three equations in (1.69).

1.1.6. The ideal operation equations in orthodromic, geocentric, and geographical coordinates

The orthodromic coordinates r, S, z are defined by Eqs. (1.17) and the matrix of direction cosines (1.16). If we take $\varkappa^1 = r$, $\varkappa^2 = S$, $\varkappa^3 = z$, we obtain for the Lamé coefficients

$$h_1 = 1, \quad h_2 = r \cos z, \quad h_3 = r. \tag{1.71}$$

Using (1.70), (1.71) and matrix (1.16), we get

$$\left.\begin{aligned} &\frac{\partial \boldsymbol{r}}{\partial t} \cdot \boldsymbol{r}^1 = 0, \\ &\frac{\partial \boldsymbol{r}}{\partial t} \cdot \boldsymbol{r}^2 = \frac{u}{\cos z} (\beta_{13} \cos z - \beta_{32} \sin z \sin S - \beta_{31} \sin z \cos S), \\ &\frac{\partial \boldsymbol{r}}{\partial t} \cdot \boldsymbol{r}^3 = u (\beta_{21} \sin S - \beta_{22} \cos S). \end{aligned}\right\} \tag{1.72}$$

The standard expressions* for the nonvanishing Christoffel symbols in terms of the Lamé coefficients give

$$\left.\begin{aligned} \Gamma_{21,2} = \Gamma_{12,2} = -\Gamma_{22}^1 = -\Gamma_{22}^1 = r \cos^2 z, \\ \Gamma_{21}^2 = \Gamma_{12}^2 = \Gamma_{31}^3 = \Gamma_{13}^3 = \frac{1}{r}, \\ -\Gamma_{23,2} = -\Gamma_{32,2} = \Gamma_{22,3} = r \sin z \cos z, \\ \Gamma_{23}^3 = \Gamma_{32}^2 = -\operatorname{tg} z, \\ \Gamma_{22}^3 = \sin z \cos z, \\ \Gamma_{31,3} = \Gamma_{13,3} = -\Gamma_{33,1} = -\Gamma_{33}^1 = r. \end{aligned}\right\} \tag{1.73}$$

* See, e.g., Lur'e, A.I. Analiticheskaya mekhanika (Analytical Mechanics). — Fizmatgiz, 1961.

According to (1.17), (1.67) and matrix (1.16), the following symbols are also different from zero:

$$\left.\begin{aligned}
&\Gamma_{00,1} = -ru^2[1-(\beta_{31}\cos z\cos S+\beta_{32}\cos z\sin S+\beta_{33}\sin z)^2],\\
&\Gamma_{00,2} = r^2u^2\cos z(\beta_{31}\cos z\cos S+\beta_{32}\cos z\sin S+\\
&\qquad +\beta_{33}\sin z)(-\beta_{31}\sin S+\beta_{32}\cos S),\\
&\Gamma_{00,3} = r^2u^2(\beta_{31}\cos z\cos S+\beta_{32}\cos z\sin S+\beta_{33}\sin z)\times\\
&\qquad \times(-\beta_{31}\sin z\cos S-\beta_{32}\sin z\sin S+\beta_{33}\cos z),\\
&\Gamma_{01,2} = -\Gamma_{02,1} = ru\cos z(-\beta_{31}\sin z\cos S-\beta_{32}\sin z\sin S+\beta_{33}\cos z),\\
&\Gamma_{01,3} = -\Gamma_{03,1} = ru(\beta_{31}\sin S-\beta_{32}\cos S),\\
&\Gamma_{02,3} = -\Gamma_{03,2} = r^2u\cos z(\beta_{31}\cos z\cos S+\beta_{32}\cos z\sin S+\beta_{33}\sin z).
\end{aligned}\right\} \tag{1.74}$$

Now, using Eqs. (1.52), (1.71), and (1.59), we get

$$\left.\begin{aligned}
&\Gamma_{00}^1 = \Gamma_{00,1},\quad \Gamma_{00}^2 = \frac{1}{r^2\cos^2 z}\Gamma_{00,2},\quad \Gamma_{00}^3 = \frac{1}{r^2}\Gamma_{00,3},\\
&\Gamma_{01}^2 = \frac{1}{r^2\cos^2 z}\Gamma_{01,2},\quad \Gamma_{01}^3 = \frac{1}{r^2}\Gamma_{01,3},\\
&\Gamma_{02}^1 = -\Gamma_{01,2},\quad \Gamma_{03}^1 = -\Gamma_{01}^3,\\
&\Gamma_{02}^3 = \frac{1}{r^2}\Gamma_{02,3},\quad \Gamma_{03}^2 = -\frac{1}{r^2\cos^2 z}\Gamma_{02,3}.
\end{aligned}\right\} \tag{1.75}$$

For the projections $\omega_{(s)}$ we have from (1.64), (1.71), (1.73), and (1.74):

$$\left.\begin{aligned}
&\omega_{(1)} = \dot S\sin z+u(\beta_{31}\cos z\cos S+\beta_{32}\cos z\sin S+\beta_{33}\sin z),\\
&\omega_{(2)} = -\dot z-u(\beta_{31}\sin S-\beta_{32}\cos S),\\
&\omega_{(3)} = \dot S\cos z+u(-\beta_{31}\sin z\cos S-\beta_{32}\sin z\sin S+\beta_{33}\cos z).
\end{aligned}\right\} \tag{1.76}$$

The projections $v_{(s)}$ are obtained from (1.68), (1.71), (1.72), and (1.76):

$$v_{(1)} = \dot r,\quad v_{(2)} = r\omega_{(3)},\quad v_{(3)} = -r\omega_{(2)}. \tag{1.77}$$

Comparison of the first and third expressions in (1.76) gives a further equality

$$\omega_{(1)} = \omega_{(3)}\operatorname{tg} z+\frac{u}{\cos z}(\beta_{31}\cos S+\beta_{32}\sin S). \tag{1.78}$$

Let us find the projections $g_{(s)}$. The vectors $\boldsymbol{r}_2$, $\boldsymbol{r}_3$, $\boldsymbol{r}_1$ are directed in the present case along the axes of the co-moving orthodromic trihedron $Ox_3y_3z_3$, defined in 1.1.2. Therefore,

$$g_{(1)} = g_{z_3},\quad g_{(2)} = g_{x_3},\quad g_{(3)} = g_{y_3}, \tag{1.79}$$

or, using (1.19), (1.20),

$$\left.\begin{aligned}
&g_{(1)} = g_{z_2}(r,\varphi),\\
&g_{(2)} = g_{y_2}(r,\varphi)\frac{1}{\cos\varphi}(-\beta_{31}\sin S+\beta_{32}\cos S),\\
&g_{(3)} = g_{y_2}(r,\varphi)\frac{1}{\cos\varphi}(-\beta_{31}\sin z\cos S-\beta_{32}\sin z\sin S+\beta_{33}\cos z).
\end{aligned}\right\} \tag{1.80}$$

where $g_{y_2}(r,\varphi)$, $g_{z_2}(r,\varphi)$ are expressed by (1.26) or (1.27). The geocentric latitude φ entering these equalities is expressed in terms of z and S by the third equation in (1.18)

Turning to Eqs. (1.67), (1.69) and using the above intermediate relations, we obtain the following ideal operation equations for an inertial navigation system which measures orthodromic coordinates and uses a controlled gyroframe:

$$
\left.
\begin{aligned}
v_{x_3} &= \int_0^t \Big[n_{x_3} + v_{y_3}\omega_{z_3} - \dot{r}\omega_{y_3} + \\
&\quad + g_{y_3}(r, \varphi) \frac{1}{\cos\varphi} (-\beta_{31}\sin S + \beta_{32}\cos S) \Big] dt + v_{x_3}(0), \\
v_{y_3} &= \int_0^t \Big[n_{y_3} - v_{x_3}\omega_{z_3} + \dot{r}\omega_{x_3} + \\
&\quad + g_{y_3}(r, \varphi) \frac{1}{\cos\varphi} (-\beta_{31}\sin z\cos S - \\
&\quad - \beta_{32}\sin z\sin S + \beta_{33}\cos z) \Big] dt + v_{y_3}(0), \\
\dot{r} &= \int_0^t [n_{z_3} + v_{x_3}\omega_{y_3} - v_{y_3}\omega_{x_3} + g_{z_3}(r, \varphi)]\, dt + \dot{r}(0), \\
r &= \int_0^t \dot{r}\, dt + r(0), \\
\omega_{x_3} &= -\frac{v_{y_3}}{r}, \quad \omega_{y_3} = \frac{v_{x_3}}{r}, \\
\omega_{z_3} &= \omega_{y_3}\operatorname{tg} z + \frac{u}{\cos z}(\beta_{31}\cos S + \beta_{32}\sin S), \\
S &= \int_0^t \frac{1}{\cos z} [\omega_{y_3} + u(\beta_{31}\sin z\cos S + \\
&\quad + \beta_{32}\sin z\sin S - \beta_{33}\cos z)]\, dt + S(0), \\
z &= -\int_0^t [\omega_{x_3} + u(\beta_{31}\sin S - \beta_{32}\cos S)]\, dt + z(0), \\
\sin\varphi &= \beta_{31}\cos z\cos S + \beta_{32}\cos z\sin S + \beta_{33}\sin z, \\
M_y^2 &= H_2\omega_{x_3}, \quad M_x^1 = H_1\omega_{y_3}, \quad M_x^3 = -H_3\omega_{z_3}.
\end{aligned}
\right\} \tag{1.81}
$$

The subscripts 2, 3, 1 have been consistently replaced by x_3, y_3, z_3, i.e., the notation of the co-moving geocentric axes. A block diagram of the inertial system corresponding to ideal operation equations (1.81) is shown in Figure 1.7.

If the inertial system uses a gyroscopically-stabilized platform, the expressions for the control torques M_y^2, M_x^1, M_x^3 are omitted from Eqs. (1.81) and a matrix of the direction cosines between the axes x_3, y_3, z_3 and the axes ξ^1, ξ^2, ξ^3 of the stabilized platform is added. These direction cosines are obtained either from the first group of relations in (1.62) or directly from the definition of the trihedron $Ox_3y_3z_3$.

Eqs. (1.81) were obtained using Eqs. (1.67), (1.69). They can be replaced with Eqs. (1.61), seeing that

$$h_s \operatorname{grad}^i V\eta_i^s = g_{(s)}. \tag{1.82}$$

Eqs. (1.81) for orthodromic coordinates give as a particular case the ideal work equations in geocentric coordinates. To this end, z and S are replaced with φ and λ, putting $\beta_{ii} = 1$, $\beta_{ij} = 0$ for $i \neq j$, and replacing the

subscripts x_3, y_3, z_3 with the subscripts x_2, y_2, z_2, in accordance with the notation of the geocentric co-moving axes introduced in 1.1.2.

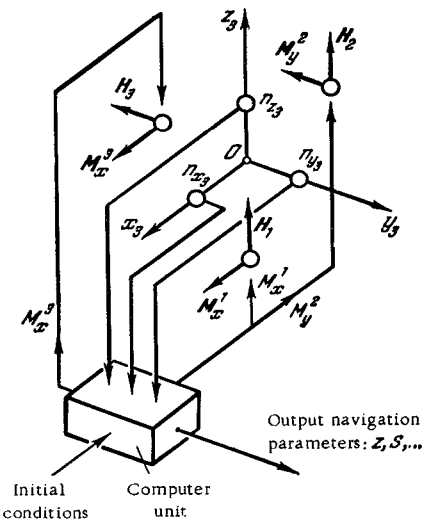

FIGURE 1.7.

Let us now derive the ideal operation equations for the geographical coordinates. The geographical coordinates are defined by Eqs. (1.8a). Taking $\varkappa^1 = h$, $\varkappa^2 = \lambda$, $\varkappa^3 = \varphi'$, we find the corresponding Lamé coefficients:

$$\left.\begin{aligned} h_1 &= 1, \\ h_2 &= \left(\frac{a}{\sqrt{1 - e^2 \sin^2 \varphi'}} + h\right) \cos \varphi', \\ h_3 &= \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \varphi')^{3/2}} + h. \end{aligned}\right\} \tag{1.83}$$

Now from (1.8a) and (1.70) we get

$$\frac{\partial r}{\partial t} \cdot r^1 = 0, \quad \frac{\partial r}{\partial t} \cdot r^2 = u, \quad \frac{\partial r}{\partial t} \cdot r^3 = 0. \tag{1.84}$$

For the nonvanishing Christoffel symbols we have

$$\left.\begin{aligned} -\Gamma_{22,1} &= -\Gamma_{22}^1 = \Gamma_{12,2} = \Gamma_{21,2} = h_2 \cos \varphi', \\ -\Gamma_{33,1} &= -\Gamma_{33}^1 = \Gamma_{31,3} = \Gamma_{13,3} = h_3, \\ \Gamma_{22}^3 &= -\Gamma_{23,2} = -\Gamma_{32,2} = h_2 h_3 \sin \varphi', \\ \Gamma_{12}^2 &= \Gamma_{21}^2 = \frac{1}{h_2} \cos \varphi', \quad \Gamma_{23}^2 = \Gamma_{32}^2 = -\frac{h_3}{h_2} \sin \varphi', \\ \Gamma_{31}^3 &= \Gamma_{13}^3 = \frac{1}{h_3}, \qquad \Gamma_{22}^3 = \frac{h_2}{h_3} \sin \varphi', \\ \Gamma_{33,3} &= \frac{3ae^2(1 - e^2) h_3 \sin \varphi' \cos \varphi'}{(1 - e^2 \sin^2 \varphi')^{5/2}}, \\ \Gamma_{33}^3 &= \frac{3ae^2(1 - e^2) \sin \varphi' \cos \varphi'}{h_3 (1 - e^2 \sin^2 \varphi')^{5/2}}. \end{aligned}\right\} \tag{1.85}$$

According to Eqs. (1.52), (1.59), (1.83), and (1.67), other Christoffel symbols which are different from zero include

$$\left.\begin{aligned}
&\Gamma_{(2)(1)}^{1}=\Gamma_{(1)(2)}^{1}=-u^{2}h_{2}\cos\varphi', \quad \Gamma_{(3)(1)}^{1}=u^{2}h_{2}h_{3}\sin\varphi',\\
&\Gamma_{(1)(3)}^{3}=u^{2}\frac{h_{2}}{h_{3}}\sin\varphi',\\
&\Gamma_{(1)(1)}^{2}=-\Gamma_{(2)(1)}^{1}=-\Gamma_{(1)(2)}^{1}=uh_{2}\cos\varphi',\\
&\Gamma_{(2)(3)}^{3}=-\Gamma_{(3)(2)}^{2}=uh_{2}h_{3}\sin\varphi',\\
&\Gamma_{(3)(1)}^{1}=\frac{1}{h_{2}}u\cos\varphi',\\
&\Gamma_{(1)(2)}^{2}=u\frac{h_{2}}{h_{3}}\sin\varphi', \quad \Gamma_{(3)(2)}^{3}=-u\frac{h_{3}}{h_{2}}\sin\varphi'.
\end{aligned}\right\} \tag{1.86}$$

The projections $\omega_{(i)}$ of the absolute angular velocity of the trihedron $r_1r_2r_3$ on its axes are obtained using Eqs. (1.64), (1.83), (1.85), and (1.86):

$$\left.\begin{aligned}
\omega_{(1)}&=(\dot{\lambda}+u)\sin\varphi', \quad \omega_{(2)}=-\dot{\varphi}',\\
\omega_{(3)}&=(\dot{\lambda}+u)\cos\varphi'.
\end{aligned}\right\} \tag{1.87}$$

From (1.68), (1.84), and (1.87), we further have

$$\left.\begin{aligned}
v_{(1)}&=\dot{h}, \quad v_{(2)}=\left(\frac{a}{\sqrt{1-e^{2}\sin^{2}\varphi'}}+h\right)\omega_{(3)}=r_{2}\omega_{(3)},\\
v_{(3)}&=-\left[\frac{a(1-e^{2})}{(1-e^{2}\sin^{2}\varphi')^{3/2}}+h\right]\omega_{(2)}=-r_{3}\omega_{(2)},
\end{aligned}\right\} \tag{1.88}$$

where r_2, r_3 are the radii of curvature (1.11) of the principal normal sections of the surface $h = \text{const}$.

Comparison of the first and third relations in (1.87) gives

$$\omega_{(1)}=\omega_{(3)}\operatorname{tg}\varphi' \tag{1.89}$$

It now remains to find the projections $g_{(1)}$, $g_{(2)}$, $g_{(3)}$ of the Earth's gravitational field which enter the integrands in (1.67a). If the gravitational field is expressed by expansion (1.27), we have, using the matrix of direction cosines (1.14),

$$\left.\begin{aligned}
g_{(1)}&=g_{z_1}=g_{z_2}\cos(\varphi'-\varphi)-g_{y_2}\sin(\varphi'-\varphi),\\
g_{(2)}&=g_{y_1}=g_{z_2}\sin(\varphi'-\varphi)+g_{y_2}\cos(\varphi'-\varphi),\\
g_{(3)}&=g_{x_1}=0
\end{aligned}\right\} \tag{1.90}$$

Now, using Eqs. (1.67a), (1.69), we obtain the ideal operation equations of an inertial navigation system which measures the geographic coordinates:

$$
\left.
\begin{aligned}
v_{x_1} &= \int_0^t (n_{x_1} + v_{y_1}\omega_{z_1} - \dot{h}\omega_{y_1})\,dt + v_{x_1}(0),\\
v_{y_1} &= \int_0^t (n_{y_1} + \dot{h}\omega_{x_1} - v_{x_1}\omega_{z_1} + g_{y_1})\,dt + v_{y_1}(0),\\
\dot{h} &= \int_0^t (n_{z_1} + v_{x_1}\omega_{y_1} - v_{y_1}\omega_{x_1} + g_{z_1})\,dt + \dot{h}(0),\\
h &= \int_0^t \dot{h}\,dt + h(0),\\
\varphi' &= -\int_0^t \omega_{x_1}dt + \varphi(0),\quad \lambda = \int_0^t \left(\frac{\omega_{y_1}}{\cos\varphi'} - u\right)dt + \lambda(0),\\
\omega_{y_1} &= \frac{v_{x_1}}{r_2},\quad \omega_{x_1} = -\frac{v_{y_1}}{r_3},\quad \omega_{z_1} = \omega_{y_1}\operatorname{tg}\varphi',\\
r_2 &= \frac{a}{\sqrt{1 - e^2\sin^2\varphi'}} + h,\\
r_3 &= \frac{a(1-e^2)}{(1 - e^2\sin^2\varphi')^{3/2}} + h,\\
M_y^2 &= H_2\omega_x,\quad M_x^1 = H_1\omega_{y_1},\quad M_x^4 = -H_3\omega_{z_1},\\
g_{y_1} &= g_{z_2}\sin(\varphi' - \varphi) + g_{y_2}\cos(\varphi' - \varphi),\\
g_{z_1} &= g_{z_2}\cos(\varphi' - \varphi) - g_{y_2}\sin(\varphi' - \varphi).
\end{aligned}
\right\} \qquad (1.91)
$$

In Eqs. (1.91) the subscripts x_1, y_1, z_1 correspond to the notation of the geographical co-moving axes introduced in 1.1.2. The last two relations in (1.91) contain φ and r. These variables are found from (1.13) and (1.15).

§1.2. THE EQUATIONS OF ERRORS FOR INERTIAL NAVIGATION SYSTEMS

1.2.1. Perturbed inertial systems. Main instrumental errors

The ideal operation equations of the previous section only describe the operation of an inertial navigation system with ideal elements and devices (without instrumental errors), when the initial conditions of system operation exactly correspond to the initial conditions of motion of the object.

In real systems these requirements are satisfied only to some approximation. The ideal operation equations no longer describe the actual operation of the system, and the navigation parameters are determined by a system with errors. We are thus dealing with perturbed operation of a navigation system.

The equations for the deviations of the variables describing the state of an inertial system from their values corresponding to the ideal operation

equations will be referred to as the equations of errors. These equations determine the functional stability of the system as a whole. They also provide a relation between the instrumental errors and the inaccuracies in the initial conditions, on the one hand, and the errors of determination of the navigation parameters, on the other. The properties of the error equations thus largely determine the functional precision of the inertial system.

By solving the equations of errors, we find the requirements to be met by the system characteristics if the desired accuracy is to be attained or establish the accuracy with which the navigation parameters are determined by a system with given characteristics. Analysis of the equations of errors further enables us to approach the selection of a functional algorithm of the inertial system. Without analyzing these equations, we can never decide on the applicability of various simplified forms of the ideal operation equations. Finally, investigation of the properties of the error equations is essential for choosing correction techniques and evaluating their relative efficiencies.

Any inertial navigation system generally contains a great number of elements and devices. Each of these components introduces a certain error to system operation. Our aim, however, is not to write error equations for as many individual elements as possible. Conversely, the best policy is to reduce the errors of the various components to a limited number of most typical and characteristic errors.

The principal characteristic instrumental errors in inertial navigation systems are the errors of its sensitive elements — the accelerometers and gyroscopes. In systems of constant structure, the errors of all other elements and devices clearly can be reduced to some equivalent errors in primary input information, i.e., errors of the sensitive elements of the system.

1.2.2. Position error equations

The error equations of an inertial system measuring the Cartesian coordinates of an object (a system of this kind was considered in 1.1.4) are obtained by varying the ideal operation equations (1.36), (1.38)—(1.40). This gives the following set of vector equations:*

$$\frac{d^2\delta r}{dt^2} + \frac{\mu\delta r}{r^3} - \frac{\mu r}{r^5}\frac{3r\cdot\delta r}{r^2} = \Delta n - \Delta m \times \frac{dr}{dt} - \frac{d}{dt}(\Delta m \times r), \tag{1.92}$$

$$\frac{d\theta}{dt} = \Delta m, \tag{1.93}$$

$$\delta r_1 = \theta \times r, \quad \delta r_2 = \delta r + \delta r_1, \tag{1.94}$$

where Δn and Δm are the vector instrumental errors of the accelerometers and the gyroscopes, δr_2 is the vector total error in the position of the object

* Andreev, V.D. General equations of inertial navigation. — Prikladnaya Matematika i Mekhanika, Vol. 28, No. 2, 1964. Andreev, V.D. Theory of Inertial Navigation: Autonomous System. — Nauka, 1966.

in the axes $O_1\xi\eta\zeta$. The derivatives in Eqs. (1.92) and (1.93) are with respect to the fundamental Cartesian axes $O_1\xi_*\eta_*\zeta_*$.

Projecting Eqs. (1.92)—(1.94) onto the xyz axes which rotate with angular velocity ω (these axes may be fixed to the stabilized platform, say), we obtain

$$\left.\begin{aligned}
&\delta\ddot{x} + \left[\frac{\mu}{r^5}(y^2+z^2-2x^2) - \omega_y^2 - \omega_z^2\right]\delta x + \\
&+ \left(\omega_x\omega_y - \dot{\omega}_z - \frac{3\mu xy}{r^5}\right)\delta y - 2\omega_z\delta\dot{y} + \\
&+ \left(\omega_x\omega_z + \dot{\omega}_y - \frac{3\mu xz}{r^5}\right)\delta z + 2\omega_y\delta\dot{z} = \\
&= \Delta m_x - 2(\Delta m_y\dot{z} - \Delta m_z\dot{y}) - \Delta\dot{m}_y z + \Delta\dot{m}_z y - \\
&- \omega_x(\Delta m_y y + \Delta m_z z) - \Delta m_x(\omega_y y + \omega_z z) + \\
&+ 2x(\omega_y\Delta m_y + \omega_z\Delta m_z), \\
&\delta\ddot{y} + \left[\frac{\mu}{r^5}(z^2+x^2-2y^2) - \omega_z^2 - \omega_x^2\right]\delta y + \\
&+ \left(\omega_y\omega_z - \dot{\omega}_x - \frac{3\mu yz}{r^5}\right)\delta z - 2\omega_x\delta\dot{z} + \\
&+ \left(\omega_y\omega_x + \dot{\omega}_z - \frac{3\mu yx}{r^5}\right)\delta x + 2\omega_z\delta\dot{x} = \\
&= \Delta m_y - 2(\Delta m_z\dot{x} - \Delta m_x\dot{z}) - \Delta\dot{m}_z x + \Delta\dot{m}_x z - \\
&- \omega_y(\Delta m_z z + \Delta m_x x) - \Delta m_y(\omega_z z + \omega_x x) + \\
&+ 2y(\omega_z\Delta m_z + \omega_x\Delta m_x), \\
&\delta\ddot{z} + \left[\frac{\mu}{r^5}(x^2+y^2-2z^2) - \omega_x^2 - \omega_y^2\right]\delta z + \\
&+ \left(\omega_z\omega_x - \dot{\omega}_y - \frac{3\mu zx}{r^5}\right)\delta x - 2\omega_y\delta\dot{x} + \\
&+ \left(\omega_z\omega_y + \dot{\omega}_x - \frac{3\mu zy}{r^5}\right)\delta y + 2\omega_x\delta\dot{y} = \\
&= \Delta m_z - 2(\Delta m_x\dot{y} - \Delta m_y\dot{x}) - \Delta\dot{m}_x y + \Delta\dot{m}_y x - \\
&- \omega_z(\Delta m_x x + \Delta m_y y) - \Delta m_z(\omega_x x + \omega_y y) + \\
&+ 2z(\omega_x\Delta m_x + \omega_y\Delta m_y);
\end{aligned}\right\} \tag{1.95}$$

$$\left.\begin{aligned}
\dot{\theta}_x + \omega_y\theta_z - \omega_z\theta_y &= \Delta m_x, \\
\dot{\theta}_y + \omega_z\theta_x - \omega_x\theta_z &= \Delta m_y, \\
\dot{\theta}_z + \omega_x\theta_y - \omega_y\theta_x &= \Delta m_z;
\end{aligned}\right\} \tag{1.96}$$

$$\left.\begin{aligned}
\delta x_1 = \theta_y z - \theta_z y,\ \ \delta y_1 &= \theta_z x - \theta_x z, \\
\delta z_1 &= \theta_x y - \theta_y x, \\
\delta x_2 = \delta x + \delta x_1,\ \ \delta y_2 &= \delta y + \delta y_1, \\
\delta z_2 &= \delta z + \delta z_1.
\end{aligned}\right\} \tag{1.97}$$

The error equations of inertial systems measuring general curvilinear coordinates are also obtained by varying the corresponding ideal operation equations. If the system accelerometers n_{e_s} are oriented along the contra-

variant base vectors r^i, i.e., when the ideal operation equations are in the form (1.54)—(1.58), the first group of error equations takes the form*

$$\begin{aligned}
&\delta\ddot{\varkappa}^k + \frac{\mu}{r^3}\delta\varkappa^k + 2\left(\Gamma_{mj}^k\dot{\varkappa}^m + \Gamma_{0j}^k\right)\delta\dot{\varkappa}^j + \\
&+\left[\dot{\varkappa}^m\dot{\varkappa}^j\left(\frac{\partial}{\partial\varkappa^s}\Gamma_{mn}^k + \Gamma_{mn}^i\Gamma_{si}^k\right) + 2\dot{\varkappa}^n\left(\frac{\partial}{\partial\varkappa^s}\Gamma_{0n}^k + \Gamma_{0n}^i\Gamma_{si}^k\right) + \right. \\
&\left. +\left(\ddot{\varkappa}^i + \Gamma_{ij}^i\dot{\varkappa}^j\right)\Gamma_{si}^k + \frac{\partial}{\partial\varkappa^s}\Gamma_{00}^k - \frac{3\mu}{4r^5}a^{kn}\frac{\partial r_\cdot}{\partial\varkappa^n}\frac{\partial r_\cdot}{\partial\varkappa^s}\right]\delta\varkappa^s = \\
&= \Delta n^k - \epsilon^{ikl}\left\{2\Delta m^p a_{sp}a_{il}\left(\dot{\varkappa}^l + \frac{\partial r}{\partial t}\cdot r^l\right) + \right. \\
&\left. + a_{ns}\left[\Delta\dot{m}^n + \Delta m^l\left(\Gamma_{lm}^n\dot{\varkappa}^m + \Gamma_{0l}^n\right)\right]\frac{1}{2}\frac{\partial r_\cdot}{\partial\varkappa^i}\right\}. \qquad (1.98)
\end{aligned}$$

These equations are fundamentally similar to Eqs. (1.95). Here Δn^k, Δm^n are the contravariant components of the vector instrumental errors Δn, Δm in the fundamental base.

The second group of differential error equations is reduced to

$$\dot{\theta}^k + \theta^l\left(\Gamma_{ij}^k\dot{\varkappa}^j + \Gamma_{0l}^k\right) = \Delta m^k, \qquad (1.99)$$

where θ^k are the contravariant components of the vector θ in the fundamental base. Eqs. (1.99) are clearly equivalent to vector equation (1.93) or to corresponding scalar equations (1.96)

The total errors in the curvilinear coordinates $\varkappa^s$ are obtained from the relations

$$\delta\varkappa_2^k = \delta\varkappa^k + \delta\varkappa_1^k, \qquad \delta\varkappa_1^k = \epsilon^{sik}a_{ns}\theta^n\frac{1}{2}\frac{\partial r^2}{\partial\varkappa^i}, \qquad (1.100)$$

which are analogous to Eqs. (1.94), (1.97).

Eqs. (1.98)—(1.100) and the resulting equations for particular coordinate grids are cumbersome and complex. As a rule, they have no distinct symmetry which could simplify the analysis. These equations in general cannot be simplified by dropping some of the terms, since different terms predominate under different operating conditions. Eqs. (1.98)—(1.100) are linear, but their coefficients, which are determined by the trajectory of the moving object, are complicated functions of time.

Eqs. (1.98)—(1.100) take different forms in different coordinate systems. This circumstance, combined with the dependence of the coefficients of these equations on the trajectory of the object, makes for a virtually unlimited number of alternatives.

Note, however, that generally we do not have to express the error equations in terms of variations of the very coordinates that the inertial system measures. Indeed, one of the main problems in the analysis of an unperturbed inertial system (ideal operation) is the appropriate choice of the coordinate system, and it is in these coordinates that the ideal operation equations are written. The aim of the error equations, on the other hand, is primarily to investigate such system characteristics as stability and dependence of navigation errors on instrumental errors and deviations in the initial setting. To determine these characteristics, we need not

* Andreev, V.D. Equations of errors of an inertial system measuring general curvilinear coordinates of a moving object. — Izvestiya AN SSSR, Mechanics, No.4. 1965.

confine the analysis of the error equations to the coordinates in which the inertial system works; any other suitable coordinate system will do for this purpose. Of course, the instrumental errors and the initial setting errors should be converted to the system of coordinates chosen for the analysis of the error equations.

As Eqs. (1.98)—(1.100) are variational forms of the fundamental equation of inertial navigation (1.3), which is clearly invariant in relation to the particular coordinate system chosen for its solution, they can be transformed without difficulty from one system of axes to another. Furthermore, the tensor form of Eqs. (1.98)—(1.100) also shows that their properties are independent of the particular choice of the coordinate system.

In particular, it can be shown that introduction of the vector error

$$\delta r = \delta\varkappa^k r_k \tag{1.101}$$

reduces Eqs. (1.98)—(1.100) to Eqs. (1.92)—(1.94) or to the corresponding Eqs. (1.95)—(1.97). The relation of the errors δx, δy, δz to $\delta\varkappa^s$ follows from the definition (1.50) of the vectors r_k and the matrix of direction cosines a_{ij} between the axes x, y, z and ξ^1, ξ^2, ξ^3. This relation is expressed in the form

$$\left.\begin{aligned} \delta x &= \frac{\partial\xi^1}{\partial\varkappa^k} a_{s1}\,\delta\varkappa^k, \\ \delta y &= \frac{\partial\xi^2}{\partial\varkappa^k} a_{s2}\,\delta\varkappa^k, \\ \delta z &= \frac{\partial\xi^3}{\partial\varkappa^k} a_{s3}\,\delta\varkappa^k. \end{aligned}\right\} \tag{1.102}$$

For orthogonal curvilinear coordinates $\varkappa^s$, the xyz axes may be represented by the trihedron $r_2 r_3 r_1$. For orthodromic coordinates we have in this case from (1.102)

$$\delta x = r\,\delta S\cos z, \qquad \delta y = r\,\delta z, \qquad \delta z = \delta r. \tag{1.103}$$

Similar expressions are obtained for geocentric coordinates: it suffices to replace S and z in (1.103) with λ and φ.

For the geographical coordinates we get

$$\left.\begin{aligned} \delta x &= \delta\lambda\cos\varphi'\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}+h\right), \\ \delta y &= \delta\varphi'\left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{3/2}}+h\right], \\ \delta z &= \delta h. \end{aligned}\right\} \tag{1.104}$$

1.2.3. Alternative forms of the position error equations

In what follows, Eqs. (1.92)—(1.94) or the equivalent scalar equations (1.95)—(1.97) are generally used as the coordinate error equations.

Eqs. (1.95)—(1.97) are obtained by projecting the vector equations (1.92)—(1.94) onto the axes of some quite arbitrary trihedron xyz rotating with

absolute angular velocity ω. Error equations (1.95)—(1.97) are often analyzed using xyz axes of some special orientation.

Let the xyz axes have invariable orientation in space. In Eqs. (1.95)—(1.97) we should then put $\omega_x = \omega_y = \omega_z = 0$, and Eqs. (1.95)—(1.96) take a relatively simple form

$$\left.\begin{aligned}
&\delta\ddot{x} + \frac{\mu}{r^5}(y^2 + z^2 - 2x^2)\,\delta x - 3xy\,\delta y - 3xz\,\delta z = \\
&\quad = \Delta n_x - 2(\Delta m_y \dot{z} - \Delta m_z \dot{y}) - \Delta\dot{m}_y z + \Delta\dot{m}_z y, \\
&\delta\ddot{y} + \frac{\mu}{r^5}(z^2 + x^2 - 2y^2)\,\delta y - 3yz\,\delta z - 3yx\,\delta x = \\
&\quad = \Delta n_y - 2(\Delta m_z \dot{x} - \Delta m_x \dot{z}) - \Delta\dot{m}_z x + \Delta\dot{m}_x z, \\
&\delta\ddot{z} + \frac{\mu}{r^5}(x^2 + y^2 - 2z^2)\,\delta z - 3zx\,\delta x - 3zy\,\delta y = \\
&\quad = \Delta n_z - 2(\Delta m_x \dot{y} - \Delta m_y \dot{x}) - \Delta\dot{m}_x y + \Delta\dot{m}_y x;
\end{aligned}\right\} \tag{1.105}$$

$$\dot{\theta}_x = \Delta m_x, \quad \dot{\theta}_y = \Delta m_y, \quad \dot{\theta}_z = \Delta m_z. \tag{1.106}$$

Eqs. (1.97) are not affected

If xyz are the co-moving axes on a sphere around the Earth and the axis z is directed along the vector $\boldsymbol{r}$, we should take $x = y = 0$, $z = r$ in Eqs. (1.95), (1.97), which give

$$\left.\begin{aligned}
&\delta\ddot{x} + \left(\frac{\mu}{r^3} - \omega_y^2 - \omega_z^2\right)\delta x + (\omega_x\omega_y - \dot{\omega}_z)\,\delta y - \\
&\quad - 2\omega_z\,\delta\dot{y} + (\omega_x\omega_z + \dot{\omega}_y)\,\delta z + 2\omega_y\,\delta\dot{z} = \\
&\quad = \Delta n_x - 2\,\Delta m_y \dot{r} - \Delta\dot{m}_y r - \Delta m_z r\omega_x - \Delta m_x r\omega_z, \\
&\delta\ddot{y} + \left(\frac{\mu}{r^3} - \omega_z^2 - \omega_x^2\right)\delta y + (\omega_y\omega_z - \dot{\omega}_x)\,\delta z - \\
&\quad - 2\omega_x\,\delta\dot{z} + (\omega_y\omega_x + \dot{\omega}_z)\,\delta x + 2\omega_z\,\delta\dot{x} = \\
&\quad = \Delta n_y + 2\,\Delta m_x \dot{r} + \Delta\dot{m}_x r - \Delta m_z r\omega_y - \Delta m_y r\omega_z, \\
&\delta\ddot{z} - \left(2\frac{\mu}{r^3} + \omega_x^2 + \omega_y^2\right)\delta z + (\omega_z\omega_x - \dot{\omega}_y)\,\delta x - \\
&\quad - 2\omega_y\,\delta\dot{x} + (\omega_z\omega_y + \dot{\omega}_x)\,\delta y + 2\omega_x\,\delta\dot{y} = \\
&\quad = \Delta n_z + 2r(\Delta m_x\omega_x + \Delta m_y\omega_y);
\end{aligned}\right\} \tag{1.107}$$

$$\left.\begin{aligned}
\delta x_1 &= \theta_y r, & \delta y_1 &= -\theta_x r, & \delta z_1 &= 0, \\
\delta x_2 &= \delta x + \delta x_1, & \delta y_2 &= \delta y + \delta y_1, & \delta z_2 &= \delta z.
\end{aligned}\right\} \tag{1.108}$$

Eqs. (1.96) are not affected

Eqs. (1.107), (1.108), (1.96) are the projections of Eqs. (1.92)—(1.94) on the co-moving axes xyz. The orientation of the axes x and y in the horizontal plane remains quite arbitrary, and we may consider various particular cases.

The xyz axes, for example, can be identified with the orthodromic or geocentric axes. The axes x and y can be directed so as to make $\omega_z = 0$.

Eqs. (1.95), (1.96) then take the form

$$\left.\begin{aligned}
&\delta\ddot{x} + \left(\frac{\mu}{r^3} - \omega_y^2\right)\delta x + \omega_x\omega_y\,\delta y + \dot{\omega}_y\,\delta z + 2\omega_y\,\delta\dot{z} = \\
&\qquad = \Delta n_x - 2\,\Delta m_y\dot{r} - \Delta\dot{m}_y r - \Delta m_z r\omega_x, \\
&\delta\ddot{y} + \left(\frac{\mu}{r^3} - \omega_x^2\right)\delta y - \dot{\omega}_x\,\delta z - 2\omega_x\,\delta\dot{z} + \omega_y\omega_x\,\delta x = \\
&\qquad = \Delta n_y + 2\,\Delta m_x\dot{r} + \Delta\dot{m}_x r - \Delta m_z r\omega_y, \\
&\delta\ddot{z} - \left(2\,\frac{\mu}{r^3} + \omega_x^2 + \omega_y^2\right)\delta z - \dot{\omega}_y\,\delta x - 2\omega_y\,\delta\dot{x} + \\
&\quad + \dot{\omega}_x\,\delta y + 2\omega_x\delta\dot{y} = \Delta n_z + 2r\,(\Delta m_x\omega_x + \Delta m_y\omega_y);
\end{aligned}\right\} \tag{1.109}$$

$$\left.\begin{aligned}
\dot{\theta}_x + \omega_y\theta_z &= \Delta m_x, \\
\dot{\theta}_y - \omega_x\theta_z &= \Delta m_y, \\
\dot{\theta}_z + \omega_x\theta_y - \omega_y\theta_x &= \Delta m_z.
\end{aligned}\right\} \tag{1.110}$$

Eqs. (1.97) reduce to Eqs. (1.108).

The coefficients in Eqs. (1.109), (1.110) contain only ω_x, ω_y, which are related to v_x, v_y by the equalities

$$\omega_x = -\frac{v_y}{r}, \qquad \omega_y = \frac{v_x}{r}. \tag{1.111}$$

Since the projections v_x, v_y (and $\dot{v}_x$, $\dot{v}_y$) of the absolute velocity (and acceleration) of the object are always bounded, ω_x, ω_y (and their derivatives) are also bounded. Various approximate methods can thus be applied to Eqs. (1.109), (1.110).

In conclusion, note that if the xyz axes, to which Eqs. (1.95)—(1.97) apply, are made to coincide with the co-moving geographical axes (directing the y axis northward), the coefficients of Eqs. (1.95), (1.97), according to equalities (1.8a) and matrix (1.10), take the form

$$\left.\begin{aligned}
x &= 0, \\
y &= -\frac{ae^2\sin\varphi'\cos\varphi'}{(1 - e^2\sin^2\varphi')^{1/2}}, \\
z &= h + a\,(1 - e^2\sin^2\varphi')^{1/2}.
\end{aligned}\right\} \tag{1.112}$$

Expanding the expressions for y and z in powers of e and retaining only terms up to e^2, we find

$$\left.\begin{aligned}
y &= -\,ae^2\sin\varphi'\cos\varphi', \\
z &= h + a\left(1 - \frac{e^2}{2}\sin^2\varphi'\right).
\end{aligned}\right\} \tag{1.113}$$

By (1.13)

$$r = h + a\left(1 - \frac{e^2}{2}\sin^2\varphi'\right). \tag{1.114}$$

From Eqs. (1.112)—(1.114) it follows that if the terms containing products of e^2 and δx, δy, δz, θ_x, θ_y, and instrumental errors are dropped in Eqs. (1.95),

(1.97), these equations in projection onto the geographical co-moving axes take the same form as Eqs. (1.107), (1.198), and for $\omega_z = 0$ Eqs. (1.95)—(1.97) reduce to Eqs. (1.109), (1.110), and (1.108), respectively. The accuracy of the resulting equations is not impaired, since similar assumptions were made in the derivation of Eqs. (1.95) (e.g., the variation of the nonspherical components of the Earth's gravitational field, containing a factor e^2, was omitted).

1.2.4. Orientation error equations for the sensitive axes and fixed directions in space

The inertial system is expected to fix the orientation of the object in space, as well as its coordinates and velocities. In the final analysis, this clearly reduces to a determination of the orientation of the sensitive elements of the inertial system.

In a perturbed system, the orientation of the sensitive axes of the accelerometers and gyroscopes differs from their orientation according to the ideal operation equations. This leads to errors in the attitude of the object and in the determination of fixed directions in space.

If the inertial system uses a spatial absolute angular velocity meter with accelerometers along the xyz axes of the platform, the errors of platform orientation are characterized by a small rotation vector θ satisfying Eq. (1.93). This vector also characterizes the orientation error of the sensitive elements in systems using a free or controlled gyroframe (or a gyroscopically-stabilized platform) when the control torques are functions of time only.

The situation is somewhat more complex when the orientation of the xyz axes (coinciding with the accelerometer sensitive axes) is a function of both time and position. Eqs. (1.93) in this case are insufficient to describe the perturbed state of the xyz axes. Indeed, Δm_x, Δm_y, Δm_z are pure instrumental errors. Further deviations of the axes due to errors introduced by the coordinate system are not accounted for by Eq. (1.93), although they implicitly enter Eqs. (1.92)—(1.94) on the whole.

If the orientation of the axes x, y, z relative to η^1, η^2, η^3 (or, equivalently, relative to ξ, η, ζ) is specified by the direction cosines $\beta_{ij}(\eta^1, \eta^2, \eta^3)$, the orientation errors δx, δy, δz of the unit vectors x, y, z along the axes x, y, z are given by*

$$\left.\begin{aligned} \delta x &= \left[(\theta \times \eta_i) \cdot x + \frac{\partial \beta_{i1}}{\partial \eta^s}(\theta \times r + \delta r) \cdot \eta_s\right]\eta_i, \\ \delta y &= \left[(\theta \times \eta_i) \cdot y + \frac{\partial \beta_{i2}}{\partial \eta^s}(\theta \times r + \delta r) \cdot \eta_s\right]\eta_i, \\ \delta z &= \left[(\theta \times \eta_i) \cdot z + \frac{\partial \beta_{i3}}{\partial \eta^s}(\theta \times r + \delta r) \cdot \eta_s\right]\eta_i. \end{aligned}\right\} \qquad (1.115)$$

Here η_i are the unit vectors along η^s, and the vectors δr and θ are defined by Eqs. (1.92), (1.93); summation from 1 to 3 over s and i is implied.

* Andreev, V.D. Theory of Inertial Navigation: Autonomous Systems — Nauka, 1966.

Introducing the small rotation vector θ_1, which determines the orientation of the trihedron $x+\delta x$, $y+\delta y$, $z+\delta z$ relative to the trihedron xyz, we write (1.115) in the form

$$\left.\begin{aligned}
\theta_{1x} &= -\theta_x + \beta_{i1}\frac{\partial\beta_{i2}}{\partial\eta^s}[\delta\boldsymbol{r}\cdot\boldsymbol{\eta}_s + (\theta_y z - \theta_z y)\beta_{s1} + \\
&\quad + (\theta_z x - \theta_x z)\beta_{s2} + (\theta_x y - \theta_y x)\beta_{s3}], \\
\theta_{1y} &= -\theta_y + \beta_{i1}\frac{\partial\beta_{i3}}{\partial\eta^s}[\delta\boldsymbol{r}\cdot\boldsymbol{\eta}_s + (\theta_z x - \theta_x z)\beta_{s2} + \\
&\quad + (\theta_x y - \theta_y x)\beta_{s3} + (\theta_y z - \theta_z y)\beta_{s1}], \\
\theta_{1z} &= -\theta_z + \beta_{i2}\frac{\partial\beta_{i1}}{\partial\eta^s}[\delta\boldsymbol{r}\cdot\boldsymbol{\eta}_s + (\theta_x y - \theta_y x)\beta_{s3} + \\
&\quad + (\theta_y z - \theta_z y)\beta_{s1} + (\theta_z x - \theta_x z)\beta_{s2}].
\end{aligned}\right\} \tag{1.116}$$

Consider some important particular cases of Eqs. (1.116).

If the direction cosines β_{ij} do not depend explicitly on the coordinates η^s, we have $\partial\beta_{ij}/\partial\eta^s \equiv 0$ and by (1.116)

$$\theta_{1x} = -\theta_x, \quad \theta_{1y} = -\theta_y, \quad \theta_{1z} = -\theta_z. \tag{1.117}$$

If the unperturbed axis z is directed along the radius-vector $\boldsymbol{r}$, we have

$$\left.\begin{aligned}
\theta_{1x} &= -\frac{\delta y}{r}, \quad \theta_{1y} = \frac{\delta x}{r}, \\
\theta_{1z} &= -\theta_z + \beta_{i2}\frac{\partial\beta_{i1}}{\partial\eta^s}[\delta\boldsymbol{r}\cdot\boldsymbol{\eta}_s + r(\theta_y\beta_{s1} - \theta_x\beta_{s2})],
\end{aligned}\right\} \tag{1.118}$$

and if further the axes x and y are so oriented in the horizontal plane that $\omega_z = 0$, we have

$$\theta_{1z} = -\theta_z; \tag{1.119}$$

if, on the other hand, the axis y points northward

$$\theta_{1z} = -\theta_z + \left(\frac{\delta x}{r} + \theta_y\right)\operatorname{tg}\varphi. \tag{1.120}$$

For orthodromic co-moving axes, φ should be replaced with z in Eq. (1.120).

Eqs. (1.115) and the ensuing relations are valid for the case when the accelerometer sensitive axes form an orthogonal trihedron, regardless of whether the system measures Cartesian or curvilinear coordinates. If the directions $\boldsymbol{e}_s$ of the sensitive axes are not orthogonal, the expressions for $\delta\boldsymbol{e}_s$ are analogous to (1.115). Indeed,

$$\delta\boldsymbol{e}_s = \left(-\boldsymbol{e}_s\cdot\delta\boldsymbol{\xi}_k + \frac{\partial e_{sk}}{\partial x^\sigma}\,\delta x_1^\sigma\right)\boldsymbol{\xi}_k. \tag{1.121}$$

Note that Eqs. (1.115), (1.121) and the ensuing relations remain valid for orientation errors of specific directions in space which are not caged to the accelerometer sensitive axes. These may be the directions along which a projectile is fired from a moving object or directions to earthbound markers and celestial objects used in navigation corrections. In particular, it follows from (1.116) that the error $\delta\rho$ in the direction of a unit vector ρ pointing to a distant star as fixed on board the moving object from the readings of the

inertial system (and in the axes of the platform), i.e., a fixed direction in the $O_1\xi_1\eta_1\zeta_1$ system, depends only on the gyroscope instrumental errors, since

$$\Delta\varphi = \theta - \rho. \tag{1.122}$$

§1.3. SOME RESULTS OF THE ANALYSIS OF ERROR EQUATIONS OF AUTONOMOUS INERTIAL SYSTEMS

1.3.1. The second group of error equations

We give here the principal results of the analysis of error equations of autonomous inertial systems:* coordinate error equations (1.95)—(1.97) and orientation error equations (1.116).

The second group of coordinate error equations (1.96) is solved, at least in quadratures, for an arbitrary motion of the object (arbitrary $\omega_x(t)$, $\omega_y(t)$, $\omega_z(t)$) and disregarding the particular form of $\Delta m_x(t)$, $\Delta m_y(t)$, $\Delta m_z(t)$. Indeed, Eqs. (1.96) are equivalent to vector equation (1.93) which in projection onto fixed-orientation axes gives three independent first-order equations (1.106), which can be immediately integrated. Changing over to projections onto moving axes (rotating with angular velocity ω), we obtain the solution of Eqs. (1.96) in the form

$$\left.\begin{aligned}
\theta_x &= \sum_{i=1}^{3} a_{i1}\left[\int_0^t (\Delta m_x a_{i1} + \Delta m_y a_{i2} + \Delta m_z a_{i3})\,dt + \right.\\
&\qquad \left. + \theta_x^0 a_{i1}^0 + \theta_y^0 a_{i2}^0 + \theta_z^0 a_{i3}^0\right],\\
\theta_y &= \sum_{i=1}^{3} a_{i2}\left[\int_0^t (\Delta m_x a_{i1} + \Delta m_y a_{i2} + \Delta m_z a_{i3})\,dt + \right.\\
&\qquad \left. + \theta_x^0 a_{i1}^0 + \theta_y^0 a_{i2}^0 + \theta_z^0 a_{i3}^0\right],\\
\theta_z &= \sum_{i=1}^{3}\left[\int_0^t (\Delta m_x a_{i1} + \Delta m_y a_{i2} + \Delta m_z a_{i3})\,dt + \right.\\
&\qquad \left. + \theta_x^0 a_{i1}^0 + \theta_y^0 a_{i2}^0 + \theta_z^0 a_{i3}^0\right],
\end{aligned}\right\} \tag{1.123}$$

where a_{ij} are the direction cosines between the axes x, y, z and ξ, η, ζ, and a_{ij}^0 are the initial values of these direction cosines. For a given law of motion of the object, these are known functions of time.

Solution (1.123) can be found directly from Eqs. (1.96), without changing over to axes of fixed orientation, if we remember that the general solution of the homogeneous equations (1.96) is

$$\left.\begin{aligned}
\theta_x &= \sum_{i=1}^{3} C_i a_{i1}, \qquad \theta_y = \sum_{i=1}^{3} C_i a_{i2},\\
\theta_z &= \sum_{i=1}^{3} C_i a_{i3},
\end{aligned}\right\} \tag{1.124}$$

* A detailed analysis will be found in "Theory of Inertial Navigation: Autonomous Systems." — Nauka. 1966.

where C_i are arbitrary constants. That (1.124) are indeed the general solution of the homogeneous equations (1.96) can be verified by direct substitution. Note that

$$\left.\begin{aligned}\omega_x &= \sum_{i=1}^{3} \dot{\alpha}_{i2}\alpha_{i3} = -\sum_{i=1}^{3} \dot{\alpha}_{i3}\alpha_{i2},\\ \omega_y &= \sum_{i=1}^{3} \dot{\alpha}_{i3}\alpha_{i1} = -\sum_{i=1}^{3} \dot{\alpha}_{i1}\alpha_{i3},\\ \omega_z &= \sum_{i=1}^{3} \dot{\alpha}_{i1}\alpha_{i3} = -\sum_{i=1}^{3} \dot{\alpha}_{i3}\alpha_{i1}.\end{aligned}\right\} \tag{1.125}$$

Solution (1.123) leads to the estimate

$$\sqrt{\theta_x^2+\theta_y^2+\theta_z^2} \leqslant \sqrt{\theta_x^{0^2}+\theta_y^{0^2}+\theta_z^{0^2}} + \int_0^t \sqrt{\Delta m_x^2+\Delta m_y^2+\Delta m_z^2}\,dt. \tag{1.126}$$

For $\Delta m_x = 0$, $\Delta m_y = 0$, $\Delta m_z = 0$, relation (1.126) reduces to an equality. This indicates that the homogeneous equautions (1.96) have the first integral

$$\theta_x^2+\theta_y^2+\theta_z^2 = \text{const}, \tag{1.127}$$

which is obtained when the first equation in (1.96) is multiplied by θ_x, the second equation by θ_y, the third by θ_z, and the results are added. From (1.93) we also see that $\Delta m = 0$, so that

$$\theta = \theta^0. \tag{1.128}$$

This equality incorporates Eq. (1.127).

Eqs. (1.127), (1.128) for a gyroscopically stabilized platform have an obvious meaning: if the instrumental errors (gyroscope drift) are zero, the platform preserves its original attitude. Estimate (1.126) for platform drift can be similarly interpreted.

Note that Eqs. (1.127), (1.128) lead to nonasymptotic stability of the second group of homogeneous equations (1.96). These are, however, first-approximation equations. In our case the first-approximation equations have zero roots, and no definite conclusion concerning the stability of the system can be made.

The exact equations for the perturbations $\delta\alpha_{ij}$ of the direction cosines α_{ij} between the axes x, y, z and ξ_*, η_*, ζ_* are obtained in the form*

$$\left.\begin{aligned}&\delta\dot{\alpha}_{i1} - \delta\alpha_{i2}\omega_z + \delta\alpha_{i3}\omega_y = 0,\\ &\delta\dot{\alpha}_{i2} - \delta\alpha_{i3}\omega_x + \delta\alpha_{i1}\omega_z = 0,\\ &\delta\dot{\alpha}_{i3} - \delta\alpha_{i1}\omega_y + \delta\alpha_{i2}\omega_x = 0\\ &\qquad (i = 1,\ 2,\ 3).\end{aligned}\right\} \tag{1.129}$$

They can be reduced to the equations

$$\left.\begin{aligned}&\frac{d}{dt}(\delta\alpha_{11}^2+\delta\alpha_{12}^2+\delta\alpha_{13}^2) = 0,\\ &\frac{d}{dt}(\delta\alpha_{21}^2+\delta\alpha_{22}^2+\delta\alpha_{23}^2) = 0,\\ &\frac{d}{dt}(\delta\alpha_{31}^2+\delta\alpha_{32}^2+\delta\alpha_{33}^2) = 0,\end{aligned}\right\} \tag{1.130}$$

* See footnote on previous page.

whence follows nonasymptotic stability with respect to the perturbations Δa_{ij}.

1.3.2. Stability of the first group of equations for latitudinal motion

Consider the first group of equations (1.95). In the general case their analysis involves insurmountable difficulties. The analysis can be carried out to completion only for objects moving with constant velocity along the parallel, when an appropriate choice of the xyz axes will reduce Eqs. (1.95) to equations with constant coefficients, and for the case of Keplerian motion.

For an object moving with constant velocity along a parallel at a constant distance r from the Earth's center, we can choose xyz as the co-moving geocentric axes, so that

$$\left.\begin{aligned} &\omega_x = 0, \quad \omega_y = u\cos\varphi + \frac{v}{r} = \text{const}, \\ &\omega_z = u\sin\varphi + \frac{v}{r}\operatorname{tg}\varphi = \text{const}, \\ &\frac{u}{r^3} = \omega_0^2 = \text{const}, \end{aligned}\right\} \tag{1.131}$$

where v is the velocity of motion and φ is the latitude. From (1.107) we obtain the first group of error equations in the form

$$\left.\begin{aligned} &\delta\ddot{x} + (\omega_0^2 - \omega_y^2 - \omega_z^2)\,\delta x - 2\omega_z\,\delta\dot{y} + \\ &\quad + 2\omega_y\,\delta\dot{z} = \Delta n_x - r\,\Delta\dot{m}_y - r\omega_z\,\Delta m_x, \\ &\delta\ddot{y} + (\omega_0^2 - \omega_z^2)\,\delta y + \omega_y\omega_z\,\delta z + 2\omega_z\delta\dot{x} = \\ &\quad = \Delta n_y + r\,\Delta\dot{m}_x - r\omega_y\,\Delta m_z - r\omega_z\,\Delta m_y, \\ &\delta\ddot{z} - (2\omega_0^2 + \omega_y^2)\,\delta z - 2\omega_y\,\delta\dot{x} + \\ &\quad + \omega_z\omega_y\,\delta y = \Delta n_z + 2r\omega_y\,\Delta m_y. \end{aligned}\right\} \tag{1.132}$$

The constant coefficients of Eqs. (1.132) are defined by Eqs. (1.131). The characteristic equation of this system is a complete cubic equation for the square of the unknown $q = p^2$. This cubic equation has the form

$$\begin{aligned} &q^3 + 2q^2(\omega_y^2 + \omega_z^2) + q\left[-3\omega_0^4 + 3\omega_0^2(\omega_y^2 - 2\omega_z^2) + \right. \\ &\left. + (\omega_y^2 + \omega_z^2)^2\right] - \omega_0^2(\omega_0^2 - \omega_y^2 - \omega_z^2)(2\omega_0^2 + \omega_y^2 - 2\omega_z^2) = 0. \end{aligned} \tag{1.133}$$

To ensure (nonasymptotic) stability, Eq. (1.133) should have negative or zero roots; multiple roots of the characteristic equation should correspond to linear elementary divisors of the characteristic matrix of Eqs. (1.132).

The stability in this case can also be determined if we interpret Eqs. (1.132) as describing the motion of a point mass about its equilibrium position ($\delta x = 0$, $\delta y = 0$, $\delta z = 0$) in a force field which is a superposition of a potential force function

$$U = -\frac{1}{2}\left[(\omega_0^2 - \omega_y^2 - \omega_z^2)(\delta x)^2 + (\omega_0^2 - \omega_z^2)(\delta y)^2 - (2\omega_0^2 + \omega_y^2)(\delta z)^2 + 2\omega_y\omega_z\,\delta y\,\delta z\right] \tag{1.134}$$

and the gyroscopic forces

$$\left.\begin{array}{c} 2\omega_z\,\delta\dot{y} - 2\omega_y\,\delta\dot{z}, \quad -2\omega_z\,\delta\dot{x}, \\ 2\omega_y\,\delta\dot{x}. \end{array}\right\} \tag{1.135}$$

The homogeneous system (1.132) has the energy integral

$$(\delta\dot{x})^2 + (\delta\dot{y})^2 + (\delta\dot{z})^2 - 2U = \text{const}, \tag{1.136}$$

which is obtained when Eqs. (1.132) are respectively multiplied by $\delta\dot{x}$, $\delta\dot{y}$, $\delta\dot{z}$.

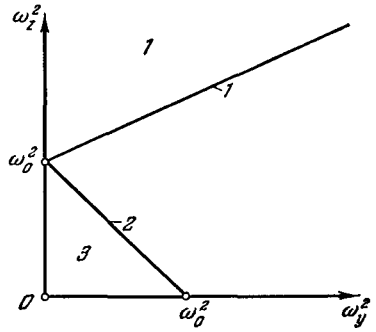

FIGURE 1.8.

If the gyroscopic forces (1.135) are dropped from Eqs. (1.132), we are left with potential forces only. To allow equilibrium in a potential force field, the force function should have an isolated maximum at the equilibrium point. Since the force function (1.134) is a quadratic form of the relevant variables, the conditions of a maximum are Sylvester's conditions for a positive definite quadratic form. In our case these are the inequalities

$$\omega_0^2 - \omega_y^2 - \omega_z^2 > 0, \qquad 2\omega_0^2 - 2\omega_z^2 + \omega_y^2 < 0. \tag{1.137}$$

In Figure 1.8, lines 1 and 2 correspond to the equalities

$$\omega_0^2 - \omega_y^2 - \omega_z^2 = 0 \text{ and } 2\omega_0^2 - 2\omega_z^2 + \omega_y^2 = 0. \tag{1.138}$$

We see that the regions defined by inequalities (1.137) do not intersect, and the force function has no maximum. Since the force function is a homogeneous function of second degree, Lyapunov's theorem definitely states that the system is unstable, without going into higher-order terms.

Let us return to gyroscopic forces (1.135). In regions 1 and 3 (see Figure 1.8), where the degree of instability (the number of negative Poincaré stability coefficients) of a conservative system is odd, the gyroscopic forces according to Thomson and Tate's theorem, cannot stabilize the equilibrium.* In region 2, where the degree of instability is even, gyroscopic forces in

*Chetaev, N.G. Ustoichivost' dvizheniya (Stability of Motion). — Gostekhizdat. 1955.

principle may stabilize the system. The resulting stability, however, is transient, as it is destroyed by the internal dissipation forces.

1.3.3. Solution of the first group of error equations for a stationary object and for an object moving along the arc of a fixed-attitude great circle

We will now solve the first group of error equations (1.132) for two particular cases: the case of an object which is stationary in the $O_1\xi\eta\zeta$ system and an object which moves with constant velocity at a constant distance from the Earth's center in a plane through the Earth's center (motion along the arc of a great circle on a fixed sphere around the Earth). The general case of motion along a parallel will not be considered here and the reader is referred to my "Theory of Inertial Navigation: Autonomous Systems." These two particular cases are quite sufficient for the following treatment.

For a stationary object in the $O_1\xi\eta\zeta$ coordinate system, fixing the orientation of the xyz axes so that the axis z points along the radius-vector r, we obtain the equation of errors in the form (in (1.132) the absolute angular velocity projections ω_x, ω_y are taken equal to zero)

$$\left.\begin{aligned} \delta\ddot{x} + \omega_0^2\delta x &= \Delta n_x - \Delta\dot{m}_y r, \\ \delta\ddot{y} + \omega_0^2\delta y &= \Delta n_y + \Delta\dot{m}_x r, \\ \delta\ddot{z} - 2\omega_0^2\delta z &= \Delta n_z. \end{aligned}\right\} \tag{1.139}$$

The solution of these equations is obviously

$$\left.\begin{aligned} \delta x &= \delta x^0\cos\omega_0 t + \frac{\delta\dot{x}^0}{\omega_0}\sin\omega_0 t + \\ &\quad + \frac{1}{\omega_0}\int_0^t (\Delta n_x - \Delta\dot{m}_y r)\sin\omega_0(t-\tau)\,d\tau, \\ \delta y &= \delta y^0\cos\omega_0 t + \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t + \\ &\quad + \frac{1}{\omega_0}\int_0^t (\Delta n_y + \Delta\dot{m}_x r)\sin\omega_0(t-\tau)\,d\tau, \\ \delta z &= \delta z^0\operatorname{ch}\omega_0\sqrt{2}\,t + \frac{\delta\dot{z}^0}{\omega_0\sqrt{2}}\operatorname{sh}\omega_0\sqrt{2}\,t + \\ &\quad + \frac{1}{\omega_0\sqrt{2}}\int_0^t \Delta n_z\operatorname{sh}\omega_0\sqrt{2}(t-\tau)\,d\tau. \end{aligned}\right\} \tag{1.140}$$

For constant instrumental errors and zero initial conditions, we have

$$\left.\begin{aligned} \delta x &= \frac{\Delta n_x}{\omega_0^2}(1-\cos\omega_0 t), \\ \delta y &= \frac{\Delta n_y}{\omega_0^2}(1-\cos\omega_0 t), \\ \delta z &= \frac{\Delta n_z}{2\omega_0^2}(\operatorname{ch}\omega_0\sqrt{2}\,t - 1). \end{aligned}\right\} \tag{1.141}$$

Note that the analysis of the inertial navigation errors over the propulsion flight trajectory of a ballistic missile reduces to Eqs. (1.139).

Indeed, if we write the error equations in the form (1.106) and place the axis y in the plane of the propulsion trajectory, we have $x=0$, and the first three equations in (1.106) take the form

$$\left.\begin{aligned}
&\delta\ddot{x}+\tfrac{\mu}{r^3}\delta x=\Delta n_x-2(\Delta m_y\dot{z}-\Delta m_z\dot{y})-\Delta\dot{m}_y z+\Delta\dot{m}_z y,\\
&\delta\ddot{y}+\tfrac{\mu}{r^5}[(z^2-2y^2)\delta y-3yz\,\delta z]=\Delta n_y+2\Delta m_x\dot{z}+\Delta\dot{m}_x z,\\
&\delta\ddot{z}+\tfrac{\mu}{r^5}[(y^2-2z^2)\delta z-3zy\,\delta y]=\Delta n_z-2\Delta m_x\dot{y}-\Delta\dot{m}_x y.
\end{aligned}\right\}\qquad(1.142)$$

Directing the axis z along the vector r at the starting point and seeing that y/r is a small quantity, we obtain from the last two equations in (1.142) to terms linear in y/r

$$\left.\begin{aligned}
&\delta\ddot{y}+\tfrac{\mu}{r^3}\delta y-3\tfrac{\mu}{r^3}\tfrac{y}{r}\delta z=\Delta n_y+2\Delta m_x\dot{z}+\Delta\dot{m}_x z,\\
&\delta\ddot{z}-2\tfrac{\mu}{r^3}\delta z-3\tfrac{\mu}{r^3}\tfrac{y}{r}\delta y=\Delta n_z-2\Delta m_x\dot{y}-\Delta\dot{m}_x y.
\end{aligned}\right\}\qquad(1.143)$$

If we further omit the terms with y/r ($\lesssim 0.1$) and take $\mu/r^3=\omega_0^2=\text{const}$, we find from Eqs. (1.142) and (1.143)

$$\left.\begin{aligned}
&\delta\ddot{x}+\omega_0^2\delta x=\Delta n_x-2(\Delta m_y\dot{z}-\Delta m_z\dot{y})-\Delta\dot{m}_y z,\\
&\delta\ddot{y}+\omega_0^2\delta y=\Delta n_y+2\Delta m_x\dot{z}+\Delta\dot{m}_x z,\\
&\delta\ddot{z}-2\omega_0^2\delta z=\Delta n_z-2\Delta m_x\dot{y}.
\end{aligned}\right\}\qquad(1.144)$$

These equations differ from (1.139) only in their right-hand sides.

Consider motion with constant velocity along the arc of a fixed-orientation great circle. Directing the axis z along the radius-vector r and putting the axis x in the plane of the great circle, we have $\omega_z=0$ and Eqs. (1.132) take the form

$$\left.\begin{aligned}
&\delta\ddot{x}+(\omega_0^2-\omega_y^2)\delta x+2\omega_y\delta\dot{z}=\Delta n_x-\Delta\dot{m}_y r,\\
&\delta\ddot{y}+\omega_0^2\delta y=\Delta n_y+\Delta\dot{m}_x r-\omega_y r\,\Delta m_z,\\
&\delta\ddot{z}-(2\omega_0^2+\omega_y^2)\delta z-2\omega_y\delta\dot{x}=\Delta n_z+2r\omega_y\Delta m_y.
\end{aligned}\right\}\qquad(1.145)$$

The second equation, which gives the error in the plane normal to the plane of motion, is separated. Its solution is

$$\delta y=\delta y^0\cos\omega_0 t+\frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t+$$
$$+\frac{1}{\omega_0}\int_0^t(\Delta n_y+\Delta\dot{m}_x r-\omega_y\Delta m_z r)\sin\omega_0(t-\tau)\,d\tau.\qquad(1.146)$$

The remaining system — the first and third equations in (1.145) — describe errors in the plane of motion. The characteristic equation of this system is

$$p^4+p^2(-\omega_0^2+2\omega_y^2)-(\omega_0^2-\omega_y^2)(2\omega_0^2+\omega_y^2)=0.\qquad(1.147)$$

If $\omega_0^2 > \omega_y^2$, i.e., the velocity of the object is less than the circular velocity, Eq. (1.147) has two real and two complex conjugate roots

$$\left.\begin{aligned} p_{1,2} &= \pm\mu, \quad p_{3,4} = \pm j\nu, \\ \mu,\ \nu &= \sqrt{\frac{1}{2}\left(\pm\omega_0^2 \mp 2\omega_y^2 + \omega_0\sqrt{9\omega_0^2 - 8\omega_y^2}\right)}. \end{aligned}\right\} \tag{1.148}$$

For $\omega_y = 0$, the root $\pm\mu$ goes over to $\pm\omega_0\sqrt{2}$, the root $\pm j\nu$ to $\pm j\omega_0$. For $\omega_y = \omega_0$, i.e., for a satellite in circular orbit, $\mu = 0$, $\nu = \omega_0$. The plot of μ and ν as a function of ω_y is shown in Figure 1.9. ν is virtually invariable in the entire range $0 < \omega_y < \omega_0$. It is equal to ω_0 for $\omega_y = 0$ and $\omega_y = \omega_0$ and reaches its maximum for $\omega_y = \omega_0\sqrt{5/8}$. This maximum is equal to $\omega_0\sqrt{9/8} \approx 1.06\omega_0$. μ monotonically decreases from $\mu = \omega_0\sqrt{2}$ to 0. It is equal to ν at the point $\omega_y = \omega_0\sqrt{3/2}$. At this point $\mu = \nu = \omega_0\sqrt{\sqrt{5}-2}$. Note that for $0 < \omega_y < \omega_0/2$, μ decreases hardly by 0.15 of its value at the point $\omega_y = 0$. The remaining variation of μ is in the range $\omega_0/2 < \omega_y < \omega_0$, where μ falls off steeply.

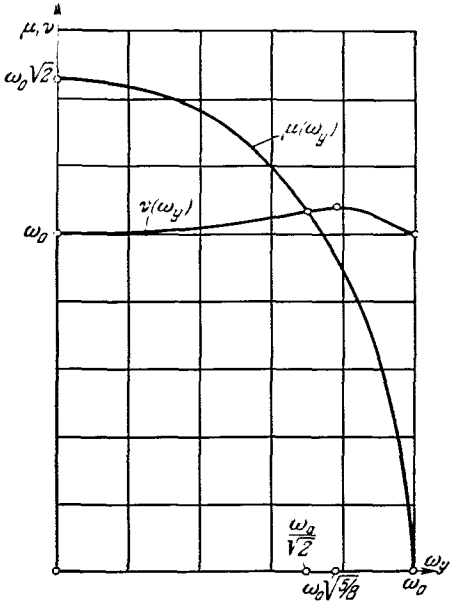

FIGURE 1.9.

The general solution of the first and second equations in (1.145) is

$$\left.\begin{aligned} \delta x = \frac{1}{(\mu^2+\nu^2)(\omega_0^2-\omega_y^2)} \int_0^t \Delta n_x \big[\mu(\omega_0^2 - \omega_y^2 - \nu^2) \times \\ \times \operatorname{sh}\mu(t-\tau) + \nu(\omega_0^2 - \omega_y^2 + \mu^2)\sin\nu(t-\tau)\big]\,d\tau + \\ + \frac{r}{\mu^2+\nu^2}\int_0^t \Delta m_y \big[(\omega_0^2 - \omega_y^2 - \nu^2)\operatorname{ch}\mu(t-\tau) - \\ - (\omega_0^2 - \omega_y^2 + \mu^2)\cos\nu(t-\tau)\big]\,d\tau - \end{aligned}\right\} \tag{1.149}$$

$$
\begin{aligned}
&-\frac{2\omega_y}{\mu^2+\nu^2}\int_0^t \Delta n_z\,[\operatorname{ch}\mu(t-\tau)-\cos\nu(t-\tau)]\,d\tau+\\
&+\frac{\delta x^0}{\mu^2+\nu^2}\left[(\omega_0^2-\omega_y^2+\mu^2)\cos\nu t-\right.\\
&\left.-(\omega_0^2-\omega_y^2-\nu^2)\operatorname{ch}\mu t\right]+\frac{\delta\dot{x}^0+r\,\Delta m_y^0}{(\omega_0^2-\omega_y^2)(\mu^2+\nu^2)}\times\\
&\times\left[\nu(\omega_0^2-\omega_y^2+\mu^2)\sin\nu t+\mu(\omega_0^2-\omega_y^2-\nu^2)\operatorname{sh}\mu t\right]+\\
&+\frac{\delta z^0 2\omega_y\mu\nu}{(\omega_0^2-\omega_y^2)(\mu^2+\nu^2)}(\mu\sin\nu t-\nu\operatorname{sh}\mu t)+\frac{\delta\dot{z}^0 2\omega_y}{\mu^2+\nu^2}(\cos\nu t-\operatorname{ch}\mu t),\\
&\delta z=\frac{2\omega_y}{\mu^2+\nu^2}\int_0^t \Delta n_x\,[\operatorname{ch}\mu(t-\tau)-\cos\nu(t-\tau)]\,d\tau+\\
&+\frac{1}{\mu\nu(\mu^2+\nu^2)}\int_0^t \Delta n_z\left[\nu(\omega_0^2-\omega_y^2+\mu^2)\operatorname{sh}\mu(t-\tau)-\right.\\
&\left.-\mu(\omega_0^2-\omega_y^2-\nu^2)\sin\nu(t-\tau)\right]d\tau-\\
&-\frac{2r\omega_y(\omega_0^2-\omega_y^2)}{\mu\nu(\mu^2+\nu^2)}\int_0^t \Delta m_y\,[\mu\sin\nu(t-\tau)-\\
&-\nu\operatorname{sh}\mu(t-\tau)]\,d\tau+\frac{\delta x^0 2\omega_y(\omega_0^2-\omega_y^2)}{\mu\nu(\mu^2+\nu^2)}\times\\
&\times(\mu\sin\nu t-\nu\operatorname{sh}\mu t)-\frac{(\delta\dot{x}^0+r\,\Delta m_y^0)2\omega_y}{\mu^2+\nu^2}\times\\
&\times(\cos\nu t-\operatorname{ch}\mu t)+\frac{\delta z^0}{(\omega_0^2-\omega_y^2)(\mu^2+\nu^2)}\times\\
&\times\left[\mu^2(\omega_0^2-\omega_y^2-\nu^2)\cos\nu t+\nu^2(\omega_0^2-\omega_y^2+\mu^2)\operatorname{ch}\mu t\right]-\\
&-\frac{\delta\dot{z}^0}{(\mu^2+\nu^2)\mu\nu}\left[\mu(\omega_0^2-\omega_y^2-\nu^2)\sin\nu t-\nu(\omega_0^2-\omega_y^2+\mu^2)\operatorname{sh}\mu t\right].
\end{aligned}
\tag{1.149}
$$

For constant instrumental errors and zero initial conditions, we have from (1.146), (1.149)

$$
\begin{aligned}
\delta x&=\frac{\Delta n_x}{\omega_0^2-\omega_y^2}\left\{1-\frac{1}{\mu^2+\nu^2}\left[(\omega_0^2-\omega_y^2+\mu^2)\cos\nu t+\right.\right.\\
&\qquad\left.\left.+(\omega_0^2-\omega_y^2-\nu^2)\operatorname{ch}\mu t\right]\right\}+\\
&\quad+\frac{2\omega_y\mu\nu(\Delta n_z+2r\omega_y\,\Delta m_y)}{(\omega_0^2-\omega_y^2)(2\omega_0^2+\omega_y^2)(\mu^2+\nu^2)}(\mu\sin\nu t-\nu\operatorname{sh}\mu t),\\
\delta y&=\frac{\Delta n_y-r\omega_y\,\Delta m_z}{\omega_0^2}(1-\cos\omega_0 t),\\
\delta z&=\frac{\Delta n_z+2r\omega_y\,\Delta m_y}{2\omega_0^2+\omega_y^2}\left\{-1+\frac{1}{(\omega_0^2-\omega_y^2)(\mu^2+\nu^2)}\times\right.\\
&\qquad\left.\times\left[\mu^2(\omega_0^2-\omega_y^2-\nu^2)\cos\nu t+\right.\right.\\
&\quad\left.\left.+\nu^2(\omega_0^2-\omega_y^2+\mu^2)\operatorname{ch}\mu t\right]\right\}-\frac{2\omega_y\,\Delta n_x}{(\mu^2+\nu^2)\mu\nu}(\mu\sin\nu t-\nu\operatorname{sh}\mu t).
\end{aligned}
\tag{1.150}
$$

If $\omega_y^2\ll\omega_0^2$, the solution of Eqs. (1.145) for constant instrumental errors can be simplified

$$
\begin{aligned}
\delta x&=\frac{\Delta n_x}{\omega_0^2}+\left(\delta x^0-\frac{\Delta n_x}{\omega_0^2}\right)\cos\omega_0 t+\frac{\delta\dot{x}^0}{\omega_0}\sin\omega_0 t-\\
&-\frac{\omega_y\sqrt{2}}{\omega_0}\left[\left(\frac{2}{3}\delta z^0+\frac{\Delta n_z}{3\omega_0^2}\right)\operatorname{sh}\omega_0\sqrt{2}\,t+\frac{\sqrt{2}}{3\omega_0}\delta\dot{z}^0\operatorname{ch}\omega_0\sqrt{2}\,t\right],
\end{aligned}
\tag{1.151}
$$

$$\left.\begin{aligned}
\delta y &= \frac{\Delta n_y}{\omega_0^2} + \left(\delta y^0 - \frac{\Delta n_y}{\omega_0^2}\right)\cos\omega_0 t + \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t,\\
\delta z &= -\frac{\Delta n_z}{2\omega_0^2} + \left(\frac{\Delta n_z}{2\omega_0^2} + \delta z^0\right)\operatorname{ch}\omega_0\sqrt{2}\,t + \frac{\delta\dot{z}^0}{\sqrt{2}\,\omega_0}\operatorname{sh}\omega_0\sqrt{2}\,t.
\end{aligned}\right\} \qquad (1.151)$$

From (1.146), (1.149) we can obtain expressions for $\delta x, \delta y, \delta z$ for a satellite in circular orbit. Taking the limit $\omega_y \to \omega_0$, we obtain for this case

$$\left.\begin{aligned}
\delta x &= \frac{1}{\omega_0^2}\int_0^t \Delta n_x\,[-3\omega_0(t-\tau) + 4\sin\omega_0(t-\tau)]\,d\tau -\\
&- r\int_0^t \Delta m_z\,d\tau - \frac{2}{\omega_0}\int_0^t \Delta n_z\,[1-\cos\omega_0(t-\tau)]\,d\tau +\\
&+ \delta x^0 + \frac{\delta\dot{x}^0 + r\,\Delta m_y^0}{\omega_0^2}(4\sin\omega_0 t - 3\omega_0 t) +\\
&+ 6\delta z^0(\sin\omega_0 t - \omega_0 t) + \frac{2\delta\dot{z}^0}{\omega_0^2}(\cos\omega_0 t - 1),\\
\delta y &= \frac{1}{\omega_0^2}\int_0^t (\Delta n_y - \omega_0 r\,\Delta m_z)\sin\omega_0(t-\tau)\,d\tau +\\
&+ r\int_0^t \Delta m_x\cos\omega_0(t-\tau)\,d\tau + \delta y^0\cos\omega_0 t + \frac{\delta\dot{y}^0 - r\,\Delta m_x^0}{\omega_0}\sin\omega_0 t,\\
\delta z &= \frac{2}{\omega_0^2}\int_0^t \Delta n_x\,[1-\cos\omega_0(t-\tau)]\,d\tau + \frac{1}{\omega_0^2}\int_0^t \Delta n_z\sin\omega_0(t-\tau)\,d\tau +\\
&+ \frac{2(\delta\dot{x}^0 + r\,\Delta m_y^0)}{\omega_0^2}(1-\cos\omega_0 t) + \delta z^0(4 - 3\cos\omega_0 t) + \frac{\delta\dot{z}^0}{\omega_0}\sin\omega_0 t.
\end{aligned}\right\} \qquad (1.152)$$

For constant instrumental errors and zero initial conditions, we have

$$\left.\begin{aligned}
\delta x &= \frac{\Delta n_x}{\omega_0^2}\left[-\frac{3}{2}\omega_0^2 t^2 + 4(1-\cos\omega_0 t)\right] +\\
&\quad + \left(\frac{4r\,\Delta m_y}{\omega_0^2} + \frac{2\,\Delta n_z}{\omega_0^3}\right)(\sin\omega_0 t - \omega_0 t),\\
\delta y &= \frac{1}{\omega_0^2}(\Delta n_y - r\omega_0\,\Delta m_z)(1-\cos\omega_0 t),\\
\delta z &= \frac{2\,\Delta n_x}{\omega_0^3}(\omega_0 t - \sin\omega_0 t) +\\
&\quad + \frac{\Delta n_z}{\omega_0^2}(1-\cos\omega_0 t) + \frac{2r\,\Delta m_y}{\omega_0}(1-\cos\omega_0 t).
\end{aligned}\right\} \qquad (1.153)$$

1.3.4. Integration of the first group of equations for Keplerian motion

Consider the case of Keplerian motion. Turning to the vector equation (1.92), we note that the corresponding homogeneous equation can be written in the form

$$\delta\left(\frac{d^2\mathbf{r}}{dt^2} + \frac{\mu\mathbf{r}}{r^3}\right) = 0. \qquad (1.154)$$

Thus, if the point O where the accelerometer sensitive masses are located coincides with the center of mass of the object, the homogeneous equation (1.92) is the variational equation of Keplerian motion

$$\frac{d^2 r}{dt^2} + \frac{\mu r}{r^3} = 0. \tag{1.155}$$

The general integral of Eq. (1.155), containing six arbitrary constants, is well known. According to Poincaré's theorem, the particular solutions of Eq. (1.92) are the derivatives of the general integral of Eq. (1.155) with respect to the arbitrary constants.* The general solution of the inhomogeneous equation can be obtained by Lagrange's method of variation of the constants.

With the orbital plane of the object we associate the axes $O_1\xi'_*\eta'_*\zeta'_*$, so that the plane $O_1\xi'_*\eta'_*$ is coplanar with the orbit. The axis $O_1\zeta'_*$ is perpendicular to the orbital plane, so that the object moves counterclockwise if viewed from the end of this axis. Keplerian (elliptical) motion in the orbital plane is described by the equalities**

$$\left.\begin{aligned} &M = \nu(t - t^0) + M^0, \qquad \nu = \mu^{1/2} a^{-3/2}, \\ &E - e\sin E = M, \\ &r = a(1 - e\cos M), \quad \operatorname{tg}\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\operatorname{tg}\frac{E}{2}, \\ &\sigma = v + \omega, \\ &\sin E = \frac{2}{e}\sum_{k=1}^{\infty}\frac{J_k(ke)}{k}\sin kM, \end{aligned}\right| \tag{1.156}$$

where a is the semimajor axis, e is the eccentricity of the orbit, M, E, v are the mean, eccentric, and true anomaly, ν is the mean orbital frequency, σ is the angle between the axis ξ'_* and the current radius-vector r of the object, ω is the angle between the axis ξ'_* and the direction to the perigee, t^0 is the time of perigee passage, J_k is Bessel's function.

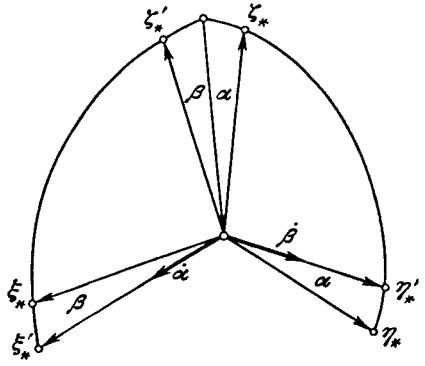

FIGURE 1.10.

* See, e.g., Goursat, E. Cours d'Analyse. — Paris, Gauthier—Villars. 1905.

** See, e.g., Duboshin, G.N. Nebesnaya mekhanika. Osnovnye zadachi i metody (Celestial Mechanics Problems and Methods). — Fizmatgiz. 1963.

Eqs. (1.156) contain four arbitrary constants, t^0, e, a, ω. The two missing constants enter the definition of the orbital plane relative to the $O_1\xi_*\eta_*\zeta_*$ axes. The relative orientation of the $O_1\xi_*\eta_*\zeta_*$ and $O_1\xi'_*\eta'_*\zeta'_*$ axes (Figure 1.10) is defined by the following matrix of direction cosines:

$$\begin{array}{c|ccc} & \xi'_* & \eta'_* & \zeta'_* \\ \hline \xi_* & \cos\beta & 0 & \sin\beta \\ \eta_* & \sin\alpha\sin\beta & \cos\alpha & -\sin\alpha\cos\beta \\ \zeta_* & -\cos\alpha\sin\beta & \sin\alpha & \cos\alpha\cos\beta. \end{array} \tag{1.157}$$

Then

$$\left.\begin{gathered} r = \xi_*\xi_* + \eta_*\eta_* + \zeta_*\zeta_*, \\ \xi_* = r\cos\sigma\cos\beta, \quad \eta_* = r(\cos\sigma\sin\alpha\sin\beta + \sin\sigma\cos\alpha), \\ \zeta_* = r(-\cos\sigma\cos\alpha\sin\beta + \sin\sigma\sin\alpha), \end{gathered}\right\} \tag{1.158}$$

where ξ_*, η_*, ζ_* are the unit vectors of the corresponding axes.

Eqs. (1.156), (1.158) give the general integral of Eq. (1.155) with six arbitrary constants, $t^0, e, a, \omega, \alpha, \beta$. Without loss of generality, we may take

$$t^0 = 0, \quad \alpha = \beta = 0, \quad \omega = \sigma(0) = 0, \tag{1.159}$$

so as to simplify further notation.

We form the following linear combinations of the derivatives of the radius-vector r from Eqs. (1.156), (1.158) with respect to the arbitrary constants:

$$\left.\begin{gathered} q_1 = \frac{\partial r}{\partial a}, \quad q_2 = \frac{1}{a}\frac{\partial r}{\partial e}, \\ q_3 = -\frac{1}{a^2\sqrt{1-e^2}}\frac{\partial r}{\partial \omega} - \frac{\sqrt{1-e^2}}{aev}\frac{\partial r}{\partial t^0}, \\ q_4 = \frac{1}{a}\frac{\partial r}{\partial \omega}, \quad q_5 = -\frac{1}{a}\frac{\partial r}{\partial \beta}, \quad q_6 = \frac{1}{a}\frac{\partial r}{\partial \alpha}. \end{gathered}\right\} \tag{1.160}$$

Let p_i be the total derivatives of the vectors q_i with respect to time. The vectors q_i, p_i clearly constitute a system of particular solutions of Eqs. (1.2).

Taking the projections of the vectors q_i, p_i on the orbital axes x, y, z, with the z axis pointing along the vector r and y directed along the axis $O_1\zeta'_*$, we form two matrices A and B with the following elements:

$$\left.\begin{gathered} A_{1i} = q_i \cdot x, \quad A_{2i} = q_i \cdot z, \quad A_{3i} = p_i \cdot x, \quad A_{4i} = p_i \cdot z \\ (i = 1, 2, 3, 4), \\ E_{1i} = q_{i+4} \cdot y, \quad B_{2i} = p_{i+4} \cdot y \quad (i = 1, 2), \end{gathered}\right\} \tag{1.161}$$

where x, y, z are the unit vectors of the orbital axes. In calculating the matrix elements of A and B, we use the following equalities:

$$\left.\begin{gathered} p_i = \frac{dq_i}{dt}, \quad \frac{dx}{dt} = -\omega_y z, \quad \frac{dy}{dt} = 0, \quad \frac{dz}{dt} = \omega_y x \\ \omega_y = \dot{v} = v\sqrt{1-e^2}\,\frac{a^2}{r^2}, \quad r = \frac{a(1-e^2)}{1+e\cos v}, \\ \dot{r} = \frac{vea\sin v}{\sqrt{1-e^2}}. \end{gathered}\right\} \tag{1.162}$$

The projections of Eq. (1.92) on the orbital axes xyz are obtained from Eqs. (1.107) taking $\omega_x = 0$. They can be written in Cauchy form as a fourth-order system

$$\left.\begin{aligned} \dot{x}_1 &= -\omega_y x_2 + x_3, \\ \dot{x}_2 &= \omega_y x_1 + x_4, \\ \dot{x}_3 &= -\omega_y x_4 - \frac{\mu x_1}{r^3} + \Delta n_x - 2\dot{r}\,\Delta m_y - r\,\Delta\dot{m}_y, \\ \dot{x}_4 &= \omega_y x_3 + \frac{2\mu x_2}{r^3} + \Delta n_z + 2r\omega_y \Delta m_y, \end{aligned}\right\} \tag{1.163}$$

where $x_1 = \delta x$, $x_2 = \delta z$, and a second-order system

$$\left.\begin{aligned} \dot{x}_5 &= x_6, \\ \dot{x}_6 &= -\frac{\mu x_5}{r^3} + \Delta n_y + 2\dot{r}\,\Delta m_x + r\,\Delta\dot{m}_x - r\omega_y \Delta m_z, \end{aligned}\right\} \tag{1.164}$$

where $x_5 = \delta y$.

The variables x_1, x_5, x_2 are the projections of the vector δr on the axes x, y, z, and x_3, x_6, x_4 are the corresponding projections of the vector dr/dt. The matrix elements of A and B constitute linearly independent solutions of the homogeneous systems (1.163), (1.164), since the determinants of A and B are the Wronskians of these solutions and do not vanish:

$$|A| = -\nu^2/2, \quad |B| = \nu\sqrt{1-e^2}.$$

The general solution of homogeneous systems (1.163), (1.164) can now be written in the form

$$x_l = \sum_{j=1}^{4} A_{lj} C_j \; (l = 1, 2, 3, 4), \quad x_i = \sum_{j=1}^{2} B_{ij} C_{4+j} \quad (i = 1, 2). \tag{1.165}$$

The solution of the inhomogeneous equations (1.163), (1.164) is obtained from (1.165) by variation of constants. Introducing the matrices $D = A^{-1}$ and $G = B^{-1}$ and changing over from x_1, x_5, x_2 to δx, δy, δz, we obtain for the projections of the vector δr on the orbital axes:*

$$\left.\begin{aligned} \delta x &= \sum_{i=1}^{4} A_{1i} \left\{ \int_0^t [(\Delta n_x - 2\dot{r}\,\Delta m_y - r\,\Delta\dot{m}_y) D_{i3} + \right. \\ &\qquad \left. + (\Delta n_z + 2r\omega_y \Delta m_y) D_{i4}]\, dt + \sum_{j=1}^{4} D^0_{ij} x^0_j \right\}, \\ \delta z &= \sum_{i=1}^{4} A_{2i} \left\{ \int_0^t [(\Delta n_x - 2\dot{r}\,\Delta m_y - r\,\Delta\dot{m}_y) D_{i3} + \right. \\ &\qquad \left. + (\Delta n_z + 2r\omega_y \Delta m_y) D_{i4}]\, dt + \sum_{j=1}^{4} D^0_{ij} x^0_j \right\}, \\ \delta y &= \sum_{i=1}^{2} B_{1i} \left[\int_0^t (\Delta n_y + 2\dot{r}\,\Delta m_x + r\,\Delta\dot{m}_x - \right. \\ &\qquad \left. - r\omega_y \Delta m_z)\, G_{i2}\, dt + \sum_{j=1}^{2} G^0_{ij} x^0_{j+4} \right]. \end{aligned}\right\} \tag{1.166}$$

* Andreev, V.D. Integration of the equations of errors of an inertial navigation system for Keplerian motion of the object. — Prikladnaya Matematika i Mekhanika, Vol. 29, No. 2. 1965.

The matrix elements of A, B and D, G entering (1.166) are*

$$\left.\begin{aligned}
&A_{11}=\frac{-3\nu t(1+e\cos v)}{2\sqrt{1-e^2}},\\
&A_{12}=\frac{2+e\cos v}{1+e\cos v}\sin v,\\
&A_{13}=\frac{2+e\cos v}{1+e\cos v}\cos v,\qquad A_{14}=\frac{r}{a},\\
&A_{21}=\frac{r}{a}-\frac{3\nu t e\sin v}{2\sqrt{1-e^2}},\qquad A_{22}=-\cos v,\\
&A_{23}=\sin v,\qquad A_{24}=0;\\
&B_{11}=\frac{r}{a}\cos v,\qquad B_{12}=\frac{r}{a}\sin v;\\
&D_{13}=\frac{2(1+e\cos v)}{\nu\sqrt{1-e^2}},\\
&D_{23}=\frac{\sqrt{1-e^2}}{\nu}\,\frac{e+2\cos v+e\cos^2 v}{1+e\cos v},\\
&D_{33}=\frac{1}{\nu}\left[-\frac{\sqrt{1-e^2}}{1+e\cos v}(2+e\cos v)\sin v+\frac{3\nu t}{1-e^2}(1+e\cos v)e\right],\\
&D_{43}=\frac{1}{\nu}\left[-e\sin v\,\frac{2+e\cos v}{\sqrt{1-e^2}\,(1+e\cos v)}+\frac{3\nu t}{(1-e^2)^2}(1+e\cos v)\right],\\
&D_{14}=\frac{2e\sin v}{\nu\sqrt{1-e^2}},\qquad D_{24}=\frac{\sqrt{1-e^2}}{\nu}\sin v,\\
&D_{34}=\frac{1}{\nu}\left[\frac{3\nu t}{1-e^2}e^2\sin v-\frac{\sqrt{1-e^2}}{1+e\cos v}(2e-\cos v-e\cos v)\right],\\
&D_{44}=\frac{1}{\nu}\left[\frac{3\nu t}{(1-e^2)^2}e\sin v+\frac{e\cos v+e^2\cos^2 v-2}{\sqrt{1-e^2}\,(1+e\cos v)}\right];\\
&G_{12}=-\frac{\sqrt{1-e^2}\sin v}{\nu(1+e\cos v)},\qquad G_{22}=\frac{\sqrt{1-e^2}\cos v}{\nu(1+e\cos v)}.
\end{aligned}\right\}\tag{1.167}$$

For C_i^0, C_{i+4}^0 we get

$$C_i^0=\sum_{j=1}^{4}D_{ij}^0x_j^0,\qquad C_{i+4}^0=\sum_{j=1}^{2}G_{ij}^0x_{i+4}^0.\tag{1.168}$$

D_{i3}^0, D_{i4}^0, G_{12}^0, G_{22}^0 are obtained from Eqs. (1.167) putting $t=0$, $v=0$. D_{i1}^0, D_{i2}^0, G_{i1}^0 are given by

$$\left.\begin{aligned}
&D_{11}^0=D_{21}^0=0,\quad D_{31}^0=1,\quad D_{41}^0=-\frac{1}{\sqrt{1-e^2}},\\
&D_{12}^0=\frac{2}{(1-e)^2},\quad D_{22}^0=\frac{1+e}{1-e},\quad D_{32}^0=D_{42}^0=0,\\
&G_{11}^0=\frac{1}{1-e},\quad G_{21}^0=0.
\end{aligned}\right\}\tag{1.169}$$

In Eqs. (1.168), by (1.163), (1.164), (1.162),

$$\left.\begin{aligned}
&x_1^0=\delta x^0,\quad x_2^0=\delta z^0,\quad x_3^0=\delta\dot{x}^0+\frac{\nu}{\sqrt{1-e^2}}\delta z^0,\\
&x_4^0=\delta\dot{z}^0-\frac{\nu}{\sqrt{1-e^2}}\delta x^0,\quad x_5^0=\delta y^0,\quad x_6^0=\delta\dot{y}^0.
\end{aligned}\right\}\tag{1.170}$$

For $e=0$, i.e., a circular orbit,

$$r=a,\quad \dot{r}=0,\quad \nu=\omega_y=\omega_0,\quad v=\omega_0 t,\quad \dot{\omega}_y=0,\tag{1.171}$$

and Eqs. (1.166) reduce to previously obtained Eqs. (1.152).

* Lur'e, A.I. Free fall of a point mass on board a satellite. — Prikladnaya Matematika i Mekhanika, Vol. 27, No. 1. 1963.

1.3.5. Position and orientation errors as a function of instrumental and initial setting errors

For an object which is stationary in the $O_1\xi_*\eta_*\zeta_*$ system, in case of constant instrumental errors, the total position errors δx_2, δy_2, δz_2 are expressed in terms of the solutions (1.140) of the first group of equations and in terms of the solutions

$$\theta_x = \int_0^t \Delta m_x\, dt + \theta_x^0,\quad \theta_y = \int_0^t \Delta m_y\, dt + \theta_y^0,\quad \theta_z = \int_0^t \Delta m_z\, dt + \theta_z^0 \tag{1.171a}$$

of the second group of equations, which follow from (1.123) and (1.108). For constant instrumental errors, we have from (1.108), (1.171a), (1.140), and (1.141)

$$\left.\begin{aligned}
\delta x_2 &= \frac{\Delta n_x}{\omega_0^2} + r\theta_y^0 + \left(\delta x^0 - \frac{\Delta n_x}{\omega_0^2}\right)\cos\omega_0 t + \\
&\quad + \frac{\delta\dot{x}^0}{\omega_0}\sin\omega_0 t + r\,\Delta m_y t,\\
\delta y_2 &= \frac{\Delta n_y}{\omega_0^2} - r\theta_x^0 + \left(\delta y^0 - \frac{\Delta n_y}{\omega_0^2}\right)\cos\omega_0 t + \\
&\quad + \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t - r\,\Delta m_x t,\\
\delta z_2 &= -\frac{\Delta n_z}{2\omega_0^2} + \left(\delta z^0 + \frac{\Delta n_z}{2\omega_0^2}\right)\operatorname{ch}\omega_0\sqrt{2}\,t + \\
&\quad + \frac{\delta\dot{z}^0}{\omega_0\sqrt{2}}\operatorname{sh}\omega_0\sqrt{2}\,t.
\end{aligned}\right\} \tag{1.172}$$

The errors δx_2, δy_2 have a constant component, which fluctuates with Schuler's frequency ω_0 (with a period of about 84 min) and a monotonically increasing component, which is proportional to time. Quantitatively, partial errors of 1 km are caused by accelerometer errors Δn_x, $\Delta n_y \approx 8\cdot 10^{-4}$ m/sec^2 (i.e., $\approx 8\cdot 10^{-5}\,g$), initial orientation errors θ_x^0, $\theta_y^0 \approx 1.6\cdot 10^{-6}$ rad ($\approx 0'.55$), errors in the initial coordinates δx^0, $\delta y^0 = 1$ km and initial velocity errors $\delta\dot{x}^0$, $\delta\dot{y}^0 \approx 1.25$ m/sec. For the error due to the inaccuracies Δm_x, Δm_y (gyroscope drift with minus sign) not to exceed 1 km in 1 hour, Δm_x, Δm_y should be at most $4.4\cdot 10^{-8}$ sec^{-1} (i.e., 0.009 deg/hr).

From the third equation in (1.172) it follows that in prolonged operation the error δz_2 increases exponentially. Thus the inertial system is capable of autonomously measuring the distance r from the Earth with acceptable accuracy only over relatively short times. For instance, if the acceptable error is $\delta z_2 = 1$ km, then for the previously cited inaccuracies of accelerometers and initial setting, we have here times of about 10—15 min. Note that since the object is stationary, the errors δz_2 do not affect the errors δx_2, δy_2.

The orientation errors in fixed-attitude axes with constant gyroscope errors are, by (1.117), (1.171a),

$$\left.\begin{aligned}
\theta_{1x} &= -\Delta m_x t - \theta_x^0,\quad \theta_{1y} = -\Delta m_y t - \theta_y^0,\\
\theta_{1z} &= -\Delta m_z t - \theta_z^0,
\end{aligned}\right\} \tag{1.173}$$

i.e., they are proportional to time.

Orientation errors in azimuthally free axes are given by (1.118), (1.119). For constant instrumental errors, we have

$$\left.\begin{aligned}
\theta_{1x} &= -\frac{\Delta n_y}{r\omega_0^2} - \frac{1}{r}\left(\delta y^0 - \frac{\Delta n_y}{\omega_0^2}\right)\cos\omega_0 t - \frac{\delta\dot{y}^0}{r\omega_0}\sin\omega_0 t,\\
\theta_{1y} &= \frac{\Delta n_x}{r\omega_0^2} + \frac{1}{r}\left(\delta x^0 - \frac{\Delta n_x}{\omega_0^2}\right)\cos\omega_0 t + \frac{\delta\dot{x}^0}{r\omega_0}\sin\omega_0 t,\\
\theta_{1z} &= -\Delta m_z t - \theta_z^0.
\end{aligned}\right\} \tag{1.174}$$

It follows from these expressions that the angles θ_{1x}, θ_{1y} which characterize the errors in the direction of the vertical contain only a constant component and one which oscillates with frequency ω_0. A partial error of 1' in the vertical is due to accelerometer errors Δn_x, $\Delta n_y \approx 1.5 \cdot 10^{-3}\,\mathrm{m/sec^2}$, and setting errors δx^0, $\delta y^0 \approx 1.8\,\mathrm{km}$, $\delta\dot{x}^0$, $\delta\dot{y}^0 \approx 2.3\,\mathrm{m/sec}$. The angle θ_{1z} which characterizes the error in the determination of the azimuth contains a growing component proportional to time. This explains the rigid requirements to be met by the azimuthal gyroscopes in inertial systems (directional gyroscopes).

Orientation errors in the geocentric co-moving axes differ from the errors (1.174) only in that the third relation is replaced in accordance with Eq. (1.120), and so

$$\theta_{1z} = -\Delta m_z t - \theta_z^0 + \frac{\delta x_2}{r}\operatorname{tg}\varphi. \tag{1.175}$$

where δx_2 is taken from the first relation in (1.172). The geometrical meaning of the last term in (1.175) is obvious: near the pole small errors in position determination lead to substantial errors in the determination of the northward direction.

Let us now consider an object moving along the arc of a great circle. The total position errors in this case are found from (1.108), where δx, δy, δz and θ_x, θ_y are inserted. The values of δx, δy, δz are taken from (1.146), (1.149); for the errors θ_x, θ_y, θ_z we obtain from (1.123)

$$\left.\begin{aligned}
\theta_x &= \theta_x^0\cos\omega_y t - \theta_z^0\sin\omega_y t + \\
&+ \int_0^t [\Delta m_x\cos\omega_y(t-\tau) - \Delta m_z\sin\omega_y(t-\tau)]\,d\tau,\\
\theta_y &= \theta_y^0 + \int_0^t \Delta m_y\,d\tau,\\
\theta_z &= \theta_x^0\sin\omega_y t + \theta_z^0\cos\omega_y t + \\
&+ \int_0^t [\Delta m_x\sin\omega_y(t-\tau) + \Delta m_z\cos\omega_y(t-\tau)]\,d\tau.
\end{aligned}\right\} \tag{1.176}$$

In case of constant instrumental errors and constant ω_y, we have

$$\left.\begin{aligned}
\theta_x &= \left(\theta_x^0 + \frac{\Delta m_z}{\omega_y}\right)\cos\omega_y t + \left(-\theta_z^0 + \frac{\Delta m_x}{\omega_y}\right)\sin\omega_y t - \frac{\Delta m_z}{\omega_y},\\
\theta_y &= \theta_y^0 + \Delta m_y t,\\
\theta_z &= \left(\theta_x^0 + \frac{\Delta m_z}{\omega_y}\right)\sin\omega_y t - \left(-\theta_z^0 + \frac{\Delta m_x}{\omega_y}\right)\cos\omega_y t + \frac{\Delta m_x}{\omega_y}.
\end{aligned}\right\} \tag{1.177}$$

For velocities much less than the circular velocity, we obtain from (1.108), (1.177), (1.151) the following approximate expressions:

$$\left.\begin{aligned}
\delta x_2 &= \frac{\Delta n_x}{\omega_0^2} + \left(\delta x^0 - \frac{\Delta n_x}{\omega_0^2}\right)\cos\omega_0 t + \frac{\delta\dot{x}^0}{\omega_0}\sin\omega_0 t - \\
&\quad - \frac{\omega_y\sqrt{2}}{3\omega_0}\left[\left(2\delta z^0 + \frac{\Delta n_z}{\omega_0^2}\right)\operatorname{sh}\omega_0\sqrt{2}\,t + \right. \\
&\quad \left. + \frac{\sqrt{2}\,\delta\dot{z}^0}{\omega_0}\operatorname{ch}\omega_0\sqrt{2}\,t\right] + r\theta_y^0 + r\,\Delta m_y t, \\
\delta y_2 &= \frac{\Delta n_y}{\omega_0^2} + \frac{r\,\Delta m_z}{\omega_y} + \left(\delta y^0 - \frac{\Delta n_y}{\omega_0^2}\right)\cos\omega_0 t + \\
&\quad + \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t - r\left(\theta_x^0 + \frac{\Delta m_z}{\omega_y}\right)\cos\omega_y t + r\left(\theta_z^0 - \frac{\Delta m_x}{\omega_y}\right)\sin\omega_y t, \\
\delta z_2 &= -\frac{\Delta n_z}{2\omega_0^2} + \left(\delta z^0 + \frac{\Delta n_z}{2\omega_0^2}\right)\operatorname{ch}\omega_0\sqrt{2}\,t + \frac{\delta\dot{z}^0}{\omega_0\sqrt{2}}\operatorname{sh}\omega_0\sqrt{2}\,t.
\end{aligned}\right\} \qquad (1.178)$$

The expression for δz_2 coincides with the third expression in (1.172). The expression for δx_2 differs from the corresponding expression in (1.172) only in the fourth term, which shows that in this case, as distinct from the stationary case, the position error in the direction of motion contains exponentially growing components. They are multiplied by the ratio ω_y/ω_0. The quantitative effect of these components on δx_2 becomes clear from the following example. Let $\left(\delta z^0 + \frac{\Delta n_z}{2\omega_0^2}\right)$, $\frac{\delta\dot{z}^0}{\omega_0}$ be of the same order of magnitude as $\left(\delta x^0 - \frac{\Delta n_x}{\omega_0^2}\right)$, $\frac{\delta\dot{x}^0}{\omega_0}$. The errors $(\delta z^0 + \Delta n_z/2\omega_0^2)$, $\delta\dot{z}^0/\omega_0$ in this case will naturally have a marked effect on δx_2 only if $(2\sqrt{2}\,\omega_y \operatorname{sh}\omega_0\sqrt{2}t)/3\omega_0$, $(2\omega_y \operatorname{ch}\omega_0\sqrt{2}\,t)/3\omega_0$ become comparable to unity following the increase in $\operatorname{sh}\omega_0\sqrt{2}\,t$, $\operatorname{ch}\omega_0\sqrt{2}\,t$. The time t_1 required for this is readily estimated. If ω_y is of the same order of magnitude as the Earth's spin ($\omega_0/\omega_y = \omega_0/u \approx 18$), we have $t_1 \approx 35$ min, for $\omega_0/\omega_y \approx 6$ (linear velocity $\approx 1{,}000$ m/sec) this time drops to $t_1 \approx 25$ min, for $\omega_0/\omega_y \approx 3$ ($\approx 2{,}000$ m/sec), we have $t_1 \approx 20$ min. It is readily seen that the approximate expression (1.178) for δx_2 is valid for $\omega_y/\omega_0 = 1/3$. If we recall the variation of the roots μ and ν of the characteristic equation (1.147) as a function of ω_y (Figure 1.9), it can readily be shown that Eq. (1.178) for δx_2 gives satisfactory accuracy up to $\omega_y/\omega_0 \approx 1/2$.

Eq. (1.178) for δy_2 differs from the corresponding expression in the stationary case in that it is free from terms proportional to time. However, the last two terms oscillate with the frequency ω_y. The amplitudes of these oscillations are determined by the ratios $r\,\Delta m_x/\omega_y$, $r\,\Delta m_z/\omega_y$. If we stipulate that these amplitudes do not exceed 1 km, the ratios $\Delta m_x/\omega_y$, $\Delta m_z/\omega_y$ should not exceed a figure of the order of $3 \cdot 10^{-4}$. For $\omega_y = u$ we then have Δm_x, $\Delta m_y \approx$ ≈ 0.005 deg/hr, for $\omega_y = 6u$ ($\approx 2{,}000$ m/sec), Δm_x, $\Delta m_y \approx 0.025$ deg/hr, and for $\omega_y = \omega_0$, Δm_x, $\Delta m_y \approx 0.075$ deg/hr.

The orientation errors are specified by Eqs. (1.117)—(1.120), (1.151), (1.177), (1.178). The dependence of these errors on instrumental and setting inaccuracies is the same as for δx, δy, δx_2, θ_x, θ_y, θ_z.

Consider the position and orientation errors for motion in a Keplerian orbit.

For Keplerian motion, from Eqs. (1.123), (1.125), seeing that $\alpha_{11} = -\sin\sigma$, $\alpha_{12} = 0$, $\alpha_{13} = \cos\sigma$, $\alpha_{21} = \cos\sigma$, $\alpha_{22} = 0$, $\alpha_{23} = \sin\sigma$, $\alpha_{31} = 0$, $\alpha_{32} = 1$, $\alpha_{33} = 0$, $\dot{\sigma} = \omega_y$, or

directly solving Eqs. (1.96), we find

$$\left.\begin{aligned}
\theta_x &= \theta_x^0 \cos\int_0^t \omega_y\,d\tau - \theta_z^0 \sin\int_0^t \omega_y\,d\tau + \\
&+ \int_0^t \left[\Delta m_x(\tau)\cos\int_0^{t-\tau}\omega_y\,d\tau - \Delta m_z(\tau)\sin\int_0^{t-\tau}\omega_y\,d\tau\right]d\tau, \\
\theta_y &= \int_0^t \Delta m_y\,dt + \theta_y^0, \\
\theta_z &= \theta_x^0 \sin\int_0^t \omega_y\,d\tau + \theta_z^0\cos\int_0^t \omega_y\,d\tau + \\
&+ \int_0^t \left[\Delta m_x(\tau)\sin\int_0^{t-\tau}\omega_y\,d\tau + \Delta m_z(\tau)\cos\int_0^{t-\tau}\omega_y\,d\tau\right]d\tau.
\end{aligned}\right\} \qquad (1.179)$$

For a circular orbit* and constant instrumental errors, seeing that $\omega_y = \nu = \text{const}$, we find

$$\left.\begin{aligned}
\theta_x &= \theta_x^0\cos\nu t - \theta_z^0\sin\nu t + \frac{\Delta m_x}{\nu}\sin\nu t - \frac{\Delta m_z}{\nu}(1-\cos\nu t), \\
\theta_y &= \Delta m_y t + \theta_y^0, \\
\theta_z &= \theta_x^0\sin\nu t + \theta_z^0\cos\nu t + \left|\frac{\Delta m_x}{\nu}(1-\cos\nu t) + \frac{\Delta m_z}{\nu}\sin\nu t.\right.
\end{aligned}\right\} \qquad (1.180)$$

The total position errors for elliptical motion are obtained from Eqs. (1.108), (1.166), and (1.179). For motion in a circular orbit with constant instrumental errors, we have from Eqs. (1.108), (1.180), and (1.152) ($\nu = \omega_0$)

$$\left.\begin{aligned}
\delta x_2 &= r\theta_y^0 + \delta x^0 + \frac{\delta\dot{x}^0}{\nu}(4\sin\nu t - 3\nu t) + \\
&+ 6\,\delta z^0(\sin\nu t - \nu t) + \frac{2\delta\dot{z}^0}{\nu}(\cos\nu t - 1) + \\
&+ \frac{\Delta n_x}{\nu^2}\left[4(1-\cos\nu t) - \frac{3\nu^2 t^2}{2}\right] + \\
&+ \frac{r\,\Delta m_y}{\nu}(4\sin\nu t - 3\nu t) + \frac{2\Delta n_z}{\nu^2}(\sin\nu t - \nu t), \\
\delta y_2 &= \frac{\Delta n_y}{\nu^2} + \left(\delta y^0 - \frac{\Delta n_y}{\nu^2} - r\theta_x^0\right)\cos\nu t + \left(\frac{\delta\dot{y}^0}{\nu} + r\theta_z^0\right)\sin\nu t, \\
\delta z_2 &= \frac{2(\delta\dot{x}^0 + r\,\Delta m_y)}{\nu}(1-\cos\nu t) + \\
&+ \delta z^0(4 - 3\cos\nu t) + \frac{\delta\dot{z}^0}{\nu}\sin\nu t + \\
&+ \frac{2\Delta n_x}{\nu^2}(\nu t - \sin\nu t) + \frac{\Delta n_z}{\nu^2}(1-\cos\nu t).
\end{aligned}\right\} \qquad (1.181)$$

The right-hand sides of (1.181) are free from exponentially growing terms. They only contain terms proportional to t and t^2, and the terms with t^2 are multiplied only by the instrumental error Δn_x of the longitudinal accelerometer.

From Eqs. (1.181), (1.180), (1.152), (1.117)—(1.119) we find the orientation errors relative to the fixed and orbital axes. These orientation errors are fully described by Eqs. (1.180) and (1.152).

* The solution for elliptical orbits will be found in my "Theory of Inertial Navigation: Autonomous Systems."

Chapter 2

AIDED INERTIAL SYSTEMS: ALTIMETER CORRECTION

§2.1. GENERAL CONSIDERATIONS PERTAINING TO AIDED INERTIAL SYSTEMS

In the previous chapter we discussed the main properties of autonomous inertial systems. These systems determine the navigation and guidance parameters from the information supplied by the readings of inertial sensitive elements, namely accelerometers and gyroscopes. No additional information is used, except the initial conditions which are assumed known.

The equations of errors given in the previous chapter determine the functional stability of autonomous inertial navigation systems. Their solution links up the errors in the navigation parameters, on the one hand, with instrumental errors and inaccuracies in the initial conditions, on the other.

The solutions of the error equations show that an autonomous inertial system cannot function for a long time with satisfactory precision. The total position and orientation errors inevitably build up with time. For motion with low velocity, the position errors increase exponentially, for Keplerian motion the errors are a quadratic (parabolic) function of time. Orientation errors at their best are proportional to time.

Consider an inertial system which is required to fix the position with a definite accuracy during a certain period of continuous operation. Then, given the dependence of navigation errors on instrumental errors and setting inaccuracies, we can in principle design the system elements and set the initial conditions in such a way as to meet the tolerance standards, the error buildup notwithstanding. However, in practice, the resulting tolerances of the elements and the initial conditions may be so stringent as to make the system impracticable.

A possible way out of this dilemma is provided by aided systems using additional guidance information, i.e., inertial systems with correction from external sources of information. This aiding information may include the altitude of the object above the Earth's surface, measured barometrically or with a radio altimeter; the velocity of the object relative to the Earth's surface, determined from the Doppler effect or air log readings; the position of the object relative to the Earth determined by a radio navigation system or radar, etc. The gyroscopic elements of the inertial system may use astronomic (or celestial) correction methods, which in the final analysis involve comparison of the orientation of the gyroscope axes with fixed

directions to stars and planets, or correction using the direction to Earth markers or moving objects (e.g., artificial Earth satellites) whose motion relative to the Earth is known with the desired accuracy.

The simplest aided system can be visualized as follows. The readings of the inertial system are periodically compared with the values of the navigation parameters obtained from other sources of guidance information and appropriate corrections are introduced. The sources of errors and transient processes in the inertial system are not considered. The interval between corrections is determined by the time during which the system errors remain within permissible limits. Since the errors of an inertial system increase at least as a quadratic function of time, the correction interval should rapidly decrease with time for equal error increments. The aided inertial system is thus essentially a device which stores for some time the exact information on the navigation parameters supplied by external sources. Continuous correction is meaningless in this case. Simple periodic correction of the inertial system naturally does not introduce any fundamentally new effects not covered by the theory of autonomous inertial systems.

Further development of the correction principle can take the following course. The readings of the inertial system are periodically corrected, and the inertial system is reset as if the correction time were the initial (starting) time of its operation. The errors built up by the inertial system during the correction interval are thus cleared. The correction intervals remain constant. This correction procedure in practice amounts to another initial setting of the system (we mean here, naturally, its setting on board the moving object). Continuous correction is again meaningless in this case.

The second correction technique, like the first, does not affect the fundamental structure of the inertial system, its operation algorithm, the error equations (except the setting errors), sources of error, and error dynamics. Both these correction techniques do not introduce any new properties into the inertial system. The analysis of the system operation in these cases does not differ from the analysis of autonomous inertial systems.

Much more interesting are the correction techniques which affect the functional algorithm of the inertial system. As a result of the change in the algorithm (the ideal operation equations), the structure of the error equations also changes, i.e., in the final analysis the dependence of system errors on instrumental and setting errors is changed. These correction techniques naturally presuppose availability of the external guidance information for a sufficiently long time or even during the entire course of operation of the inertial system.

Suppose the inertial system measures some coordinates x^1, x^2, x^3 of the object. Let further the external sources supply information from which one of the coordinates, x^1 say, can be computed continuously and independently of the inertial system. In this case we can clearly use the computed value of the coordinate x^1 to form the terms of the ideal operation equations which include this coordinate. By differentiating x^1, we can further form the terms which contain the rate of change $\dot{x}^1$ of the coordinate x^1. Conversely, if the external sources provide information on the rate of change $\dot{x}^1$, integration gives x^1.

Two possibilities arise in this case. We may form all the terms containing $\varkappa^1, \dot{\varkappa}^1$ in the ideal operation equations. The inertial system is then reduced to a two-dimensional setup. The accelerometer whose sensitive axis is along the normal to the coordinate surface $\varkappa^1 = \text{const}$ can be dispensed with in this case. The equation containing $\delta\ddot{\varkappa}^1$ drops out from the set of error equations. In the remaining equations, the terms with $\delta\varkappa^1$, $\delta\dot{\varkappa}^1$ are moved to the right-hand side. They are now known functions of time, provided that the errors in the determination of $\varkappa^1$, $\dot{\varkappa}^1$ from the external guidance information are known functions of time (whether determinate or random).

An alternative approach is to use the external information only for some of the terms with $\varkappa^1, \dot{\varkappa}^1$ in the ideal operation equations. We naturally choose those terms which are responsible for the particularly disturbing and undesirable contributions with $\delta\varkappa^1$, $\delta\dot{\varkappa}^1$ in the error equations. A three-dimensional system, with all the three accelerometers, is retained in this case. The order of the coordinate error equations does not change, the only difference being that some of the terms with $\delta\varkappa^1$, $\delta\dot{\varkappa}^1$ are moved to the right and are dropped out from the homogeneous error equations. As we shall see in the following, both methods of continuous correction using additional information on $\varkappa^1, \dot{\varkappa}^1$ lead to very interesting results.

If the additional guidance sources supply two coordinates, $\varkappa^1$ and $\varkappa^2$ say, as well as their time derivatives, the above procedure can be extended to cover all the coordinates. By forming all the terms in the ideal operation equations from the external information, we reduce our system to a one-dimensional setup, whereas if some of the terms are left in, we have a two-dimensional system. Finally, the three-dimensional arrangement may naturally be preserved.

We have so far assumed the coordinates $\varkappa^1, \varkappa^2$ to be known. The situation hardly changes if the external information sources do not supply the coordinates $\varkappa^1, \varkappa^2$ but one or two relations of the form

$$F_j(\varkappa^1, \varkappa^2, \varkappa^3, t) = 0 \qquad (2.1)$$

among the three coordinates. Furthermore, we have so far dealt with external information on the coordinates of the object, which may permit reducing the number of accelerometers in the system to two or even one. Additional information on orientation can be similarly employed. In this case the system will require only two or possibly one gyroscope. Note, however, that two gyroscopes are quite sufficient for fixing the orientation of the object even without any external information.

A fundamental feature of this aiding technique for inertial systems is the continuous assimilation of the external information in forming the ideal operation equations and the dependence of the structure of the error equations on the particular mode of application of the additional information. This is no longer simple correction of inertial systems, but rather complete operation of some complex mixed system which includes "aiding" devices that supply additional guidance information. In general, these systems can no longer be considered autonomous.

One more point should be borne in mind. When we discussed additional information on the coordinates $\varkappa^1, \varkappa^2$ we tacitly implied that the system was linked up to the outside world. The additional information was obtained over the radio, say. There are, however, sources of information which do

not require communication with the outside world. In this case the system remains autonomous and may even be regarded as purely inertial.

As an example, consider the motion of a ship in the ocean. To find its position in the coordinate system fixed to the Earth, and hence in the $O_1\xi^1\xi^2\xi^3$ axes (the motion of the Earth relative to these axes is known), it suffices to specify two coordinates on the surface of the Earth, the geographical latitude and longitude, say. The third coordinate — the distance of the ship from the Earth's center — is a function of the other two coordinates. If, however, the system measures some other coordinates x^1, x^2, x^3, such that the level ellipsoid does not coincide with any of the coordinate surfaces

$$x^1 = \text{const}, \quad x^2 = \text{const}, \quad x^3 = \text{const}, \tag{2.2}$$

the equation of the level ellipsoid will provide a relation of the form (2.1) among these coordinates. The inertial system can therefore be so designed as to make use of the information on the figure of the Earth, close to which the ship moves. One of the variables can be eliminated from the equation of the system, and it can be reduced to a two-dimensional system. At the same time, the system remains autonomous and purely inertial.

The above example can be developed somewhat further. Let the object move so that its altitude above the Earth's surface is a known function of time. The inertial system also remains autonomous. Finally, if the variation of the altitude with time is not known in advance, but the altitude can be autonomously measured in the course of motion (e.g., with a barometric altimeter), the mixed system comprising the inertial system and the altimeter is again autonomous.

This example naturally leads to more general considerations. Let the approximate trajectory of the object or the equation of the surface near which it moves be known. This information can be used to aid the navigation system. Indeed, some terms in the ideal operation equations of the system may prove to be small compared to other terms. They can therefore either be omitted or formed into functions of time using the approximate values of the parameters which specify the motion of the object. Variations of the corresponding terms then drop out from among the homogeneous error equations. The right-hand sides now contain differences between their true and approximate values. These differences can be regarded as known functions of time in the sense that they can be specified in some way during the analysis of error equations.

Smallness of some terms compared to other terms is obviously insufficient for omitting them from the ideal operation equations. We must estimate the direct error that the omission of each particular term introduces. If the resulting error is substantially less than the system tolerance, or anyhow substantially less than the instrumental errors, the corresponding term of the ideal operation equations may be dropped out. It may prove, however, that none of the terms entering the position or orientation equations gives a small error when omitted, but the variations of some terms are significantly less than the instrumental errors. In this case, the error equations can be considerably simplified by omitting the variations of these terms.

Note that the argument of this section concerning the use of additional guidance information shows that the change in the structure of error equations always has one common property, namely certain terms drop out from the homogeneous error equations, while no new terms are added. In

the present chapter we will consider such cases only. Naturally, the additional information can be put to work in such a way that the error equations do acquire new terms, which do not enter the error equations of purely inertial systems. Analysis of these cases is deferred until later chapters.

The main problem to be considered in the following sections of this chapter is concerned with the analysis of various effects arising in systems with additional guidance information in the form of relation (2.1) among the object coordinates.

We will mainly concentrate on the case when this relation is recovered from additional information on the object's distance from the surface of the Earth. The simplest relation is that which expresses the distance r of the object from the Earth's center as a function of time. This is not a significant restriction in our analysis, while on the other hand it leads to a number of interesting and valuable effects which cannot be realized by other methods.

The particular choice of the distance r is traceable to the fact that we are dealing with the operation of inertial systems in the gravitational field of the Earth, which is nearly spherical. Its properties are determined mainly by the distance of the object from the Earth's center. Furthermore, the analysis of error equations has established that the most dangerous conflict with functional stability arises when the inertial system itself is required to measure the distance r. The terms of ideal operation equations using the r measured by the inertial system itself are the ones responsible for the rapidly growing exponential and power functions of time in the solutions of the error equations.

The use of alternative sources of guidance information for the distance from the Earth's surface largely reduces these difficulties.

§2.2. IDEAL OPERATION EQUATIONS AND ERROR EQUATIONS FOR INERTIAL SYSTEMS WITH ALTIMETER CORRECTION

2.2.1. Three-accelerometer system

Let the distance between the moving object and the surface of the level ellipsoid, i.e., the altitude h above ocean level, be known. From Eqs. (1.8a) then

$$r=\sqrt{\left(\frac{a}{\sqrt{1-e^2\sin^2\varphi'}}+h\right)^2\cos^2\varphi'+\left[\frac{a(1-e^2)}{\sqrt{1-e^2\sin^2\varphi'}}+h\right]^2\sin^2\varphi'}. \qquad (2.3)$$

Specifying the altitude h is thus equivalent to specifying the distance r from the Earth's center as a function of the geographical latitude φ'.

Consider an inertial navigation system in which all the three accelerometers have been retained but the spherical component of the Earth's gravitational field entering the ideal operation equations is formed using Eq. (2.3).

The ideal operation equations for this system are essentially similar to the ideal operation equations of autonomous inertial systems listed in §1.1. They are only supplemented by Eq. (2.3) and appropriately chosen relations expressing φ' in terms of the coordinates that the system measures. Thus,

to Eqs. (1.36), (1.38)—(1.40) we add Eq. (2.3) and the third equality from (1.8a)

$$\zeta = \left[\frac{a(1-e^2)}{\sqrt{1-e^2\sin^2\varphi'}} + h\right]\sin\varphi'. \tag{2.4}$$

Similar changes are introduced in the ideal operation equations (1.54)—(1.58) of inertial systems measuring curvilinear coordinates. To these equations we add Eq. (2.4) where

$$\zeta = \zeta(x^1, x^2, x^3, t). \tag{2.5}$$

The simplest case is obviously that of a system which measures the geographical coordinates. Since h is one of the geographical coordinates, the ideal operation equations (1.91) have to be supplemented by a single relation, Eq. (2.3), which is moreover essential only if the projections g_{y_i}, g_{z_i} entering Eqs. (1.91) are given as functions of r, and not h.

If the inertial system measures the geocentric coordinates, φ' is obtained from the first equality in (1.13). For orthodromic coordinates, the third relation from (1.18) is also added.

For a system measuring spherical curvilinear coordinates with the accelerometers aligned along the contravariant base vectors r^s, taking

$$x^1 = r, \tag{2.6}$$

we get

$$\xi^s = r f^s(x^2, x^3, t). \tag{2.7}$$

where the functions f^s are the direction cosines of the radius-vector r relative to the axes ξ^s, and x^2, x^3 are the curvilinear coordinates on a sphere of radius r. From Eqs. (1.54) we get for $s = 1$:

$$\left.\begin{aligned} \dot{x}^1 &= \int_0^t \left[n_{e_1} - \left(\Gamma^1_{mn}\dot{x}^m\dot{x}^n + 2\Gamma^1_{0n}\dot{x}^n + \Gamma^1_{00} - \frac{\mu}{r^3} - \operatorname{grad}^1\varepsilon\eta^1_l\right)\right] dt + \dot{x}^1(0), \\ x^1 &= \int_0^t \dot{x}^1\, dt + x^1(0). \end{aligned}\right\} \tag{2.8}$$

The function ε is defined by the equality

$$\boldsymbol{g} = \operatorname{grad} V = -\frac{\mu \boldsymbol{r}}{r^3} + \operatorname{grad}\varepsilon, \tag{2.9}$$

i.e., $\varepsilon(\eta^1, \eta^2, \eta^3)$ is the force function of the nonspherical component of the Earth's gravitational field.

For $s = 2, 3$, the only change required in Eqs. (1.54) is the substitution of $\operatorname{grad}^s\varepsilon\eta^s_l$ for $\operatorname{grad}^s V'\eta^s_l$.

Let us now consider the changes in the error equations of inertial systems when additional information on the altitude h is received. We recall that the first group of error equations of autonomous inertial systems, Eqs. (1.95), is obtained by projecting onto the moving axes xyz the vector equation (1.92):

$$\frac{d^2\delta\boldsymbol{r}}{dt^2} + \frac{\mu\,\delta\boldsymbol{r}}{r^3} - \frac{\mu\boldsymbol{r}}{r^3}\frac{3\boldsymbol{r}\cdot\delta\boldsymbol{r}}{r^2} = \Delta\boldsymbol{n} - \Delta\boldsymbol{m}\times\frac{d\boldsymbol{r}}{dt} - \frac{d}{dt}(\Delta\boldsymbol{m}\times\boldsymbol{r}). \tag{2.10}$$

If the r obtained from Eq. (2.3) is used in the ideal operation equations only for forming the ratio μ/r^3 (and it is with this particular case that we are concerned here), only the last term in the left-hand side of Eq. (2.10) is affected, as it contains the factor

$$-\frac{3\boldsymbol{r}\cdot\delta\boldsymbol{r}}{r^5} = \delta\left(\frac{1}{r^3}\right) = \frac{1}{|\boldsymbol{r}+\delta\boldsymbol{r}|^3} - \frac{1}{r^3}. \tag{2.11}$$

This expression now takes the form

$$\delta\left(\frac{1}{r^3}\right) = -\frac{3\Delta r}{r^4}, \tag{2.12}$$

and

$$\Delta r = \frac{\partial r}{\partial h}\Delta h + \frac{\partial r}{\partial \varphi'}\delta\varphi', \tag{2.13}$$

where the derivatives $\frac{\partial r}{\partial h}$ and $\frac{\partial r}{\partial \varphi'}$ are evaluated using (2.3).

Since e is small, Eq. (2.3) can be approximately written as

$$r = a + h - \frac{1}{2}ae^2\sin^2\varphi', \tag{2.14}$$

whence

$$\Delta r = \Delta h - ae^2\sin\varphi'\cos\varphi'\delta\varphi' \tag{2.15}$$

$ae^2\delta\varphi'$ is of the same order of smallness as $e^2\delta x$, $e^2\delta y$, $e^2\delta z$. In the derivation of Eq. (2.10), which is now used as the starting equation, we ignored the variation of the nonspherical field component. This variation also gives a contribution proportional to $e^2\delta x$, $e^2\delta y$, $e^2\delta z$. To be consistent, the second term in Eq. (2.15) thus can also be ignored.

In view of the preceding, we obtain from Eq. (2.10)

$$\left.\begin{aligned} &\frac{d^2\delta r}{dt^2} + \frac{\mu\delta r}{r^3} = \Delta n - \Delta m \times \frac{dr}{dt} - \frac{d}{dt}(\Delta m \times r) + \frac{3\mu r\,\Delta r}{r^4} \\ &(\Delta r = \Delta h), \end{aligned}\right\} \tag{2.15a}$$

where Δh is the altimeter error.

Eq. (2.15a) projected onto the co-moving axes xyz, with the axis z pointing along r, gives

$$\left.\begin{aligned} &\delta\ddot{x} + \left(\frac{\mu}{r^3} - \omega_y^2 - \omega_z^2\right)\delta x + (\omega_x\omega_y - \dot{\omega}_z)\delta y - \\ &\quad - 2\omega_z\delta\dot{y} + (\omega_x\omega_z + \dot{\omega}_y)\delta z + 2\omega_y\delta\dot{z} = \\ &\quad = \Delta n_x - r\Delta\dot{m}_y - 2\dot{r}\Delta m_y - r\omega_x\Delta m_z - r\omega_z\Delta m_x, \\ &\delta\ddot{y} + \left(\frac{\mu}{r^3} - \omega_z^2 - \omega_x^2\right)\delta y + (\omega_y\omega_z - \dot{\omega}_x)\delta z - \\ &\quad - 2\omega_x\delta\dot{z} + (\omega_y\omega_x + \dot{\omega}_z)\delta x + 2\omega_z\delta\dot{x} = \\ &\quad = \Delta n_y + 2\dot{r}\Delta m_x + r\Delta\dot{m}_x - r\omega_y\Delta m_z - r\omega_z\Delta m_y, \\ &\delta\ddot{z} + \left(\frac{\mu}{r^3} - \omega_x^2 - \omega_y^2\right)\delta z + (\omega_z\omega_x - \dot{\omega}_y)\delta x - \\ &\quad - 2\omega_y\delta\dot{x} + (\omega_z\omega_y + \dot{\omega}_x)\delta y + 2\omega_x\delta\dot{y} = \\ &\quad = \Delta n_z + 2r(\omega_x\Delta m_x + \omega_y\Delta m_y) + \frac{3\mu\Delta r}{r^3} \\ &\quad (\Delta r = \Delta h). \end{aligned}\right\} \tag{2.16}$$

The second group of error equations (1.96), Eqs. (1.97), (1.116), and the ensuing relations do not change.

2.2.2. Two-accelerometer system

If the altitude h is known, two accelerometers, as we have noted above, are quite sufficient for system operation. Indeed, knowledge of h implies that a relation of the form (2.1) is available, so that one of the three coordinates of the object can be expressed in terms of the other two. Two accelerometers are clearly sufficient to measure the two independent coordinates.

The most natural approach is that using two accelerometers in a plane which is close to the tangent plane to the surface $h = \text{const}$, e.g., systems with two accelerometers arranged in the plane of the geocentric horizon.

Suppose the system measures the Cartesian coordinates ξ, η, ζ. Let the axis z of the unperturbed platform coincide with the direction of the radius vector r, and let the accelerometers n_x, n_y be aligned along the axes x, y of the platform. The ideal operation equations of the system are then obtained from the equations of the autonomous inertial system (1.36), (1.38), (1.39) in the following way. The projections of Eqs. (1.36) onto the axis z are dropped out. In the x and y projections we take $x = y = 0$, $z = r$, which gives

$$\left.\begin{aligned} v_x &= \int_0^t (n_x - m_y v_z - m_z v_y + g_x)\, dt + v_x^0, \\ v_y &= \int_0^t (n_y - m_z v_x + m_x v_z + g_y)\, dt + v_y^0, \\ v_x &= m_y r, \quad v_y = -m_x r, \quad v_z = \dot{r}. \end{aligned}\right\} \tag{2.17}$$

The vector equations (1.38) are projected on all the three axes, x, y, z. We thus obtain scalar equations for the direction cosines β_{ij}:

$$\left.\begin{aligned} \beta_{i1} &= \int_0^t [\beta_{i2}(m_z - u_z) - \beta_{i3}(m_y - u_y)]\, dt + \beta_{i1}^0, \\ \beta_{i2} &= \int_0^t [\beta_{i3}(m_x - u_x) - \beta_{i1}(m_z - u_z)]\, dt + \beta_{i2}^0, \\ \beta_{i3} &= \int_0^t [\beta_{i1}(m_y - u_y) - \beta_{i2}(m_x - u_x)]\, dt + \beta_{i3}^0 \\ & (i = 1, 2, 3). \end{aligned}\right\} \tag{2.18}$$

In these equations m_x, m_y are calculated according to the third and fourth equalities in (2.17). m_z is arbitrary, being determined by the orientation of the axes x, y in the plane perpendicular to r. Now, in (2.18),

$$u_x = u\beta_{31}, \quad u_y = u\beta_{32}, \quad u_z = u\beta_{33}. \tag{2.19}$$

Eqs. (1.39) remain valid. Using these equations we find the coordinates ξ, η, ζ defined by the equalities

$$\xi = \beta_{13} r, \quad \eta = \beta_{23} r, \quad \zeta = \beta_{33} r. \tag{2.20}$$

The projections g_x, g_y entering the integrands in the first two equations in (2.17) are found from

$$\left.\begin{aligned} g_x &= \frac{\partial \varepsilon}{\partial \xi}\beta_{11} + \frac{\partial \varepsilon}{\partial \eta}\beta_{21} + \frac{\partial \varepsilon}{\partial \zeta}\beta_{31}, \\ g_y &= \frac{\partial \varepsilon}{\partial \xi}\beta_{12} + \frac{\partial \varepsilon}{\partial \eta}\beta_{22} + \frac{\partial \varepsilon}{\partial \zeta}\beta_{32}, \\ \varepsilon &= \varepsilon(\xi, \eta, \zeta). \end{aligned}\right\} \tag{2.21}$$

Eqs. (2.17)—(2.21) are supplemented with Eqs. (2.3), (2.4). Now, if a stable platform is the main element and the xyz axes are oriented relative to this platform, we calculate the direction cosines α_{ij} between the platform axes and the axes x, y, z. To this end we use the matrix (1.7) of direction cosines between the axes ξ, η, ζ and ξ_*, η_*, ζ_*. The latter, in particular, may coincide with the stable platform axes. If, on the other hand, the

system uses a controlled gyroframe, the control torques are formed in accordance with Eqs. (1.42):

$$M_y^2 = -H_2 m_x, \quad M_x^1 = H_1 m_y, \quad M_x^3 = -H_3 m_z. \tag{2.22}$$

In the case of spherical curvilinear coordinates, the ideal operation equations of a two-accelerometer system can be also obtained from Eqs. (1.54)—(1.58) for autonomous systems. For orthogonal spherical coordinates we may clearly start also with Eqs. (1.61), (1.55), (1.57), (1.62) or (1.67), (1.69), (1.70), (1.64), (1.42). In these equations we only have to drop out those relations which determine $x^1 = r$ and in the other equations insert for x^1 its expression from Eq. (2.3). Moreover, an analog of Eq. (2.4) should be added, which relates the curvilinear spherical coordinates x^2, x^3 to the geographical latitude φ' entering Eq. (2.3).

For orthodromic coordinates, Eqs. (1.81) are replaced by the following equations, say:

$$\left.\begin{aligned}
v_x &= \int_0^t (n_x + v_y\omega_z - \dot{r}\omega_y + g_x)\,dt + v_x^0,\\
v_y &= \int_0^t (n_y - v_x\omega_z + \dot{r}\omega_x + g_y)\,dt + v_y^0,\\
\omega_x &= -\frac{v_y}{r}, \quad \omega_y = \frac{v_x}{r},\\
S &= \int_0^t \frac{1}{\cos z}\,[\omega_y + u(\beta_{31}\sin z\cos S + \\
&\qquad + \beta_{32}\sin z\sin S - \beta_{33}\cos z)]\,dt + S^0,\\
z &= \int_0^t [-\omega_x + u(-\beta_{31}\sin S + \beta_{32}\cos S)]\,dt + z^0,\\
\omega_z &= \omega_y \operatorname{tg} z + \frac{u}{\cos z}(\beta_{31}\cos S + \beta_{32}\sin S),\\
g_x &= g_{y_2}(r, \varphi)\frac{1}{\cos\varphi}(-\beta_{31}\sin S + \beta_{32}\cos S),\\
g_y &= g_{y_3}(r, \varphi)\frac{1}{\cos\varphi}(-\beta_{31}\sin z\cos S - \\
&\qquad - \beta_{32}\sin z\sin S + \beta_{33}\cos z),\\
\sin\varphi &= \beta_{31}\cos z\cos S + \beta_{32}\cos z\sin S + \beta_{33}\sin z,\\
M_y^2 &= -H_2\omega_x, \quad M_x^1 = H_1\omega_y, \quad M_x^3 = -H_3\omega_z,\\
\operatorname{tg}\varphi &= \left[1 - \frac{ae^2}{a + h(1 - e^2\sin^2\varphi')^{1/2}}\right]\operatorname{tg}\varphi',\\
r &= \left[\frac{a(1-e^2)}{(1 - e^2\sin^2\varphi')^{1/2}} + h\right]\frac{\sin\varphi'}{\sin\varphi}.
\end{aligned}\right\} \tag{2.23}$$

To simplify the notation, the co-moving geocentric axes are designated x, y, z, and not x_3, y_3, z_3 as in Eqs. (1.81). The last two equalities in (2.23) are equivalent to (2.3)—(2.5). The function g_{y_3} is the projection of the Earth's gravitational acceleration along the tangent to the geocentric meridian. This function is defined by the first equation in (1.27)

The ideal operation equations in geocentric coordinates are obtained from Eqs. (2.23) if we take

$$\left.\begin{aligned}
&z = \varphi, \quad S = \lambda,\\
&\beta_{31} = \beta_{32} = 0, \quad \beta_{33} = 1.
\end{aligned}\right\} \tag{2.24}$$

The eighth and the tenth equation in (2.23) drop out, and the remaining equations take the form

$$
\left.
\begin{aligned}
v_x &= \int_0^t (n_x + v_y\omega_z - \dot{h}\omega_y)\,dt + v_x^0,\\
v_y &= \int_0^t (n_y - v_x\omega_z + r\omega_x + g_y)\,dt + v_y^0,\\
\omega_x &= -\frac{v_y}{r}, \quad \omega_y = \frac{v_x}{r},\\
\lambda &= \int_0^t \left(\frac{\omega_y}{\cos\varphi} - u\right) dt + \lambda^0,\\
\varphi &= -\int_0^t \omega_x\,dt + \varphi'^0,\\
\omega_z &= \omega_y \operatorname{tg}\varphi, \quad g_y = g_{y_2}(r,\ \varphi),\\
M_y^2 &= -H_2\omega_x, \quad M_x^1 = H_1\omega_y,\\
M_x^3 &= -H_3\omega_z,\\
\operatorname{tg}\varphi &= \left[1 - \frac{ae^2}{a + h\,(1 - e^2\sin^2\varphi')^{1/2}}\right]\operatorname{tg}\varphi',\\
r &= \left[\frac{a\,(1-e^2)}{(1 - e^2\sin^2\varphi')^{3/2}} + h\right]\frac{\sin\varphi}{\sin\varphi'}.
\end{aligned}
\right\} \qquad (2.25)
$$

We have so far discussed the ideal operation equations of systems with two accelerometers in the plane of the geocentric horizon (normal to the radius-vector r). The accelerometers may naturally be aligned in the plane of the geographical horizon too (tangent to the surface h = const). The simplest arrangement of this kind is a system measuring the geographical coordinates which uses a controlled gyroframe with the x and y axes directed to the points of the compass. Since the altimeter reading h is one of the coordinates in this case, the corresponding ideal operation equations are obtained from Eqs. (1.91), omitting the third and the fourth equality, which give $\dot{h}$ and h. Supplementing this system with Eqs. (1.13) and changing the subscripts to x, y, z, we obtain the following set of equations:

$$
\left.
\begin{aligned}
v_x &= \int_0^t (n_x + v_y\omega_z - \dot{h}\omega_y)\,dt + v_x^0,\\
v_y &= \int_0^t (n_y + \dot{h}\omega_x - v_x\omega_z + g_y)\,dt + v_y^0,\\
\omega_x &= -\frac{v_y}{r_1}, \quad \omega_y = \frac{v_x}{r_2}, \quad \omega_z = \omega_y \operatorname{tg}\varphi',\\
\varphi &= -\int_0^t \omega_x\,dt + \varphi'^0,\\
\lambda &= \int_0^t \left(\frac{\omega_y}{\cos\varphi} - u\right) dt + \lambda^0,\\
r_2 &= \frac{a}{\sqrt{1 - e^2\sin^2\varphi}} + h,\\
r_3 &= \frac{a\,(1-e^2)}{(1 - e^2\sin^2\varphi')^{3/2}} + h,\\
M_y^2 &= H_2\omega_x, \quad M_x^1 = H_1\omega_y, \quad M_z^3 = -H_3\omega_z,\\
g_y &= -g_{y_2}(r,\ \varphi)\sin(\varphi' - \varphi) + g_{y_3}(r,\ \varphi)\cos(\varphi' - \varphi),
\end{aligned}
\right\} \qquad (2.26)
$$

$$\left.\begin{aligned} \operatorname{tg}\varphi &= \left[1 - \frac{ae^2}{a + h\,(1 - e^2\sin^2\varphi')^{1/2}}\right]\operatorname{tg}\varphi', \\ r &= \left[\frac{a\,(1-e^2)}{(1 - e^2\sin^2\varphi')^{1/2}} + h\right]\frac{\sin\varphi'}{\sin\varphi}. \end{aligned}\right\} \tag{2.26}$$

To find the difference $\varphi' - \varphi$ which enters the expression for g_y we may use Eq. (1.15), which is a consequence of the last two relations in (2.26).

Let us derive the equations of errors for systems with two accelerometers which in unperturbed state lie in the plane of the geocentric horizon.

The first group of error equations projected onto the co-moving axes xyz, with the axis z directed along the geocentric vertical (along the vector r), is clearly derived directly from Eqs. (2.16). To this end, we drop out the third equation in (2.16) and in the first two write

$$\delta z = \Delta r \ (= \Delta h). \tag{2.27}$$

This gives a set of fourth-order differential equations

$$\left.\begin{aligned} \delta\ddot{x} + \left(\tfrac{\mu}{r^3} - \omega_y^2 - \omega_z^2\right)\delta x + (\omega_x\omega_y - \dot{\omega}_z)\,\delta y - 2\omega_z\,\delta\dot{y} = \\ = -\Delta r\,(\omega_x\omega_z + \dot{\omega}_y) - 2\Delta\dot{r}\omega_y + \Delta n_x - r\Delta\dot{m}_y - \\ - 2\dot{r}\,\Delta m_y - r\omega_x\,\Delta m_z - r\omega_z\,\Delta m_x, \\ \delta\ddot{y} + \left(\tfrac{\mu}{r^3} - \omega_z^2 - \omega_x^2\right)\delta y + (\omega_y\omega_x + \dot{\omega}_z)\,\delta x + 2\omega_z\,\delta\dot{x} = \\ = \Delta r\,(\dot{\omega}_x - \omega_y\omega_z) + 2\Delta\dot{r}\omega_x + \Delta n_y + 2\dot{r}\,\Delta m_x + \\ + r\Delta\dot{m}_x - r\omega_y\,\Delta m_z - r\omega_z\,\Delta m_y. \end{aligned}\right\} \tag{2.28}$$

Note that we derived these equations by taking

$$\delta z = \Delta r$$

in the first two equations in (2.16).

This equality is valid for small δx, δy, when their squares are ignorable (together with the squares of δz and Δr). Indeed, from the identity

$$x^2 + y^2 + z^2 = r^2 \tag{2.29}$$

we have (in the unperturbed state $x = y = 0$, $z = r$)

$$(\delta x)^2 + (\delta y)^2 + (\delta z)^2 + 2r\,\delta z = 2r\,\Delta r + (\Delta r)^2, \tag{2.30}$$

whence, omitting the quadratic terms, we get $\delta z = \Delta r$.

If we are interested in higher approximations, Eqs. (2.28) should be replaced by the nonlinear system comprising the first two equations in (2.16) and Eq. (2.30):

$$\left.\begin{aligned} \delta\ddot{x} + \left(\tfrac{\mu}{r^3} - \omega_y^2 - \omega_z^2\right)\delta x + (\omega_x\omega_y - \dot{\omega}_z)\,\delta y - \\ - 2\omega_z\,\delta\dot{y} + (\omega_x\omega_z + \dot{\omega}_y)\,\delta z + 2\omega_y\,\delta\dot{z} = \\ = \Delta n_x - r\Delta\dot{m}_y - 2\dot{r}\,\Delta m_y - r\omega_x\,\Delta m_z - r\omega_z\,\Delta m_x, \\ \delta\ddot{y} + \left(\tfrac{\mu}{r^3} - \omega_z^2 - \omega_x^2\right)\delta y + (\omega_y\omega_z - \dot{\omega}_x)\,\delta z - \\ - 2\omega_x\,\delta\dot{z} + (\omega_y\omega_x + \dot{\omega}_z)\,\delta x + 2\omega_z\,\delta\dot{x} = \\ = \Delta n_y + 2\dot{r}\,\Delta m_x + r\Delta\dot{m}_x - r\omega_y\,\Delta m_z - r\omega_z\,\Delta m_y, \\ (\delta x)^2 + (\delta y)^2 + (\delta z)^2 + 2r\,\delta z = 2r\,\Delta r + (\Delta r)^2. \end{aligned}\right\} \tag{2.31}$$

It can be shown that Eqs. (2.28) also constitute the first group of error equations for systems with two accelerometers in the plane of the geographical

(and not geocentric) horizon. This follows from the fact that the corresponding ideal operation equations differ in terms of first order of smallness (containing a factor e^2). Therefore, the variational equations of errors differ in terms of second order, and consequently coincide to first approximation.

Eqs. (2.28) or Eqs. (2.31) constitute the first group of error equations for two-accelerometer systems. The second group of error equations (1.96), Eqs. (1.97) and (1.116) for the total position and orientation errors (and all the relations derived from these equations) remain as before.

In conclusion of this section let us consider an important property of Eqs. (2.28) or (2.31). These equations are invariant under rotation of the xyz axes through an angle ϑ around the axis z, i.e., they are invariant under the rotation group of the axis z. This property of Eqs. (2.28) follows directly from the arbitrary choice of $\omega_z = m_z$ in ideal operation equations (2.17), (2.18), (2.22). It can be also established by direct calculations, making the following substitution of variables in Eqs. (2.28), (2.31):

$$\left.\begin{aligned} \delta x &= \delta x' \cos\vartheta - \delta y' \sin\vartheta, \\ \delta y &= \delta x' \sin\vartheta + \delta y' \cos\vartheta. \end{aligned}\right\} \tag{2.32}$$

Expressing the variations δx, δy in Eqs. (2.28) in terms of $\delta x'$ $\delta y'$ from (2.32), we multiply the first equation by $\cos\vartheta$, the second by $\sin\vartheta$, and add the two equations. The resulting equation is identical in form to the first equation in (2.28), the only difference being that δx, δy, ω_x, ω_y, ω_z, Δn_x, Δn_y, Δm_x, Δm_y, Δm_z have been replaced by $\delta x'$, $\delta y'$, $\omega_{x'}$, $\omega_{y'}$, $\omega_{z'}$, $\Delta n_{x'}$, $\Delta n_{y'}$, $\Delta m_{x'}$, $\Delta m_{y'}$, $\Delta m_{z'}$, $\omega_{x'}$, $\omega_{y'}$ are expressed in terms of ω_x and ω_y by the relations

$$\left.\begin{aligned} \omega_{x'} &= \omega_x \cos\vartheta + \omega_y \sin\vartheta, \\ \omega_{y'} &= -\omega_x \sin\vartheta + \omega_y \cos\vartheta. \end{aligned}\right\} \tag{2.33}$$

Similar expressions are written for $\Delta n_{x'}$, $\Delta n_{y'}$ in terms of Δn_x, Δn_y and for $\Delta m_{x'}$, $\Delta m_{y'}$ in terms of Δm_x, Δm_y. Finally,

$$\omega_{z'} = \omega_z + \dot{\vartheta}, \quad \Delta m_{z'} = \Delta m_z. \tag{2.33a}$$

Further multiplying the first equation in (2.28) (after substituting from (2.32)) by $-\sin\vartheta$ and the second by $\cos\vartheta$, and adding the two equations, we obtain an equation which differs from the second equation in (2.28) only in that the subscripts x, y, z have been replaced by x', y', z'. This also proves the validity of our proposition for Eqs. (2.31), since in this case evidently

$$\delta z' = \delta z. \tag{2.34}$$

The invariance of Eqs. (2.28), (2.31) under rotations of the xyz axes about the geocentric vertical (the axis z) permits choosing the orientation of the axes x, y in the most convenient way. We will mention only two of the numerous possible orientations of the xyz axes: 1) one of the horizontal axes, the x axis, say, is coplanar with the absolute velocity vector $d\mathbf{r}/dt$; 2) the orientation of the axes x, y in the horizontal plane is such that $\omega_z = 0$.

In case 1 we shoulc take $\omega_x = 0$ in Eqs. (2.28), (2.31). Eqs. (2.28) thus take the form

$$\left.\begin{aligned} &\delta\ddot{x} + \left(\frac{\mu}{r^3} - \omega_y^2 - \omega_z^2\right)\delta x - \omega_z \delta y - 2\omega_z \delta\dot{y} = \\ &\qquad = -\Delta r\dot{\omega}_y - 2\Delta\dot{r}\omega_y + \Delta n_x - r\Delta\dot{m}_y - 2\dot{r}\,\Delta m_y - r\omega_z \Delta m_x, \\ &\delta\ddot{y} + \left(\frac{\mu}{r^3} - \omega_z^2\right)\delta y + \dot{\omega}_z \delta x + 2\omega_z \delta\dot{x} = \\ &\qquad = -\Delta r\omega_y\omega_z + 2\Delta\dot{r}\omega_x + \Delta n_y + 2\dot{r}\,\Delta m_x + r\Delta\dot{m}_x - r\omega_y \Delta m_z - r\omega_z \Delta m_y. \end{aligned}\right\} \tag{2.35}$$

In case 2, when $\omega_z = 0$, Eqs. (2.28) take the form

$$\left.\begin{aligned} &\delta\ddot{x} + \left(\frac{\mu}{r^3} - \omega_y^2\right)\delta x + \omega_x\omega_y\,\delta y = \\ &= -\Delta r\dot{\omega}_y - 2\Delta\dot{r}\omega_y + \Delta n_x - r\Delta\dot{m}_y - 2\dot{r}\,\Delta m_y - r\omega_x\,\Delta m_y, \\ &\delta\ddot{y} + \left(\frac{\mu}{r^3} - \omega_x^2\right)\delta y + \omega_x\omega_y\,\delta x = \\ &= \Delta r\dot{\omega}_x + 2\Delta\dot{r}\omega_x + \Delta n_y + 2\Delta m_x\dot{r} + r\Delta\dot{m}_x - r\omega_y\,\Delta m_z. \end{aligned}\right\} \qquad (2.36)$$

ω_x, ω_y entering these equations are related to the horizontal components v_x, v_y of the absolute velocity by the equalities $v_x = r\omega_y$, $v_y = -r\omega_x$. They are thus always bounded, and as we shall see in what follows Eqs. (2.36) consequently may be analyzed by approximate methods.

§2.3. STABILITY ANALYSIS AND INTEGRATION OF THE FIRST GROUP OF ERROR EQUATIONS FOR TWO-ACCELEROMETER SYSTEMS

2.3.1. Stability analysis in cases leading to equations with constant coefficients

As we have noted before, the second group of error equations of two-accelerometer systems is no different from the corresponding equations for autonomous inertial systems and the results of 1.3.1 therefore remain valid. The main difference between autonomous inertial systems and two-accelerometer systems is traceable to the first group of error equations, the analysis of which is our immediate task.

In the present chapter we will only consider first-approximation equations, i.e., variational equations (2.28). The nonlinear equations (2.31) will be treated in the next chapter, after we have established the analogy between two-accelerometer inertial systems and Schuler tuned systems. This analogy will greatly simplify the analysis of the nonlinear equations (2.31).

In the general case of arbitrary motion, Eqs. (2.28) constitute a fourth-order system of differential equations with variable coefficients, which are either determinate or random functions of time. General analysis of Eqs. (2.28) meets with insurmountable mathematical difficulties. Thus, even the simple case of motion along the arc of a fixed-orientation great circle leads to a Mathieu—Hill equation if variable velocity is permitted.

Eqs. (2.28) can be comprehensively analyzed only in those cases when they are reducible to equations with constant coefficients and for Keplerian motion. Eqs. (2.28) reduce to equations with constant coefficients at least in three cases. The first case is that of a stationary object in the $O_1\xi_*\eta_*\zeta_*$ frame. This case is observed, say, when an object moves along the parallel against the Earth's spin with a velocity equal to the circular velocity of the points on the Earth's surface. In this case, we take $\omega_x = 0$, $\omega_y = 0$, $\omega_z = 0$ in Eqs. (2.28) and write

$$\left.\begin{aligned} &\delta\ddot{x} + \omega_0^2\,\delta x = \Delta n_x - r\Delta\dot{m}_y, \\ &\delta\ddot{y} + \omega_0^2\,\delta y = \Delta n_y + r\Delta\dot{m}_x, \end{aligned}\right\} \qquad (2.37)$$

where

$$\omega_0^2 = \frac{\mu}{r^3} = \text{const.} \tag{2.38}$$

The second case is motion with constant velocity along the arc of a fixed great circle. The equations of errors for this case are obtained from Eqs. (2.35) if $\omega_z = 0$, or from Eqs. (2.36) if $\omega_x = 0$. This gives

$$\left.\begin{aligned} \delta\ddot{x} - (\omega_0^2 - \omega_y^2)\,\delta x &= -2\Delta\dot{r}\omega_y + \Delta n_x - r\Delta m_y, \\ \delta\ddot{y} + \omega_0^2\,\delta y &= \Delta n_y + r\Delta\dot{m}_x - r\omega_y\,\Delta m_z. \end{aligned}\right\} \tag{2.39}$$

where

$$\omega_y = \frac{v_x}{r} = \text{const.} \tag{2.40}$$

Finally, Eqs. (2.28) are reduced to equations with constant coefficients if the object moves with constant velocity at a constant distance from the Earth's center with steady circulation. The equations for this case are obtained from Eqs. (2.35) where we should take $\omega_z = \text{const}$, $\mu/r^3 = \omega_0^2 = \text{const}$, $\omega_y = v/r = \text{const}$, which gives

$$\left.\begin{aligned} \delta\ddot{x} + (\omega_0^2 - \omega_y^2 - \omega_z^2)\,\delta x - 2\omega_z\,\delta\dot{y} &= \\ &= \Delta n_x - r\Delta\dot{m}_y - r\omega_z\,\Delta m_z, \\ \delta\ddot{y} + (\omega_0^2 - \omega_z^2)\,\delta y + 2\omega_z\,\delta x &= -\Delta r\omega_z\omega_z + \\ &+ \Delta n_y + r\Delta\dot{m}_x - r\omega_y\,\Delta m_z - r\omega_z\,\Delta m_y. \end{aligned}\right\} \tag{2.41}$$

A particular case are the equations of an object moving with constant velocity along a parallel. Eqs. (2.41) are quite general, so that Eqs. (2.39) and (2.37) are particular cases of Eqs. (2.41).

The homogeneous equations (2.37) are equations of harmonic oscillations with Schuler's frequency, i.e., the motion in this case is nonasymptotically stable. Similarly, the homogeneous equations (2.39) are also equations of harmonic oscillations. However, the frequencies of these oscillations are different in the plane of motion of the object and in the normal plane through the vector r. In the plane of motion the harmonic frequency is $\sqrt{\omega_0^2 - \omega_y^2}$, and in the normal plane it is equal to Schuler's frequency ω_0. Eqs. (2.39) reveal nonasymptotic stability for

$$\omega_y^2 < \omega_0^2, \tag{2.42}$$

i.e., at velocities less than the circular velocity of the object.

Consider the homogeneous system (2.41). Here the equations are not separated. The characteristic equation of this system is biquadratic:

$$p^4 + p^2(2\omega_0^2 - \omega_y^2 + 2\omega_z^2) + (\omega_0^2 - \omega_z^2)(\omega_0^2 - \omega_y^2 - \omega_z^2) = 0. \tag{2.43}$$

For (nonasymptotic) stability, the roots of the characteristic equation (2.43) should be either zero or imaginary, and multiple roots of the characteristic equation should correspond to linear elementary divisors of the characteristic polynomial of Eqs. (2.41). Now, Eq. (2.43) can have zero or pure imaginary roots only if the roots of the quadratic

$$q^2 + q(2\omega_0^2 - \omega_y^2 + 2\omega_z^2) + (\omega_0^2 - \omega_z^2)(\omega_0^2 - \omega_y^2 - \omega_z^2) = 0 \tag{2.44}$$

are real and nonpositive. This is evidently so if the coefficients of Eq. (2.44)

are nonnegative and its discriminant Δ is nonpositive, which leads to the following three constraints:

$$\left.\begin{aligned} 2\omega_0^2 - \omega_y^2 + 2\omega_z^2 &\geqslant 0, \\ (\omega_0^2 - \omega_z^2)(\omega_0^2 - \omega_y^2 - \omega_z^2) &\geqslant 0, \\ 2\omega_z^2\omega_y^2 - \frac{\omega_y^4}{4} - 4\omega_0^2\omega_z^2 &\leqslant 0. \end{aligned}\right\} \tag{2.45}$$

The last inequality in (2.45) defines a hyperbola in the ω_y^2, ω_z^2 plane (Figure 2.1):

$$\omega_z^2 = \frac{\omega_y^4}{8(\omega_y^2 - 2\omega_0^2)} \tag{2.46}$$

with the asymptotes

$$\omega_y^2 = 2\omega_0^2, \quad \omega_z^2 = \frac{\omega_0^2}{4} + \frac{\omega_y^2}{8}. \tag{2.47}$$

There are two stable regions (diagonally hatched in Figure 2.1). The first is limited by the lines

$$\omega_y^2 = 0, \quad \omega_z^2 = 0, \quad \omega_0^2 - \omega_y^2 - \omega_z^2 = 0 \tag{2.48}$$

being defined by the single inequality

$$\omega_0^2 - \omega_y^2 - \omega_z^2 > 0. \tag{2.49}$$

The second stability region is limited by the lines

$$\omega_y^2 = 0, \quad \omega_z^2 = \omega_0^2 \tag{2.50}$$

and the part of the upper branch of the hyperbola between the points $(2\omega_0^2, \infty)$ and $(4\omega_0^2, \omega_0^2)$. Inside these regions the roots of the characteristic equation (2.43) are simple and pure imaginary.

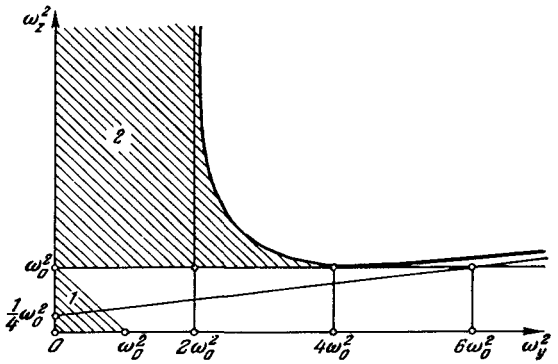

FIGURE 2.1.

At the limits of the stability regions the following situation is observed. At the point ($\omega_y^2 = 0$, $\omega_z^2 = 0$) the characteristic equation has two double roots

$$p_{1,2} = p_{3,4} = \pm j\omega_0; \tag{2.51}$$

the corresponding elementary divisors of the characteristic polynomial of Eqs. (2.41) are linear.

On the segment of the straight line $\omega_z^2 = 0$, $\omega^2 < \omega_0^2$ the imaginary roots are different:

$$p_{1,2} = \pm j\omega_0, \quad p_{3,4} = \pm j\sqrt{\omega_0^2 - \omega_y^2}. \tag{2.52}$$

At the point ($\omega_z^2 = 0$, $\omega_0 = \omega_0^2$) we have

$$p_{1,2} = \pm j\omega_0, \quad p_{3,4} = 0, \tag{2.53}$$

and the double zero root corresponds to a quadratic elementary divisor.

On the segment of the line $\omega_y^2 = 0$, $0 < \omega_z^2 < \omega_0^2$ the imaginary roots are different:

$$p_{1,2,3,4} = \pm j(\omega_0 \pm \omega_z). \tag{2.54}$$

At the point ($\omega_y^2 = 0$, $\omega_z^2 = \omega_0^2$) the roots are equal:

$$p_{1,2} = 0, \quad p_{3,4} = \pm 2j\omega_0. \tag{2.55}$$

and the double zero root corresponds to a linear elementary divisor.

On the line $\omega_0^2 - \omega_y^2 - \omega_z^2 = 0$, between the points $(0, \omega_0)$ and $(\omega_0^2, 0)$, we have

$$p_{1,2} = 0, \quad p_{3,4} = \pm j\sqrt{4\omega_0^2 - 3\omega_y^2}; \tag{2.56}$$

here the zero root has a quadratic elementary divisor.

On the ray $\omega_y^2 = 0$, $\omega_z^2 > \omega_0^2$ the imaginary roots are different:

$$p_{1,2,3,4} = \pm j(\omega \mp \omega_z). \tag{2.57}$$

On the segment $\omega_z^2 = \omega_0^2$, $0 < \omega_y^2 < 4\omega_0^2$ the imaginary roots

$$p_{1,2} = \pm j\sqrt{4\omega_0^2 - \omega_y^2} \tag{2.58}$$

are supplemented by the zero root

$$p_{3,4} = 0 \tag{2.59}$$

with a quadratic elementary divisor.

At the point ($\omega_z^2 = \omega_0^2$, $\omega_y^2 = 4\omega_0^2$) we have a quadruple zero root

$$p_{1,2,3,4} = 0, \tag{2.60}$$

which corresponds to a quadratic elementary divisor.

Finally, along the branch of the hyperbola (2.46) where $2\omega_0^2 < \omega_z^2 < 4\omega_0^2$, we have multiple roots

$$\left.\begin{aligned} p_{1,2} &= j\sqrt{\omega_0^2 + \omega_z^2 - \omega_y^2/2}, \\ p_{3,4} &= -j\sqrt{\omega_0^2 + \omega_z^2 - \omega_y^2/2} \end{aligned}\right\} \tag{2.61}$$

with quadratic elementary divisors.

A different approach to the stability of the homogeneous equations (2.41) is also possible. We can interpret these equations, as in Chapter 1 when analyzing Eqs. (1.132), as describing the motion of a unit point mass about the state of equilibrium

$$\delta x = 0, \quad \delta y = 0 \tag{2.62}$$

under the combined action of potential and gyroscopic forces.

The potential force function in this case is

$$U = -\frac{1}{2}\left[(\omega_0^2 - \omega_y^2 - \omega_z^2)(\delta x)^2 + (\omega_0^2 - \omega_y^2)(\delta y)^2\right], \tag{2.63}$$

and the gyroscopic forces are

$$-2\omega_z \delta y. \quad 2\omega_z \delta \dot{x}. \tag{2.64}$$

The equilibrium (2.62) in the field of potential forces alone is stable if the force function (2.63) has an isolated maximum at the point of equilibrium, which is evidently so when inequality (2.49) is satisfied. According to Kelvin's theorem, the system in this case will remain stable in the region (2.49) when the gyroscopic forces are added too.*

Inequality (2.49) describes only one of the two stability regions derived in the preceding analysis of the characteristic equation (2.43). The second stability region is entirely associated with gyroscopic forces (2.64), since it corresponds to a region where, in virtue of the second inequality in (2.45), the degree of instability in the potential field alone is even.

It is a well-known fact that if a system unstable in a potential force field is stabilized by gyroscopic forces (its degree of instability being even), the stability is eventually destroyed by the total internal dissipation forces; gyroscopic stability is thus transient, unlike stabilization by potential forces, which is secular and is actually enhanced by the internal dissipation forces.

Introducing in homogeneous equations (2.41) the total internal dissipation with Rayleigh's dispersion function

$$R = \frac{\varepsilon}{2}\left[(\delta \dot{x})^2 + (\delta \dot{y})^2\right], \tag{2.65}$$

where ε is some positive constant, we obtain the homogeneous equations in the form

$$\left.\begin{aligned} \delta \ddot{x} + \varepsilon \delta \dot{x} + (\omega_0^2 - \omega_y^2 - \omega_z^2)\delta x - 2\omega_z \delta \dot{y} = 0, \\ \delta \ddot{y} + \varepsilon \delta \dot{y} + (\omega_0^2 - \omega_z^2)\delta y + 2\omega_z \delta \dot{x} = 0. \end{aligned}\right\} \tag{2.66}$$

If the dissipation is small, so that only terms linear in ε need be retained in the characteristic equation of (2.66), we have

$$\begin{aligned} p^4 + 2\varepsilon p^3 + p^2(2\omega_0^2 - \omega_y^2 + 2\omega_z^2) + p\varepsilon(2\omega_0^2 - \omega_y^2 - 2\omega_z^2) + \\ + (\omega_0^2 - \omega_z^2)(\omega_0^2 - \omega_y^2 - \omega_z^2) = 0. \end{aligned} \tag{2.67}$$

The form of the coefficients of Eq. (2.67) suggests that outside the region

$$\omega_0^2 - \omega_y^2 - \omega_z^2 > 0 \tag{2.68}$$

at least one of them is negative.

Hurwitz's necessary conditions are thus broken, the roots of Eq. (2.67) have positive real parts, and this implies instability outside the region (2.68) due to the detrimental effect of the internal dissipation forces. The Hurwitz

* See Chetaev, N.G. Ustoichivost' dvizheniya (Stability of Motion). — Gostekhizdat. 1955.

inequalities in this case are

$$\left.\begin{aligned} \varepsilon > 0, \quad 2\omega_0^2 - \omega_y^2 + 6\omega_z^2 > 0, \\ 16\omega_z^2(\omega_0^2 - \omega_y^2 - \omega_z^2) + \omega_y^2(\omega_y^2 + 8\omega_z^2) > 0, \\ (\omega_0^2 - \omega_z^2)(\omega_x^2 - \omega_y^2 - \omega_z^2) > 0. \end{aligned}\right\} \tag{2.68a}$$

It follows that in the region (2.68) the system is asymptotically stabilized by the internal dissipation forces.

Sometimes regions of gyroscopic stability are ignored altogether, since in any actual displacement a material system is presumably always exposed to the action of dissipative forces. It should be noted, however, that if the dissipative forces are small and the system functions but for a limited time, the gyroscopic stability need not be entirely broken. Also note that arbitrary introduction of dissipative forces in stability analysis may lead to grave errors. An educational example will be given in §2.4 below.

We have carried out a stability analysis using Eqs. (2.28). These are first-approximation equations, however. The characteristic equation of (2.28) has zero and pure imaginary roots in the stability regions. This is a critical case and therefore, by Lyapunov's theorems, the first-approximation equations do not provide conclusive information on stability. The above conditions are only necessary. To obtain the sufficient conditions, the nonlinear equations (2.31) should be analyzed. This analysis is deferred until the next chapter, when we will examine the analogy between two-accelerometer inertial systems and Schuler tuned systems.

2.3.2. Solution of the first group of error equations in cases which lead to equations with constant coefficients

The different cases when the error equations of a two-accelerometer inertial system reduce to equations with constant coefficients were described in the preceding section. For a stationary object in the $O_1\xi\eta\zeta$ axes, the first group of error equations (2.28) reduces to Eqs. (2.37). Their solution is clearly given by

$$\left.\begin{aligned} \delta x = \delta x^0 \cos\omega_0 t + \frac{\delta\dot{x}^0 + r\,\Delta m_y^0}{\omega_0}\sin\omega_0 t + \\ + \frac{1}{\omega_0}\int_0^t \Delta n_x \sin\omega_0(t-\tau)\,d\tau - r\int_0^t \Delta m_y \cos\omega_0(t-\tau)\,d\tau, \\ \delta y = \delta y^0 \cos\omega_0 t + \frac{\delta\dot{y}^0 - r\,\Delta m_x^0}{\omega_0}\sin\omega_0 t + \\ + \frac{1}{\omega_0}\int_0^t \Delta n_y \sin\omega_0(t-\tau)\,d\tau + r\int_0^t \Delta m_x \cos\omega_0(t-\tau)\,d\tau. \end{aligned}\right\} \tag{2.69}$$

For constant instrumental errors, these expressions take the form

$$\left.\begin{aligned} \delta x = \frac{\Delta n_x}{\omega_0^2} + \left(\delta x^0 - \frac{\Delta n_x}{\omega_0^2}\right)\cos\omega_0 t + \frac{\delta\dot{x}^0}{\omega_0}\sin\omega_0 t, \\ \delta y = \frac{\Delta n_y}{\omega_0^2} + \left(\delta y^0 - \frac{\Delta n_y}{\omega_0^2}\right)\cos\omega_0 t + \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t. \end{aligned}\right\} \tag{2.70}$$

Comparison of (2.69), (2.70) with the first two relations in (1.140), (1.141) shows that Eqs. (2.69), (2.70) for δx, δy coincide with the analogous expressions for autonomous inertial systems.

Now consider the case of an object moving with constant velocity along the arc of a fixed-orientation great circle on a sphere of constant radius around the Earth. Eqs. (2.28) are then replaced by Eqs. (2.39), which can be written in the form

$$\left.\begin{aligned} \delta\ddot{x} + (\omega_0^2 - \omega_y^2)\delta x &= \Delta n_x - \Delta\dot{m}_y r - 2\Delta\dot{r}\omega_y, \\ \delta\ddot{y} + \omega_0^2\,\delta y &= \Delta n_y + \Delta\dot{m}_x r - \Delta m_z r\omega_y. \end{aligned}\right\} \tag{2.71}$$

Eqs. (2.28) have thus separated into two second-order equations. Their solution is

$$\left.\begin{aligned} \delta x &= \delta x^0 \cos\nu t + \frac{\delta\dot{x}^0}{\nu}\sin\nu t + \\ &\quad + \frac{1}{\nu}\int_0^t (\Delta n_x - \Delta\dot{m}_y r - 2\,\Delta\dot{r}\omega_y)\sin\nu(t-\tau)\,d\tau \\ &\qquad (\nu = \sqrt{\omega_0^2 - \omega_y^2}), \\ \delta y &= \delta y^0 \cos\omega_0 t + \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t + \\ &\quad + \frac{1}{\omega_0}\int_0^t (\Delta n_y + \Delta\dot{m}_x r - \Delta m_z r\omega_y)\sin\omega_0(t-\tau)\,d\tau. \end{aligned}\right\} \tag{2.72}$$

For constant instrumental errors Δn_x, Δn_y, Δm_x, Δm_y, Δm_z and constant position errors Δr (inaccuracy in the information on the distance from the Earth's center), the expressions for δx, δy take the form

$$\left.\begin{aligned} \delta x &= \frac{\Delta n_x}{\nu^2} + \left(\delta x^0 - \frac{\Delta n_x}{\nu^2}\right)\cos\nu t + \frac{\delta\dot{x}^0}{\nu}\sin\nu t, \\ \delta y &= \frac{\Delta n_y - \Delta m_z r\omega_y}{\omega_0^2} + \left(\delta y^0 - \frac{\Delta n_y - \Delta m_z r\omega_y}{\omega_0^2}\right)\cos\omega_0 t + \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t. \end{aligned}\right\} \tag{2.73}$$

For small ω_y, Eqs. (2.72), (2.73) are not greatly different from (2.69), (2.70). If $\omega_y = 0$, Eqs. (2.72), (2.73) reduce to Eqs. (2.69), (2.70).

Another interesting limiting case of solutions (2.72), (2.73) is provided by the motion of a satellite in circular orbit, when

$$\omega_y = \omega_0. \tag{2.74}$$

The solution of this case can be obtained from Eqs. (2.72), (2.73) by taking the limit $\omega_y \to \omega_0$ (compare the derivation of (1.152) from (1.149)). Taking the limit, we obtain

$$\left.\begin{aligned} \delta x &= \delta x^0 + \delta\dot{x}^0 t + \\ &\quad + \int_0^t (\Delta n_x - r\,\Delta\dot{m}_y - 2\omega_0\Delta\dot{r})\,(t-\tau)\,d\tau, \\ \delta y &= \delta y^0 \cos\omega_0 t + \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t + \\ &\quad + \frac{1}{\omega_0}\int_0^t (\Delta n_y + r\,\Delta\dot{m}_x - \Delta m_z r\omega_0)\sin\omega_0(t-\tau)\,d\tau. \end{aligned}\right\} \tag{2.75}$$

For constant Δn_x, Δn_y, Δm_x, Δm_y, Δm_z, Δr, we have

$$\left.\begin{aligned}\delta x &= \delta x^0 + \delta \dot{x}^0 t + \Delta n_x \frac{t^2}{2},\\ \delta y &= \delta y^0 \cos\omega_0 t + \frac{\delta \dot{y}^0}{\omega_0}\sin\omega_0 t + \frac{1}{\omega_0^2}(\Delta n_y - \Delta m_z r\omega_0)(1-\cos\omega_0 t).\end{aligned}\right\} \tag{2.76}$$

Let us now consider the last case, namely the case of Eqs. (2.41):

$$\left.\begin{aligned}&\delta\ddot{x} + (\omega_0^2 - \omega_y^2 - \omega_z^2)\delta x - 2\omega_z\,\delta\dot{y} = \Delta n_x - r\,\Delta\dot{m}_y - r\omega_z\,\Delta m_x,\\ &\delta\ddot{y} + (\omega_0^2 - \omega_z^2)\delta y + 2\omega_z\,\delta\dot{x} =\\ &\qquad = -\Delta r\omega_y\omega_z + \Delta n_y + r\,\Delta\dot{m}_x - r\omega_y\,\Delta m_z - r\omega_z\,\Delta m_y.\end{aligned}\right\} \tag{2.77}$$

As we have noted previously, this is the case of uniform circulation of the object over the surface of a fixed-orientation constant-radius sphere concentric with the Earth. In other words, it corresponds to motion with constant velocity along the arc of a small circle of this sphere. A particular case of this circulation is motion with constant velocity along a parallel. The xyz axes (in which Eqs. (2.77) are written) can be oriented to the points of the compass (with the axis y directed northward), and the components ω_y, ω_z entering this equation take the form

$$\omega_y = \omega\cos\varphi, \quad \omega_z = \omega\sin\varphi, \tag{2.78}$$

where

$$\omega = u + \frac{V}{r\cos\varphi}. \tag{2.79}$$

Here V is the velocity of the object relative to the Earth, and φ is the latitude at which the object moves. Further analysis of this case will be reduced, evidently without loss of generality, to motion along the parallel.

The characteristic equation of (2.77) is biquadratic:

$$p^4 + (2\omega_0^2 + 2\omega_z^2 - \omega_y^2)\,p^2 + (\omega_0^2 - \omega_z^2)(\omega_0^2 - \omega^2) = 0. \tag{2.80}$$

If (2.49) is satisfied, i.e., when

$$\omega_0^2 - \omega^2 > 0, \tag{2.81}$$

Eqs. (2.77), as we have shown in the previous section, describe nonasymptotically stable motion. When the above condition is satisfied, Eq. (2.80) has two pairs of pure imaginary conjugate roots

$$p_{1,2} = \pm j\mu, \quad p_{3,4} = \pm j\nu, \tag{2.82}$$

where

$$\left.\begin{aligned}\mu &= \sqrt{-\frac{\omega_y^2}{2} + \omega_z^2 + \omega_0^2 + \sqrt{\left(\frac{\omega_y^2}{2} - \omega_z^2 - \omega_0^2\right)^2 - (\omega_0^2 - \omega_z^2)(\omega_0^2 - \omega^2)}},\\ \nu &= \sqrt{-\frac{\omega_y^2}{2} + \omega_z^2 + \omega_0^2 - \sqrt{\left(\frac{\omega_y^2}{2} - \omega_z^2 - \omega_0^2\right)^2 - (\omega_0^2 - \omega_z^2)(\omega_0^2 - \omega^2)}}.\end{aligned}\right\} \tag{2.83}$$

The general solution of the homogeneous system (2.77) is thus written in the form

$$\left.\begin{aligned}
\delta x = {} & \frac{C_1}{b\mu - a\nu}(b\mu\cos\nu t - a\nu\cos\mu t) + \\
& + \frac{abC_2}{b\nu - a\mu}(\mu\sin\nu t - \nu\sin\mu t) + \\
& + \frac{C_3}{b\nu - a\mu}(b\sin\nu t - a\sin\mu t) + \\
& + \frac{abC_4}{b\mu - a\nu}(\cos\mu t - \cos\nu t), \\
\delta y = {} & \frac{C_1}{b\mu - a\nu}(\mu\sin\nu t - \nu\sin\mu t) + \\
& + \frac{C_2}{b\nu - a\mu}(b\nu\cos\mu t - a\mu\cos\nu t) + \\
& + \frac{C_3}{b\nu - a\mu}(\cos\mu t - \cos\nu t) + \\
& + \frac{C_4}{b\mu - a\nu}(b\sin\mu t - a\sin\nu t).
\end{aligned}\right\} \tag{2.84}$$

In Eqs. (2.84) C_1, C_2, C_3, C_4 are arbitrary constants, which can be interpreted as the initial values δx^0, δy^0, $\delta\dot{x}^0$, $\delta\dot{y}^0$. The constants a and b are expressed in terms of the moduli (2.83) of the roots of the characteristic equation (2.80) and the coefficient of the original system:

$$a = \frac{\omega_0^2 - \omega_x^2 - \mu^2}{2\mu\omega_x}, \quad b = \frac{\omega_0^2 - \omega_x^2 - \nu^2}{2\nu\omega_x}. \tag{2.85}$$

The solution of the inhomogeneous system (2.77) can be obtained from (2.84) by variation of the constants C_1, C_2, C_3, C_4. It has the form

$$\left.\begin{aligned}
\delta x = {} & \int_0^t \left\{ \frac{f_1}{b\nu - a\mu}[-a\sin\mu(t-\tau) + b\sin\nu(t-\tau)] + \right. \\
& \left. + \frac{abf_2}{b\mu - a\nu}[\cos\mu(t-\tau) - \cos\nu(t-\tau)] \right\} d\tau + \\
& + \frac{\delta x^0}{b\mu - a\nu}(b\mu\cos\nu t - a\nu\cos\mu t) + \\
& + \frac{\delta\dot{x}^0}{b\nu - a\mu}(b\sin\nu t - a\sin\mu t) + \\
& + \frac{ab\,\delta y^0}{b\nu - a\mu}(\mu\sin\nu t - \nu\sin\mu t) + \\
& + \frac{ab\,\delta\dot{y}^0}{b\mu - a\nu}(\cos\mu t - \cos\nu t), \\
\delta y = {} & \int_0^t \left\{ \frac{f_1}{b\nu - a\mu}[\cos\mu(t-\tau) - \cos\nu(t-\tau)] + \right. \\
& \left. + \frac{f_2}{b\mu - a\nu}[b\sin\mu(t-\tau) - a\sin\nu(t-\tau)] \right\} d\tau + \\
& + \frac{\delta y^0}{b\nu - a\mu}(b\nu\cos\mu t - a\mu\cos\nu t) + \\
& + \frac{\delta\dot{y}^0}{b\mu - a\nu}(b\sin\mu t - a\sin\nu t) + \\
& + \frac{\delta x^0}{b\mu - a\nu}(\mu\sin\nu t - \nu\sin\mu t) + \\
& + \frac{\delta\dot{x}^0}{b\nu - a\mu}(\cos\mu t - \cos\nu t).
\end{aligned}\right\} \tag{2.86}$$

Here $f_1(\tau)$, $f_2(\tau)$ stand for the right-hand sides of Eqs. (2.77), so that

$$\left.\begin{aligned} f_1(\tau) &= \Delta n_x - r\,\Delta\dot{m}_y - r\omega_z\,\Delta m_x - 2\Delta\dot{r}\omega_y \\ f_2(\tau) &= \Delta n_y + r\,\Delta\dot{m}_x - r\omega_y\,\Delta m_z - \\ &\quad - r\omega_z\,\Delta m_y - \Delta r\omega_y\omega_z \end{aligned}\right\} \tag{2.87}$$

Clearly, for $\omega_z = 0$, Eqs. (2.86) reduce to Eqs. (2.72). To prove this, note that for $\omega_z = 0$ we have from (2.83)

$$\mu = \omega_0, \quad \nu = \sqrt{\omega_0^2 - \omega_y^2}, \tag{2.88}$$

and from (2.85), (2.83) we see that

$$\lim_{\omega_z \to 0} a = 0, \quad \lim_{\omega_z \to 0} b = \infty. \tag{2.89}$$

For $\omega_y = 0$, substitution of variables (2.32) reduces Eqs. (2.77) to Eqs. (2.37). A reverse substitution of variables makes it possible to recover the solution (2.86) from (2.69) for $\omega_y = 0$.

The exact solution (2.86) of the error equations of the first group for motion along the parallel is highly cumbersome. If we take

$$\omega_0^2 \gg \omega^2, \tag{2.90}$$

this solution can be greatly simplified.

From (2.83) we obtain the following approximate values of μ and ν in this case:

$$\left.\begin{aligned} \mu &= \omega_0\left(1 - \frac{\omega^2}{4\omega_0^2}\cos^2\varphi + \frac{\omega^2}{4\omega_0^2}\sqrt{\omega^2\cos^2\varphi - 9\omega^2\sin^2\varphi\cos^2\varphi + 16\omega_0^2\sin^2\varphi}\right), \\ \nu &= \omega_0\left(1 - \frac{\omega^2}{4\omega_0^2}\cos^2\varphi - \frac{\omega}{4\omega_0^2}\sqrt{\omega^2\cos^2\varphi - 9\omega^2\sin^2\varphi\cos^2\varphi + 16\omega_0^2\sin^2\varphi}\right). \end{aligned}\right\} \tag{2.91}$$

Hence, to first approximation,

$$\left.\begin{aligned} \mu &= \omega_0 + \omega\sin\varphi = \omega_0 + \omega_z, \\ \nu &= \omega_0 - \omega\sin\varphi = \omega_0 - \omega_z. \end{aligned}\right\} \tag{2.92}$$

Using Eqs. (2.90)—(2.92), we simplify (2.86) and write it in the form

$$\left.\begin{aligned} \delta x &= \frac{1}{\omega_0}\int_0^t [f_1\cos\omega_z(t-\tau) + \\ &\quad + f_2\sin\omega_z(t-\tau)]\sin\omega_0(t-\tau)\,d\tau + \\ &\quad + (\delta x^0\cos\omega_z t + \delta y^0\sin\omega_z t)\cos\omega_0 t + \\ &\quad + \frac{1}{\omega_0}(\delta\dot{x}^0\cos\omega_z t + \delta\dot{y}^0\sin\omega_z t)\sin\omega_0 t, \\ \delta y &= \frac{1}{\omega_0}\int_0^t [-f_1\sin\omega_z(t-\tau) + \\ &\quad + f_2\cos\omega_z(t-\tau)]\sin\omega_0(t-\tau)\,d\tau + \\ &\quad + (-\delta x^0\sin\omega_z t + \delta y^0\cos\omega_z t)\cos\omega_0 t + \\ &\quad + \frac{1}{\omega_0}(-\delta\dot{x}^0\sin\omega_z t + \delta\dot{y}^0\cos\omega_z t)\sin\omega_0 t. \end{aligned}\right\} \tag{2.93}$$

For constant instrumental errors, we obtain the following approximate relations from (2.93), (2.87):

$$\left.\begin{aligned}
\delta x = {}& \frac{\Delta n_x}{\omega_0^2}(1-\cos\omega_0 t\cos\omega_z t) - \\
&- \frac{\Delta n_y - \Delta r\omega_y\omega_z}{\omega_0^2}\cos\omega_0 t\sin\omega_z t + \delta x^0\cos\omega_0 t\cos\omega_z t + \\
&+ \delta y^0\cos\omega_0 t\sin\omega_z t + \frac{\delta\dot{x}^0}{\omega_0}\sin\omega_0 t\cos\omega_z t + \\
&+ \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t\sin\omega_z t, \\
\delta y = {}& \frac{\Delta n_x}{\omega_0^2}\cos\omega_0 t\sin\omega_z t + \\
&+ \frac{\Delta n_y - \Delta r\omega_y\omega_z}{\omega_0^2}(1-\cos\omega_0 t\cos\omega_z t) + \\
&+ \delta y^0\cos\omega_0 t\cos\omega_z t - \delta x^0\cos\omega_0 t\sin\omega_z t + \\
&+ \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t\cos\omega_z t - \frac{\delta\dot{x}^0}{\omega_0}\sin\omega_0 t\sin\omega_z t.
\end{aligned}\right\} \qquad (2.94)$$

2.3.3. Integration of the first group of error equations for Keplerian motion

Eqs. (2.28) projected onto the orbital axes xyz (with the z axis pointing along r, the y axis normal to the orbital plane so that the object is seen to move counterclockwise when viewed from the end of this axis) take the form

$$\left.\begin{aligned}
\delta\ddot{x} + \left(\frac{\mu}{r^3} - \omega_y^2\right)\delta x = {}& \Delta n_x - \\
&- 2\dot{r}\,\Delta m_y - r\,\Delta\dot{m}_y - \Delta r\dot{\omega}_y - 2\,\Delta\dot{r}\omega_y, \\
\delta\ddot{y} + \frac{\mu}{r^3}\delta y = {}& \Delta n_y + 2\dot{r}\,\Delta m_x + r\,\Delta\dot{m}_x - r\omega_y\,\Delta m_z.
\end{aligned}\right\} \qquad (2.95)$$

where r and ω_y are expressed by Eqs. (1.162), which in their turn are obtained from Eqs. (1.156).

The coefficients in Eqs. (2.95) are variable. Moreover, Eqs. (2.95), unlike Eq. (1.92), are not the variational equations of Keplerian motion (1.155). Therefore, in order to solve Eqs. (2.95), we cannot use Poincaré's theorem, as in the solution of Eq. (1.92), the vector equation of errors of an autonomous inertial system. However, comparison of Eqs. (1.92) and (2.95) nevertheless suggests a method of integration.*

The vector equation (1.92) projected onto the orbital axes gives a set of scalar equations

$$\left.\begin{aligned}
\delta\ddot{x} + \left(\frac{\mu}{r^3} - \omega_y^2\right)\delta x + \dot{\omega}_y\delta z + 2\omega_y\delta\dot{z} = {}& \\
= \Delta n_x - 2\dot{r}\,\Delta m_y - r\,\Delta\dot{m}_y, & \\
\delta\ddot{z} - \left(\frac{2\mu}{r^3} + \omega_y^2\right)\delta z - \dot{\omega}_y\delta x - & \\
- 2\omega_y\delta\dot{x} = \Delta n_z + 2r\omega_y\,\Delta m_y &
\end{aligned}\right\} \qquad (2.96)$$

and a separated scalar equation

$$\delta\ddot{y} + \frac{\mu}{r^3}\delta y = \Delta n_y + 2\dot{r}\,\Delta m_x + r\,\Delta\dot{m}_x - r\omega_y\,\Delta m_z. \qquad (2.97)$$

* Andreev, V.D. Integration of the equations of errors of an inertial navigation system for Keplerian motion of the object. — Prikladnaya Matematika i Mekhanika, Vol. 29, No. 2. 1965.

Eqs. (2.96), (2.97) constitute the first group of error equations of an autonomous inertial system for the case of Keplerian motion. Eqs. (2.96) specify the errors in the orbital plane, and Eq. (2.97) describe the errors in a plane through the axis y at right angles to the orbital plane. The solution of Eqs. (2.96), (2.97) was given in 1.3.4. These were Eqs. (1.166).

Comparing Eq. (2.97) with the second equation in (2.95), we note perfect identity. The solution of the second equation in (2.95) is therefore the third equality in (1.166), which can be written as

$$\delta y = \sum_{i=1}^{2} B_{1i}\left[\int_0^t (\Delta n_y + 2\,\Delta m_x \dot{r} + \Delta \dot{m}_x r - r\omega_y \Delta m_z)\, G_{i2}\, dt + \sum_{j=1}^{2} G_{ij}^0 x_{j+4}^0\right]. \tag{2.98}$$

Here B_{1i}, G_{i2}, G_{ij}^0 are expressed by (1.167), (1.169), and x_5^0, x_6^0 are related to δy^0, $\delta \dot{y}^0$ by (1.170). Using these expressions for B_{1i}, G_{i2}, G_{ij}^0, x_5^0, x_6^0, we readily see that for a circular orbit, when $e = 0$, $\nu = \omega_y$, Eq. (2.98) reduces to the second relation in (2.75).

From (2.98) we can now derive an approximate expression for a small-eccentricity orbit and constant instrumental errors.

If the eccentricity e of the (elliptical) orbit is small, expansion of the first three equalities in (1.156) in powers of e, retaining only the linear terms, gives (substituting from (1.159))

$$\left.\begin{aligned} \sin E &= \sin \nu t\,(1 + e\cos \nu t), \\ \cos E &= \cos \nu t - e \sin^2 \nu t. \end{aligned}\right\} \tag{2.99}$$

Now, from the fourth equality in (1.156)

$$r = a(1 - e\cos \nu t), \tag{2.100}$$

and hence from the fifth equality in (1.162)

$$\omega_v = \dot{v} = \nu(1 + 2e\cos \nu t). \tag{2.101}$$

Therefore

$$v = \nu t + 2e \sin \nu t. \tag{2.102}$$

Eq. (1.167) for B_{1i}, G_{i2} contains the functions $\sin v$ and $\cos v$. By (2.102)

$$\left.\begin{aligned} \sin v &= \sin \nu t + e \sin 2\nu t, \\ \cos v &= \cos \nu t - 2\sin^2 \nu t. \end{aligned}\right\} \tag{2.103}$$

We thus have

$$\left.\begin{aligned} B_{11} &= \cos \nu t - e(1 + \sin^2 \nu t), \\ B_{12} &= \sin \nu t + \frac{e \sin 2\nu t}{2}, \\ G_{12} &= -\frac{1}{\nu}\left(\sin \nu t + \frac{e}{2}\sin 2\nu t\right), \\ G_{22} &= \frac{1}{\nu}[\cos \nu t - e(1 + \sin^2 \nu t)], \\ G_{11}^0 &= 1 - e,\ G_{12}^0 = 0,\ G_{21}^0 = 0,\ G_{22}^0 = \frac{1 - e}{\nu}. \end{aligned}\right\} \tag{2.104}$$

Inserting (2.104), (2.100), (2.101) in (2.98) and seeing that the instrumental errors are constant, we obtain the following expression for δy integrating and collecting the terms:

$$\delta y = \frac{\Delta n_y - a\nu\,\Delta m_z}{\nu^2}(1-\cos\nu t) + \delta y^0\cos\nu t + \frac{\delta\dot y^0}{\nu}\sin\nu t + e\left\{\frac{\Delta n_y}{\nu^2}\left(1-\cos\nu t+\sin^2\nu t - \frac{3}{2}\nu t\sin\nu t\right) + \frac{a\,\Delta m_z}{\nu}(\cos\nu t - 1 - \sin^2\nu t + \nu t\sin\nu t) + \frac{a\,\Delta m_x}{\nu}(\nu t\cos\nu t - \sin\nu t) + \delta y^0(\cos\nu t - 1 - \sin^2\nu t) + \frac{\delta\dot y^0}{\nu}(-\sin\nu t + \sin\nu t\cos\nu t)\right\}. \tag{2.105}$$

For $e = 0$ this equality reduces to Eq. (2.76) ($\nu = \omega_0$, $a = r$).

To obtain the solution of the first equation in (2.95), note that the partial solutions of the homogeneous system (2.96) found in 1.3.4 include the solution

$$\delta x = \frac{r}{a}\,,\quad \delta z = 0. \tag{2.106}$$

This is readily established by expanding the expression (1.160) for q_4. Comparing the first equation in (2.96) with the first equation in (2.95), we see that

$$\delta x = \frac{r}{a} \tag{2.107}$$

is a particular solution of the homogeneous equation corresponding to the first equation in (2.95). This can be verified also by direct substitution. The partial solution (2.107) is henceforth designated $\delta x_{(1)}$.

To obtain another partial solution, we use the Liouville formula

$$W(t) = W(0)\,e^{-\int_0^t p_1(t)\,dt}, \tag{2.108}$$

which (apart from a constant factor) is equal to the Wronskian of the equation

$$\ddot y + p_1\dot y + p_2 y = 0. \tag{2.109}$$

In our case $p_1 = 0$ and the Wronskian is constant. Choosing this constant to be unity, we get

$$\delta x_{(1)}\,\delta\dot x_{(2)} - \delta\dot x_{(1)}\,\delta x_{(2)} = 1, \tag{2.110}$$

where $\delta x_{(1)}$ is the known and $\delta x_{(2)}$ is the sought particular solution.

From (2.110) we get

$$\delta x_{(2)} = \delta x_{(1)}\left(\int\frac{dt}{\delta x_{(1)}^2} + C\right). \tag{2.111}$$

Taking $C = 0$ we insert for $\delta x_{(1)}$ in (2.111) its expression from (2.107). This gives

$$\delta x_{(2)} = \frac{r}{a}\int\frac{a^2}{r^2}\,dt. \tag{2.112}$$

Now using the fifth equality in (1.162), which gives

$$\frac{a^2}{r^2} = \frac{v}{\sqrt{1-e^2}}, \tag{2.113}$$

we find

$$\delta x_{\omega} = \frac{rv}{a\sqrt{1-e^2}}. \tag{2.114}$$

Thus, the general solution of the homogeneous equation corresponding to the first equation in (2.95) is

$$\delta x = C_1 \frac{r}{a} + C_2 \frac{rv}{a\sqrt{1-e^2}}. \tag{2.115}$$

The general solution of the inhomogeneous equation is obtained by variation of constants. Taking C_1 and C_2 in Eq. (2.115) to be functions of time, we obtain the following set of linear equations for these unknowns:

$$\left.\begin{aligned} \dot{C}_1 \frac{r}{a} + \dot{C}_2 \frac{rv}{a\sqrt{1-e^2}} &= 0, \\ \dot{C}_1 \frac{ve\sin v}{\sqrt{1-e^2}} + \dot{C}_2 \left(\frac{rv\sin v}{1-e^2} + \frac{a}{r}\right) &= \\ = \Delta n_x - 2\,\Delta m_y \dot{r} - \Delta \dot{m}_y r - \Delta r \dot{\omega}_y - 2\,\Delta \dot{r}\omega_y. & \end{aligned}\right\} \tag{2.116}$$

The determinant of this system is the Wronskian (2.110) and is thus equal to unity. From (2.116) we get

$$\left.\begin{aligned} C_1(t) &= -\frac{1}{a\sqrt{1-e^2}} \int_0^t rv(\Delta n_x - 2\,\Delta m_y \dot{r} - \Delta \dot{m}_y r - \Delta r \dot{\omega}_y - 2\,\Delta \dot{r}\omega_y)\,dt + C_1^0, \\ C_2(t) &= \frac{1}{a} \int_0^t r(\Delta n_x - 2\,\Delta m_y \dot{r} - \Delta \dot{m}_y r - \Delta r \dot{\omega}_y - 2\,\Delta \dot{r}\omega_y)\,dt + C_2^0. \end{aligned}\right\} \tag{2.117}$$

Here C_1^0, C_2^0 are determined by the initial conditions δx^0, $\delta \dot{x}^0$:

$$\left.\begin{aligned} C_1^0 &= \delta x^0 \left(\frac{ev^0 \sin v^0}{1-e^2} + \frac{a}{r^0}\right) - \frac{\delta \dot{x}^0 r^0 v^0}{av\sqrt{1-e^2}}, \\ C_2^0 &= -\frac{\delta x^0 ve \sin v^0}{\sqrt{1-e^2}} + \frac{\delta \dot{x}^0 r^0}{a}. \end{aligned}\right\} \tag{2.118}$$

Inserting (2.118) in (2.117) and further in (2.115), we obtain δx in the form

$$\begin{aligned} \delta x = \frac{r}{a}\Bigg[& -\frac{1}{av\sqrt{1-e^2}} \int_0^t rv(\Delta n_x - 2\,\Delta m_y \dot{r} - \\ & - \Delta \dot{m}_y r - \Delta r \dot{\omega}_y - 2\,\Delta \dot{r}\omega_y)\,dt + \\ & + \delta x^0 \left(\frac{ev^0 \sin v^0}{1-e^2} + \frac{a}{r_0}\right) - \frac{\delta \dot{x}^0 r^0 v^0}{av\sqrt{1-e^2}}\Bigg] + \\ & + \frac{rv}{av\sqrt{1-e^2}}\Bigg[\frac{1}{a}\int_0^t r(\Delta n_x - 2\,\Delta m_y \dot{r} - \Delta \dot{m}_y r - \\ & - \Delta r \dot{\omega}_y - 2\,\Delta \dot{r}\omega_y)\,dt - \frac{\delta x^0 ve \sin v^0}{\sqrt{1-e^2}} + \frac{\delta \dot{x}^0 r^0}{a}\Bigg]. \end{aligned} \tag{2.119}$$

It is readily seen that for a circular orbit, when $e = 0$, Eq. (2.119) reduces to the first equation in (2.75).

From Eq. (2.119) we can now obtain an expression for δx in the case of a low-eccentricity orbit with constant Δn_x, Δm_y, Δr. To simplify the treatment, suppose, without loss of generality, that the initial position of the object was at the perigee. Using expressions (2.100) and (2.102) for r and v, we obtain after some manipulations in the integrands in (2.119) and integration

$$\delta x = \delta x^0 + \delta \dot{x}^0 t + \Delta n_x \frac{t^2}{2} + e\left[\Delta n_x\left(-\frac{t^2}{2}\cos \nu t + \frac{2t}{\nu}\sin \nu t + \right.\right.$$
$$+ \frac{3}{\nu^2}\cos \nu t - \frac{3}{\nu^2}\Big) + 2\,(a\,\Delta m_y - \nu\,\Delta r)\Big(\frac{\sin \nu t}{\nu} - t\Big) +$$
$$\left. + \delta x^0 (1 - \cos \nu t) + \delta \dot{x}^0 \Big(\frac{2}{\nu}\sin \nu t - t - t\cos \nu t\Big)\right]. \qquad (2.120)$$

The first three terms in the right-hand side of Eq. (2.120) correspond to the δx in circular orbit and coincide with the first equation in (2.76).

2.3.4. Investigation of the first group of equations for arbitrary motion near the Earth

In 2.3.1 we analyzed the stability of the first group of equations (2.28) for the case of uniform circulation of the object over the surface of a fixed-orientation sphere concentric with the Earth, when the coefficients of the error equations are constant. We established the conditions of stability of the first group of error equations for this case.

In the general case, the coefficients of Eqs. (2.28) are variable and the motion described by these equations may prove to be unstable. Indeed, consider the homogeneous error equations (2.39) which correspond to motion with constant velocity along the arc of a fixed great circle. The homogeneous equation for the error in the direction of motion has the form

$$\delta \ddot{x} + (\omega_0^2 - \omega_y^2)\,\delta x = 0. \qquad (2.121)$$

Appropriate choice of ω_y as a function of time will make this equation unstable. Apart from the self-evident instability for $\omega_y^2 = \omega_0^2$ (motion in a circular orbit), a particular case of Eq. (2.121) is the famous Mathieu—Hill equation which has divergent solutions. Eq. (2.121) clearly has divergent solutions regardless of the particular relation between ω_y^2 and ω_0^2.

The simplest case is that of parametric resonance, which can be built up as follows. Let δx^0 be the initial value of δx; let further $\delta \dot{x}^0 = 0$. Let $\omega_y = 0$ for the time between 0 and $t_1 = \pi/2\omega_0$. Then $\delta x(t_1) = 0$, $\delta \dot{x}(t_1) = -\omega_0 \delta x^0$. For the time between t_1 and $t_2 = t_1 + \frac{\pi}{2\sqrt{\omega_0^2 - \omega_y^2}}$ let $\omega_y = \text{const}$. At time t_2 we clearly have $\delta \dot{x}(t_2) = 0$, $\delta x(t_2) = -\frac{\omega_0 \delta x^0}{\sqrt{\omega_0^2 - \omega_y^2}}$. Continuing this process, we have after k cycles

$$|\delta x(t_{2k})| = \delta x^0 \left(\frac{\omega_0}{\sqrt{\omega_0^2 - \omega_y^2}}\right)^k. \qquad (2.122)$$

Eq. (2.122) shows how the error amplitude δx increases with time. If $\omega_y^2 \ll \omega_0^2$, we clearly have

$$|\delta x(t_{2k})| \approx \delta x^0 \left(1 + \frac{1}{2}\frac{\omega_y^2}{\pi\omega_0} t_{2k}\right). \tag{2.123}$$

Let us now consider the case of slow motion at a constant distance from the Earth's center. The first group of the homogeneous error equations is written in the form (2.36):

$$\left.\begin{aligned} \delta\ddot{x} + (\omega_0^2 - \omega_y^2)\delta x + \omega_x\omega_y \delta y = 0, \\ \delta\ddot{y} + (\omega_0^2 - \omega_x^2)\delta y + \omega_x\omega_y \delta x = 0. \end{aligned}\right\} \tag{2.124}$$

ω_x, ω_y are related to the horizontal projections v_x, v_y of the absolute velocity by the equalities $\omega_x = -v_y/r$, $\omega_y = v_x/r$. Clearly

$$\sqrt{\omega_x^2 + \omega_y^2} \leqslant \frac{V}{r} + u = \omega, \tag{2.125}$$

where V is the maximum horizontal speed of the object relative to the Earth, and u is the angular rotation velocity of the Earth. For velocities small compared to the circular velocity, when

$$|\omega_x| \leqslant \omega_0, \quad |\omega_y| \leqslant \omega_0, \tag{2.126}$$

we can obtain* effective estimates of divergence of the solutions of Eq. (2.124) compared to the solutions of the equations

$$\delta\ddot{x} + \omega_0^2 \delta x = 0, \quad \delta\ddot{y} + \omega_0^2 \delta y = 0. \tag{2.127}$$

Changing over from t to a dimensionless independent variable

$$\tau = \omega_0 t, \tag{2.128}$$

we introduce a small parameter μ defined by the relations

$$\mu = \max \frac{\sqrt{\omega_x^2 + \omega_y^2}}{\omega_0}, \quad \frac{\omega_x}{\omega_0} = \mu C_x, \quad \frac{\omega_y}{\omega_0} = \mu C_y. \tag{2.129}$$

Clearly,

$$\mu \ll 1, \quad C_x^2 + C_y^2 \leqslant 1. \tag{2.130}$$

Using (2.128), (2.129), we write Eqs. (2.124) in the form

$$\left.\begin{aligned} \delta x'' + \delta x = \mu^2 C_y (C_y \delta x - C_x \delta y), \\ \delta y'' + \delta y = \mu^2 C_x (C_x \delta y - C_y \delta x), \end{aligned}\right\} \tag{2.131}$$

where primes denote differentiation with respect to τ.

* Andreev, V.D. A case of small oscillations of a physical pendulum with a moving suspension point. — Prikladnaya Matematika i Mekhanika, Vol. 22, No. 6, 1958. Parusnikov, N.A. and L.I. Kliger. Equations of small oscillations of the Schuler vertical. — Inzhenernyi Zhurnal, Mechanics of Rigid Bodies, No. 5, 1966.

One of the courses open to us at this stage is the following. We rewrite Eqs. (2.131) in a slightly different form

$$\left.\begin{aligned} \delta x'' + \delta x = \mu^2 F_x(\delta x,\ \delta y),\ \delta y'' + \delta y = \mu^2 F_y(\delta x,\ \delta y), \\ F_x = C_y(C_y\,\delta x - C_x\,\delta y), \\ F_y = C_x(C_x\,\delta y - C_y\,\delta x) \end{aligned}\right\} \tag{2.132}$$

and solve them by successive approximations. The iterative relations are given in the form

$$\left.\begin{aligned} \delta x_n'' + \delta x_n = F_{x,\,n-1}, \\ \delta y_n'' + \delta y_n = F_{y,\,n-1}, \end{aligned}\right\} \tag{2.133}$$

where

$$\left.\begin{aligned} F_{x,\,n-1} = F_x(\delta x_{n-1},\ \delta y_{n-1}), \\ F_{y,\,n-1} = F_y(\delta x_{n-1},\ \delta y_{n-1}), \end{aligned}\right\} \tag{2.134}$$

and the zero approximation δx_0, δy_0 is chosen as the solution of the truncated equations (2.127).

Then

$$\left.\begin{aligned} \delta x_n = \mu^2 \int_0^\tau F_{x,\,n-1}(u)\sin(\tau - u)\,du + \delta x_0, \\ \delta y_n = \mu^2 \int_0^\tau F_{y,\,n-1}(u)\sin(\tau - u)\,du + \delta y_0. \end{aligned}\right\} \tag{2.135}$$

Approximations (2.135) are in fact a modification of Picard's technique of successive approximations. When appropriate restrictions are imposed on the functions F_x, F_y (these restrictions are generally satisfied in practice), the approximations are found to converge.

Without loss of generality, we may take

$$\delta x_0 = \delta y_0 = a\cos\omega_0 t. \tag{2.136}$$

From (2.135), (2.134), (2.132), using the theorem of the mean value and seeing that $C_x^2 + C_y^2 \leqslant 1$, we obtain the following upper bound estimates for δx_n, δy_n:

$$\left|\frac{\delta x_n}{a}\right|,\ \left|\frac{\delta y_n}{a}\right| \leqslant \frac{1+\sqrt{2}}{2}\sum_{k=0}^{n}\frac{(\mu^2\tau)^k}{k!}. \tag{2.137}$$

Taking the limit as $n \to \infty$, we get

$$\left|\frac{\delta x_n}{a}\right|,\ \left|\frac{\delta y_n}{a}\right| \leqslant \frac{1+\sqrt{2}}{2}\,e^{\mu^2\tau}. \tag{2.138}$$

For $\mu \ll 1$, to high approximation,

$$\left|\frac{\delta x}{a}\right|,\ \left|\frac{\delta y}{a}\right| \leqslant 1 + \mu^2\tau\frac{1+\sqrt{2}}{2}. \tag{2.139}$$

Estimates of the form (2.138) can also be obtained by an alternative technique. Returning to Eqs. (2.131), consider the function

$$U = (\delta x)^2 + (\delta y)^2 + (\delta x')^2 + (\delta y')^2. \tag{2.140}$$

In virtue of Eqs. (2.131) we have

$$U' = 2\mu^2(C_x\delta y - C_y\delta x)(C_x\delta y' - C_y\delta x'). \tag{2.141}$$

We change over to new variables, defined by

$$\left.\begin{aligned} x_1 &= \delta x\cos\psi + \delta y\sin\psi,\\ x_2 &= -\delta x\sin\psi + \delta y\cos\psi,\\ x_3 &= \delta x'\cos\psi + \delta y'\sin\psi,\\ x_4 &= -\delta x'\sin\psi + \delta y'\cos\psi, \end{aligned}\right\} \tag{2.142}$$

where

$$\cos\psi = \frac{C_x}{\sqrt{C_x^2 + C_y^2}}, \quad \sin\psi = \frac{C_y}{\sqrt{C_x^2 + C_y^2}}. \tag{2.143}$$

In the phase space $[x_1, x_2, x_3, x_4]$, Eqs. (2.140), (2.141) take the form

$$\left.\begin{aligned} U &= x_1^2 + x_2^2 + x_3^2 + x_4^2,\\ U' &= 2\mu^2(C_x^2 + C_y^2)\,x_2x_4. \end{aligned}\right\} \tag{2.144}$$

Clearly,

$$|U'| \leqslant 2\mu^2 x_2x_4 \leqslant \mu^2 U. \tag{2.145}$$

If U^0 is the value of U at $\tau = 0$, we have from (2.145)

$$U \leqslant U^0 e^{\mu^2\tau} \tag{2.146}$$

For $\mu^2\tau \ll 1$,

$$U \leqslant U^0(1 + \mu^2\tau). \tag{2.147}$$

Inequalities (2.146), (2.147) gives upper-bound estimates of the region containing the errors δx, δy, regardless of the particular variation of ω_x, ω_y.

Estimates (2.146), (2.147) are entirely analogous to estimates (2.138), (2.139). However, the method used in the derivation of inequalities (2.146), (2.147) enables us to go one step further. It enables us to construct more exact estimates, using the time rate of change of the functions ω_x, ω_y or, equivalently, of the functions C_x, C_y.

Note that Eqs. (1.131) can be written in canonical (Hamiltonian) form:

$$\left.\begin{aligned} \frac{d\,\delta x'}{d\tau} &= \frac{\partial H}{\partial\,\delta x}, \quad \frac{d\,\delta x}{d\tau} = -\frac{\partial H}{\partial\,\delta x'},\\ \frac{d\,\delta y'}{d\tau} &= \frac{\partial H}{\partial\,\delta y}, \quad \frac{d\,\delta y}{d\tau} = -\frac{\partial H}{\partial\,\delta y'}. \end{aligned}\right\} \tag{2.148}$$

Here the Hamiltonian function H is a quadratic form of the variables δx, δy, $\delta x'$, $\delta y'$, such that

$$-2H = W = (\delta x)^2 + (\delta y)^2 + (\delta x')^2 + (\delta y')^2 - \mu^2(C_x\delta y - C_y\delta x)^2. \tag{2.149}$$

By Eqs. (2.131),

$$W' = -2\mu^2(C_x\delta y - C_y\delta x)(C'_x\delta y - C'_y\delta x). \tag{2.150}$$

Substituting the variables from (2.142), we define the angle ψ_1 as follows:

$$\left.\begin{aligned}\cos\psi_1 &= \frac{C_xC'_x + C_yC'_y}{\sqrt{(C_x^2+C_y^2)(C_x'^2+C_y'^2)}},\\ \sin\psi_1 &= \frac{C_xC'_y - C_yC'_x}{\sqrt{(C_x^2+C_y^2)(C_x'^2+C_y'^2)}}.\end{aligned}\right\} \tag{2.151}$$

After simple manipulations, we obtain instead of (2.149), (2.150)

$$\left.\begin{aligned}W &= [1-\mu^2(C_x^2+C_y^2)]\,x_1^2 + x_2^2 + x_3^2 + x_4^2,\\ W' &= -2\mu^2\sqrt{(C_x^2+C_y^2)(C_x'^2+C_y'^2)}\,x_1(x_1\cos\psi_2 + \\ &\qquad + x_2\sin\psi_1).\end{aligned}\right\} \tag{2.152}$$

Let at some time

$$x_1 = \rho\cos\psi_2, \quad x_2 = \rho\sin\psi_2, \tag{2.153}$$

so that, as is readily seen,

$$\rho^2 = x_1^2 + x_2^2 \leqslant \frac{W}{1-\mu^2}. \tag{2.154}$$

From the second equality in (2.152) we have

$$W' = -2\mu^2\sqrt{(C_x^2+C_y^2)(C_x'^2+C_y'^2)}\,\rho^2\cos\psi_2\cos(\psi_1-\psi_2). \tag{2.155}$$

Hence, using inequalities (2.154), (2.130), we arrive at the estimate

$$|W'| \leqslant \frac{\mu^2\nu}{1-\mu^2}W, \tag{2.156}$$

where

$$\nu = 2\max\sqrt{C_x'^2 + C_y'^2}. \tag{2.157}$$

Now, by analogy with (2.146),

$$W \leqslant W^0 e^{\frac{\mu^2\nu}{1-\mu^2}\tau} \approx W^0 e^{\mu^2\nu\tau}. \tag{2.158}$$

For $\mu^2\nu\tau \ll 1$ we have

$$W \leqslant W^0(1+\mu^2\nu\tau). \tag{2.159}$$

We have thus obtained estimates for the divergence with time of the amplitudes of solutions of Eqs. (2.28) relative to their initial values, which take into consideration the time rate of change of ω_x, ω_y.

Let us now consider the quantitative aspects of the problem. Take the case of motion with constant velocity along a great circle of the Earth's

surface, which is regarded as a sphere. Then, as is readily seen,

$$\nu = 2 \max \sqrt{C_x'^2 + C_y'^2} \leqslant 4\mu \frac{u}{\omega} \frac{V}{r\omega}, \tag{2.160}$$

where V is the velocity relative to the Earth, and ω is defined by Eq. (2.125). Let us estimate the possible relative growth of the error $\sqrt{\delta x^2 + \delta y^2}$ in relation to its initial value in a time

$$\tau_* = \frac{\pi r}{V} \omega_0, \tag{2.161}$$

i.e., in the time it takes the object to complete half the circuit around the Earth (assuming various velocities). Taking $V = 250$ m/sec, 500 m/sec, 1000 m/sec, 2000 m/sec, we obtain the following table.

V, m/sec	μ	ν	$e^{\frac{\mu^2\tau_*}{2}}$	$e^{\mu^2\nu\tau_*}$
250	0.09	0.08	1.49	1.07
500	0.12	0.12	1.43	1.09
1000	0.18	0.16	1.49	1.14
2000	0.31	0.19	1.82	1.26

It follows from the table that at least in the case of constant cruising velocity relative to the Earth's surface the function W remains virtually constant for a fairly long time. Therefore Eqs. (2.124) can be simplified to the form (2.127), i.e., the system can be regarded as conservative (non-asymptotically stable) with regard to the errors δx, δy.

Let us consider a few examples to illustrate the application of the estimates (2.146), (2.158). We will compare the estimates (2.122), (2.123) obtained for the simple case of parametric resonance (see the beginning of this section) with estimates (2.146), (2.147).

Taking $\tau = \omega_0 t$, $\mu^2 = \omega_y^2/\omega_0^2$, we obtain from estimate (2.123)

$$\left|\frac{\delta x}{\delta x^0}\right| \approx 1 + \frac{\mu^2\tau}{2\pi}, \tag{2.162}$$

and estimate (2.147) gives in this case

$$\left|\frac{\delta x}{\delta x^0}\right| \leqslant 1 + \frac{\mu^2\tau}{2}. \tag{2.163}$$

Consider the case of slow motion along the parallel. The equations of errors for this case are Eqs. (2.41). The roots of the characteristic polynomial (2.80) of system (2.41) are expressed in terms of ω_0, ω_y, ω_z by (2.82), (2.83). When the stability condition (2.81) is broken, one of the roots of the characteristic equation becomes real and positive. For small ω_y, when the stability is broken due to the increase in ω_z (motion with constant velocity along the parallel at high latitudes), the maximum value of the positive real root is $\approx \mu^2/4$. This figure can be used as a measure of instability of the system. The analogous measure of instability obtained from estimate (2.146) is $\mu^2/2$.

As the last example, consider an object moving with constant velocity along a terrestrial meridian. Directing the xyz axes to the points of the compass we obtain in this case

$$\omega_x = -\dot{\varphi} = \text{const}, \quad \omega_y = u\cos\varphi, \quad \omega_z = u\sin\varphi. \tag{2.164}$$

The homogeneous error equations are obtained from Eqs. (2.28) using (2.164):

$$\left.\begin{aligned} \delta\ddot{x} + \omega_0^2\delta x &= u^2\delta x + 2u\dot{\varphi}\cos\varphi\,\delta y + 2u\sin\varphi\,\delta\dot{y}, \\ \delta\ddot{y} + \omega_0^2\delta y &= -\left(\dot{\varphi}^2 + \frac{1}{2}u^2 - \frac{1}{2}u^2\cos 2\varphi\right)\delta y - 2u\sin\varphi\,\delta\dot{x}. \end{aligned}\right\} \tag{2.165}$$

Analysis of the exact computer solutions $\delta x(t)$, $\delta y(t)$ of Eqs. (2.165) obtained for $V = \dot{\varphi} = 500$ m/sec, 1000 m/sec, 2000 m/sec has shown (in complete agreement with estimate (2.158)) that the expressions for δx, δy can be written to fair approximation in the form

$$\left.\begin{aligned} \delta x &= -\left(\delta y^0\cos\omega_0 t + \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t\right)\sin\varepsilon + \left(\delta x^0\cos\omega_0 t + \frac{\delta\dot{x}^0}{\omega_0}\sin\omega_0 t\right)\cos\varepsilon, \\ \delta y &= \left(\delta y^0\cos\omega_0 t + \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t\right)\cos\varepsilon + \left(\delta x^0\cos\omega_0 t + \frac{\delta\dot{x}_0}{\omega_0}\sin\omega_0 t\right)\sin\varepsilon, \end{aligned}\right\} \tag{2.166}$$

where

$$\varepsilon = \int_0^t \omega_z\,dt = \frac{u}{\dot{\varphi}}(\cos\varphi - 1). \tag{2.167}$$

Note that Eqs. (2.166) are entirely analogous to the approximate expressions (2.94) obtained for motion with constant velocity along a parallel.

As a rule, any long trip generally follows some orthodromy on the Earth's surface with virtually constant velocity (we are referring here to cruising velocities up to 2000—3000 km/sec, i.e., sea vessels, aircraft, winged rockets). It therefore follows from the above examples and estimates that to fair approximation, which is quite sufficient for the analysis of the error equations, the projections of Eqs. (2.28) onto the free azimuthal axes, i.e., Eqs. (2.36), can be replaced by the equations

$$\left.\begin{aligned} \delta\ddot{x} + \omega_0^2\delta x &= -\Delta r\dot{\omega}_y - 2\Delta\dot{r}\omega_y + \Delta n_x - r\,\Delta\dot{m}_y - r\omega_x\Delta m_z - 2\dot{r}\,\Delta m_y, \\ \delta\ddot{y} + \omega_0^2\delta y &= \Delta r\dot{\omega}_x + 2\,\Delta\dot{r}\omega_x + \Delta n_y + r\,\Delta\dot{m}_x - r\omega_y\Delta m_z + 2\dot{r}\,\Delta m_x. \end{aligned}\right\} \tag{2.168}$$

The solution of these equations clearly completes the treatment of our problem. The expressions for δx, δy for arbitrary co-moving axes on a sphere around the Earth, i.e., the solutions of Eqs. (2.28), are obtained from the solution of Eqs. (2.168) using the inverse of Eqs. (2.33), where we should take

$$\vartheta = -\int_0^t \omega_z\,dt + \vartheta^0. \tag{2.169}$$

This was in fact the method used in writing the equations of errors δx, δy in projection onto the axes oriented to the points of the compass, Eqs. (2.166).

The estimates of this section further show that Eqs. (2.168) ensure satisfactory accuracy for the case of uniform circulation on the Earth's surface and for any short maneuver.

In conclusion note that in our analysis of the homogeneous equations (2.36), i.e., Eqs. (2.124), we considered the case of arbitrary motion with small (compared to circular) velocity at a constant distance from the Earth's center, when

$$r = \text{const}, \quad \omega_0 = \text{const}. \tag{2.170}$$

It is readily seen that the fundamental results of this analysis can be further extended to case of motion with variable r. Thus, if

$$r = r^0 + \rho(t), \quad \left|\frac{\rho}{r^0}\right| \ll 1, \tag{2.171}$$

the estimates (2.138) are replaced by

$$\left|\frac{\delta x}{a}\right|, \ \left|\frac{\delta y}{a}\right| < \frac{1+\sqrt{2}}{2} e^{(\mu^2+\sigma^2)\tau} + \frac{1-\sqrt{2}}{2} e^{\sigma^2\tau}, \tag{2.172}$$

where

$$\sigma^2 = \frac{3 \max|\rho|}{r^0}. \tag{2.173}$$

For small values of $(\mu^2+\sigma^2)\tau$ we have the inequalities

$$\left|\frac{\delta x}{a}\right|, \ \left|\frac{\delta y}{a}\right| < 1 + \left(\frac{1+\sqrt{2}}{2}\mu^2 + \sigma^2\right)\tau. \tag{2.174}$$

Estimates similar to (2.146), (2.147), (2.158), and (2.159) can also be obtained. We will not derive these estimates, and the reader can do it as an exercise. As far as $\rho(t)$ is concerned, estimates (2.172) and (2.174) are always quite sufficient.

§2.4. STABILITY ANALYSIS AND INTEGRATION OF THE FIRST GROUP OF ERROR EQUATIONS FOR A THREE-ACCELEROMETER SYSTEM WITH ALTIMETER CORRECTION

2.4.1. Stability analysis

Consider the error equations of three-accelerometer systems where the spherical component of the Earth's gravitational field entering the ideal work equations is formed using external guidance information on the distance from the Earth.

The second group of error equations in this case is the same as for autonomous inertial systems, and the results of 1.3.1 can therefore be readily applied to three-accelerometer systems with altimeter correction.

The first group of error equations, however, is different. We have shown in 2.2.1 that the first group of errors is the solution of the

vector equation (2.15)

$$\left.\begin{aligned}\frac{d^2\delta r}{dt^2}+\frac{\mu}{r^3}\delta r=\Delta n-\Delta m\times\frac{dr}{dt}+\frac{d}{dt}(\Delta m\times r)+\frac{3\mu r\,\Delta r}{r^4}\\ (\Delta r=\Delta h).\end{aligned}\right\} \tag{2.175}$$

Projected onto the co-moving axes xyz, with the z axis pointing along r, the vector equation (2.175) gives a sixth-order system of scalar differential equations (2.16).

Eq. (2.175) is best analyzed in projection onto fixed-orientation axes, since the resulting scalar equations have a particularly simple form. Choosing $\xi_*\eta_*\zeta_*$ as the fixed axes, we obtain the equations

$$\left.\begin{aligned}\delta\ddot{\xi}_*+\frac{\mu}{r^3}\delta\xi_*&=\Delta n_{\xi_*}-2(\Delta m_{\eta_*}\dot{\zeta}_*-\Delta m_{\zeta_*}\dot{\eta}_*)+\eta_*\Delta\dot{m}_{\zeta_*}-\zeta_*\Delta\dot{m}_{\eta_*}+\frac{3\mu\xi_*\Delta r}{r^4},\\ \delta\ddot{\eta}_*+\frac{\mu}{r^3}\delta\eta_*&=\Delta n_{\eta_*}-2(\Delta m_{\zeta_*}\dot{\xi}_*-\Delta m_{\xi_*}\dot{\zeta}_*)+\zeta_*\Delta\dot{m}_{\xi_*}-\xi_*\Delta\dot{m}_{\zeta_*}+\frac{3\mu\eta_*\Delta r}{r^4},\\ \delta\ddot{\zeta}_*+\frac{\mu}{r^3}\delta\zeta_*&=\Delta n_{\zeta_*}-2(\Delta m_{\xi_*}\dot{\eta}_*-\Delta m_{\eta_*}\dot{\xi}_*)+\xi_*\Delta\dot{m}_{\eta_*}-\eta_*\Delta\dot{m}_{\xi_*}+\frac{3\mu\zeta_*\Delta r}{r^4}.\end{aligned}\right\} \tag{2.176}$$

The homogeneous equations (2.176) for $r=\text{const}$ reduce to harmonic equations with Schuler's frequency $\omega_0=\sqrt{\mu/r^3}$. The nonasymptotic stability is thus obvious in this case. This immediately leads to stability of Eqs. (2.16) for arbitrary $\omega_x(t)$, $\omega_y(t)$, $\omega_z(t)$ and $r=\text{const}$, i.e., arbitrary motion on a sphere of constant radius concentric with the Earth.

Note that if $\omega_x, \omega_y, \omega_z$ in Eqs. (2.16) are constant (e.g., the case of motion along a parallel with constant velocity), stability of motion can be established without considering Eqs. (2.176), by direct inspection of Eqs. (2.16). Indeed, for motion along a parallel, when $\omega_x=0$ and ω_y, ω_z are given by Eqs. (2.78), (2.79), the homogeneous equations (2.16) take the form

$$\left.\begin{aligned}\delta\ddot{x}+(\omega_0^2-\omega_y^2-\omega_z^2)\delta x-2\omega_z\delta\dot{y}+2\omega_y\delta\dot{z}=0,\\ \delta\ddot{y}+(\omega_0^2-\omega_z^2)\delta y+\omega_y\omega_z\delta z+2\omega_z\delta\dot{x}=0,\\ \delta\ddot{z}+(\omega_0^2-\omega_y^2)\delta z+\omega_y\omega_z\delta y-2\omega_y\delta\dot{x}=0.\end{aligned}\right\} \tag{2.177}$$

These equations, like Eqs. (1.132), (2.41) in the preceding, can be interpreted as the equations of motion of a unit point mass about the equilibrium

$$\delta x=0,\quad \delta y=0,\quad \delta z=0 \tag{2.178}$$

in a force field produced by the potential force function

$$U=-\frac{1}{2}\left[(\omega_0^2-\omega_y^2-\omega_z^2)(\delta x)^2+(\omega_0^2-\omega_z^2)(\delta y)^2+(\omega_0^2-\omega_y^2)(\delta z)^2+2\omega_y\omega_z\delta y\,\delta z\right] \tag{2.179}$$

and the gyroscopic forces

$$-2\omega_z\delta\dot{y}+2\omega_y\delta\dot{z},\quad 2\omega_z\delta\dot{x},\quad -2\omega_y\delta\dot{x}. \tag{2.180}$$

For the equilibrium (2.178) to be stable, the force function should have an isolated maximum at the point of equilibrium. Application of Sylvester's condition of positive definiteness to the quadratic form in the right-hand side of (2.179) gives the inequality

$$\omega_0^2-\omega_y^2-\omega_z^2>0, \tag{2.181}$$

which coincides with inequality (2.49) for the stability of a two-accelerometer inertial system.

Outside the region (2.181) the degree of instability in the potential field is even and the equilibrium is stabilized by the gyroscopic forces. This can be shown by examining the characteristic equation of (2.177), which in terms of a new unknown $p^2 = q$ takes the form

$$q^3 + (3\omega_0^2 + 2\omega^2)q^2 + (3\omega_0^4 + \omega^4)q + (\omega_0^2 - \omega^2)^2\omega_0^2 = 0, \tag{2.182}$$

where we have written

$$\omega^2 = \omega_y^2 + \omega_z^2. \tag{2.183}$$

The polynomial (2.182) satisfies Hurwitz's conditions, since always

$$(3\omega_0^2 + 2\omega^2)(3\omega_0^4 + \omega^4) - \omega_0^2(\omega_0^2 - \omega^2)^2 > 0. \tag{2.184}$$

The discriminant Δ of the cubic equation which is obtained from (2.182) by substitution of the variable

$$y = q + \frac{1}{3}(3\omega_0^2 + 2\omega^2) \tag{2.185}$$

is nonpositive:

$$\Delta = -\frac{4}{27}\omega_0^2\omega^6(4\omega_0^2 - \omega^2)^2 \leqslant 0. \tag{2.186}$$

If $\omega \neq 0$, $4\omega_0^2 - \omega^2 \neq 0$, $\omega_0^2 - \omega^2 \neq 0$. Eq. (2.182) has three different real negative roots, and the characteristic equation of (2.177) correspondingly has three pairs of different pure imaginary roots.

For $\omega^2 = \omega_0^2$ the characteristic equation has a multiple zero root. If $\omega = 0$, the characteristic equation has a pair of triple imaginary conjugate roots: $p_{1,2,3} = j\omega_0$, $p_{4,5,6} = -j\omega_0$. Finally for $4\omega_0^2 - \omega^2 = 0$, we have the double roots $p_{1,2} = -j\omega_0$, $p_{3,4} = j\omega_0$.

It is readily seen that whenever the characteristic equation of (2.177) has multiple roots, the elementary divisors of the characteristic matrix remain linear, which completes the proof of nonasymptotic stability of Eqs. (2.177).

2.4.2. Internal dissipation forces

In our analysis of the stability of Eqs. (1.132), (2.41), (2.177) we found certain regions where the degree of instability under the action of potential forces alone was even and where the system was effectively stabilized by the gyroscopic forces. This stabilization, however, is transient, as it is destroyed by internal dissipation forces.

The following point should be borne in mind, however. Although the internal dissipation forces are unavoidable in actual displacements of a mechanical system, they can be made so small that the stability is not broken during a finite time of system operation. Moreover, arbitrary introduction of dissipative forces into the final equations (with the only justification that they are unavoidable) may actually lead to serious errors in stability analysis. To demonstrate this fact, consider the homogeneous equations (2.176), (2.177).

For simplicity, take the case of equatorial motion. Then $\omega_z = 0$ in Eqs. (2.177), the equation for δy is separated, and the remaining two equations give the simultaneous system

$$\left.\begin{aligned} \delta\ddot{x} + (\omega_0^2 - \omega_y^2)\,\delta x + 2\omega_y \delta\dot{z} = 0, \\ \delta z + (\omega_0^2 - \omega_y^2)\,\delta z - 2\omega_y \delta x = 0. \end{aligned}\right\} \tag{2.187}$$

Since the axis ζ_* coincides with the Earth's spin axis, we have taking the axes x, y to coincide initially with the axes ξ_*, η_* ($\omega_y = \text{const}$)

$$\left.\begin{aligned} \delta\xi_* = -\delta x \sin\omega_y t + \delta z \cos\omega_y t, \\ \delta\eta_* = \delta x \cos\omega_y t + \delta z \sin\omega_y t. \end{aligned}\right\} \tag{2.188}$$

If the gyroscopic forces

$$2\omega_y \delta\dot{z}, \quad -2\omega_y \delta\dot{x} \tag{2.189}$$

are dropped from Eqs. (2.187), we are left only with potential forces with the force function

$$U = -\tfrac{1}{2}(\omega_0^2 - \omega_y^2)\,[(\delta x)^2 + (\delta z)^2]. \tag{2.190}$$

and Eqs. (2.187) turn to be equations in normal coordinates.

The force function has a maximum for $\delta x = \delta z = 0$ if

$$\omega_0^2 - \omega_y^2 > 0. \tag{2.191}$$

On the other hand, the characteristic equation of (2.187) is

$$p^4 + 2(\omega_0^2 + \omega_y^2)\,p^2 + (\omega_0^2 - \omega_y^2)^2 = 0. \tag{2.192}$$

Its roots are

$$p_{1,2,3,4} = \pm i(\omega_0 \pm \omega_y). \tag{2.193}$$

i.e., Eqs. (2.187) are nonasymptotically stable for any ω_y.

The stability of Eqs. (2.187) for ω_y not satisfying inequality (2.191) is obviously ensured by the gyroscopic forces (2.189), since for $\omega_0^2 - \omega_y^2 < 0$ the degree of instability in a pure potential field is even.

We now introduce in Eqs. (2.187) small internal dissipative forces with Rayleigh's dispersion function

$$f = \tfrac{1}{2}\varepsilon^2[(\delta\dot{x})^2 + (\delta\dot{y})^2 + (\delta\dot{z})^2]. \tag{2.194}$$

The second equation in (2.177) then becomes asymptotically stable ($\omega_x = 0$, $\omega_z = 0$), and Eqs. (2.187) take the form

$$\left.\begin{aligned} \delta\ddot{x} + \varepsilon^2\delta\dot{x} + (\omega_0^2 - \omega_y^2)\,\delta x + 2\omega_y\delta\dot{z} = 0, \\ \delta\ddot{z} + \varepsilon^2\delta\dot{z} + (\omega_0^2 - \omega_y^2)\,\delta z - 2\omega_y\delta\dot{x} = 0. \end{aligned}\right\} \tag{2.195}$$

For small ε^2 the characteristic equation of (2.195) can be written in the form

$$p^4 + 2\varepsilon^2 p^3 + 2p^2(\omega_0^2 + \omega_y^2) + 2\varepsilon^2 p(\omega_0^2 - \omega_y^2) + (\omega_0^2 - \omega_y^2)^2 = 0. \tag{2.196}$$

If inequality (2.191) is not satisfied, the coefficient before p becomes negative; Hurwitz's necessary conditions are broken, and the system is therefore unstable. The stability produced by gyroscopic forces (2.189) outside the region (2.191) is thus destroyed by internal dissipation forces.

Let us now introduce internal dissipation in the homogeneous equations (2.176). The first two equations take the form

$$\left.\begin{aligned} \delta\ddot{\xi}_* + \varepsilon^2\delta\dot{\xi}_* + \omega_0^2\delta\xi_* = 0, \\ \delta\ddot{\eta}_* + \varepsilon^2\delta\dot{\eta}_* + \omega_0^2\delta\eta_* = 0. \end{aligned}\right\} \tag{2.197}$$

For $\delta\xi_*$, $\delta\eta_*$ we obtain damped solutions, and by (2.188) δx, δy are also expressed by damped solutions, irrespective of ω_y.

We have thus arrived at an apparent contradiction. This is associated with the fact that we have introduced dissipative forces in different coordinate systems. In the first case these were the coordinates δx, δy, δz, and in the second case the coordinates $\delta\xi_*$, $\delta\eta_*$, $\delta\zeta_*$. This difference in approach is responsible for the difference in the final results. To prove this assertion, we project Eqs. (2.197) onto the axes x, z using Eqs. (2.188). This gives two equations

$$\left.\begin{aligned} \delta\ddot{x} + (\omega_0^2 - \omega_y^2)\delta x + 2\omega_y\delta\dot{z} + \varepsilon^2\delta\dot{x} + \varepsilon^2\omega_y\delta z = 0, \\ \delta\ddot{z} + (\omega_0^2 - \omega_y^2)\delta z - 2\omega_y\delta\dot{x} + \varepsilon^2\delta\dot{z} - \varepsilon^2\omega_y\delta x = 0, \end{aligned}\right\} \tag{2.198}$$

which differ from Eqs. (2.195) in their last term. The characteristic equation of (2.198)

$$p^4 + 2\varepsilon^2 p^3 + 2p^2(\omega_0^2 + \omega_y^2) + 2\varepsilon p(\omega_0^2 + \omega_y^2) + (\omega_0^2 - \omega_y^2)^2 = 0 \tag{2.199}$$

satisfies Hurwitz's conditions irrespective of the value of ω_y.

The above example shows that arbitrary introduction of the dissipative forces in the variational equations may eventually (in the result of some transformations) lead to erroneous results. Dissipative forces, if at all present in the system, should be considered in all stages of the derivation of the variational equations; any transformation of the variational equations should include the dissipative terms as well.

2.4.3. Stability of the first group of equations for a close object

Let us return to Eqs. (2.176). For $r = \text{const}$ their solutions are harmonic oscillations, i.e., we are dealing with nonasymptotic stability. If the inertial system is mounted on a close object, which moves near the Earth's surface, the r in Eqs. (2.176) is some function of time. A certain choice of the function $r(t)$ will make the solution of Eqs. (2.176) unstable (divergent). If we assume that the object moves near $r = r^0 = \text{const}$, so that

$$r = r^0 + \rho(t), \quad \frac{|\rho|}{r^0} \ll 1, \tag{2.200}$$

upper-bound estimates analogous to (2.138), (2.172) can be readily constructed for the measure of divergence of the solutions of Eqs. (2.176).

From Eqs. (2.176), using (2.200), we get

$$\left.\begin{aligned}\delta\ddot{\xi}_* + \omega_0^2\delta\xi_* &= -\frac{3\omega_0^2\omega}{r^0}\delta\xi_*,\\ \delta\ddot{\eta}_* + \omega_0^2\delta\eta_* &= -\frac{3\omega_0^2\omega}{r^0}\delta\eta_*,\\ \delta\ddot{\zeta}_* + \omega_0^2\delta\zeta_* &= -\frac{3\omega_0^2\omega}{r^0}\delta\zeta_*,\end{aligned}\right\} \tag{2.201}$$

where

$$\omega_0^2 = \frac{\mu}{r^{0^3}}. \tag{2.202}$$

The solution of Eqs. (2.201) is found by successive approximations, along the same lines as the approximate solution (2.135) for Eqs. (2.124). This gives the following estimates:

$$\left.\begin{gathered}\left|\frac{\delta\xi_*}{\delta\xi_*^0}\right|, \left|\frac{\delta\eta_*}{\delta\eta_*^0}\right|, \left|\frac{\delta\zeta_*}{\delta\zeta_*^0}\right| \leqslant e^{3\sigma^2\tau},\\ \sigma^2 = \max\frac{|\rho|}{r^0}, \quad \tau = \omega_0 t,\end{gathered}\right\} \tag{2.203}$$

and for

$$3\sigma^2\tau \ll 1 \tag{2.204}$$

we have

$$\left|\frac{\delta\xi_*}{\delta\xi_*^0}\right|, \left|\frac{\delta\eta_*}{\delta\eta_*^0}\right|, \left|\frac{\delta\zeta_*}{\delta\zeta_*^0}\right| \leqslant 1 + 3\sigma^2\tau. \tag{2.205}$$

Estimates (2.203), (2.205) are analogous to estimates (2.172), (2.174). The latter, as is readily seen, reduce to estimates (2.203), (2.205) for $\mu^2 = 0$.

2.4.4. Solution of the first group of equations for arbitrary motion at a constant distance from the Earth's center

In this case the error equations projected onto fixed-orientation axes are Eqs. (2.176). Their solution is

$$\left.\begin{aligned}\delta\xi_* &= \delta\xi_*^0\cos\omega_0 t + \frac{\delta\dot{\xi}_*^0}{\omega_0}\sin\omega_0 t +\\ &+ \frac{1}{\omega_0}\int_0^t\Big[\Delta n_{\xi_*} - 2(\Delta m_{\eta_*}\dot{\zeta}_* - \Delta m_{\zeta_*}\dot{\eta}_*) +\\ &+ \eta_*\Delta\dot{m}_{\zeta_*} - \zeta_*\Delta\dot{m}_{\eta_*} + 3\omega_0^2\frac{\xi_*\Delta r}{r}\Big]\sin\omega_0(t-\tau)\,d\tau,\\ \delta\eta_* &= \delta\eta_*^0\cos\omega_0 t + \frac{\delta\dot{\eta}_*^0}{\omega_0}\sin\omega_0 t +\\ &+ \frac{1}{\omega_0}\int_0^t\Big[\Delta n_{\eta_*} - 2(\Delta m_{\zeta_*}\dot{\xi}_* - \Delta m_{\xi_*}\dot{\zeta}_*) +\\ &+ \zeta_*\Delta\dot{m}_{\xi_*} - \xi_*\Delta\dot{m}_{\zeta_*} + 3\omega_0^2\frac{\eta_*\Delta r}{r}\Big]\sin\omega_0(t-\tau)\,d\tau,\end{aligned}\right\} \tag{2.206}$$

$$\left.\begin{aligned}\delta\xi_* &= \delta\xi_*^0\cos\omega_0 t + \frac{\delta\dot\xi_*^0}{\omega_0}\sin\omega_0 t + \\ &+ \frac{1}{\omega_0}\int_0^t\left[\Delta n_{\xi_*} - 2(\Delta m_{\xi_*}\dot\eta_* - \Delta m_{\eta_*}\dot\xi_*) + \right. \\ &\left. + \xi_*\Delta\dot m_{\eta_*} - \eta_*\Delta\dot m_{\xi_*} - 3\omega_0^2\frac{\xi_*\Delta r}{r}\right]\sin\omega_0(t-\tau)\,d\tau.\end{aligned}\right\} \qquad (2.206)$$

Using the known values of the direction cosines a_{ij} between the axes ξ_*, η_*, ζ_* and x, y, z we find δx, δy, δz; in the corresponding expressions $\delta\xi_*^0$, $\delta\eta_*^0$, $\delta\dot\xi_*^0$, $\delta\dot\eta_*^0$, $\delta\dot\zeta_*^0$ can be expressed in terms of δx^0, δy^0, δz^0, $\delta\dot x^0$, $\delta\dot y^0$, $\delta\dot z^0$, and the functions Δn_{ξ_*}, Δn_{η_*}, Δn_{ζ_*}, Δm_{ξ_*}, Δm_{η_*}, Δm_{ζ_*} in terms of Δn_x, Δn_y, Δn_z, Δm_x, Δm_y, Δm_z. For an object which remains stationary in the $O_1\xi_*\eta_*\zeta_*$ axes, we may take the axes ξ_*, η_*, ζ_* to coincide with the axes x, y, z and direct the z axis along r, so that for δx, δy, δz we obtain

$$\left.\begin{aligned}\delta x &= \delta x^0\cos\omega_0 t + \frac{\delta\dot x^0}{\omega_0}\sin\omega_0 t + \\ &+ \frac{1}{\omega_0}\int_0^t(\Delta n_x - r\,\Delta\dot m_y)\sin\omega_0(t-\tau)\,d\tau, \\ \delta y &= \delta y^0\cos\omega_0 t + \frac{\delta\dot y^0}{\omega_0}\sin\omega_0 t + \\ &+ \frac{1}{\omega_0}\int_0^t(\Delta n_y + r\,\Delta\dot m_x)\sin\omega_0(t-\tau)\,d\tau. \\ \delta z &= \delta z^0\cos\omega_0 t + \frac{\delta\dot z^0}{\omega_0}\sin\omega_0 t + \\ &+ \frac{1}{\omega_0}\int_0^t(\Delta n_z + 3\omega_0^2\Delta r)\sin\omega_0(t-\tau)\,d\tau.\end{aligned}\right\} \qquad (2.207)$$

For constant instrumental errors we get

$$\left.\begin{aligned}\delta x &= \frac{\Delta n_x}{\omega_0^2} + \left(\delta x^0 - \frac{\Delta n_x}{\omega_0^2}\right)\cos\omega_0 t + \frac{\delta\dot x^0}{\omega_0}\sin\omega_0 t, \\ \delta y &= \frac{\Delta n_y}{\omega_0^2} + \left(\delta y^0 - \frac{\Delta n_y}{\omega_0^2}\right)\cos\omega_0 t + \frac{\delta\dot y^0}{\omega_0}\sin\omega_0 t, \\ \delta z &= \frac{\Delta n_z}{\omega_0^2} + 3\,\Delta r + \\ &+ \left(\delta z^0 - \frac{\Delta n_z}{\omega_0^2} - 3\,\Delta r\right)\cos\omega_0 t + \frac{\delta\dot z^0}{\omega_0}\sin\omega_0 t.\end{aligned}\right\} \qquad (2.208)$$

For an object moving with constant velocity along the arc of a fixed-orientation great circle, the plane of the great circle is made to coincide with the planes $\xi_*\eta_*$ and xy, so that using the transformation formulas (2.188) we obtain

$$\left.\begin{aligned}\delta x &= (\delta x^0\cos\omega_y t - \delta z^0\sin\omega_y t)\cos\omega_0 t + \frac{1}{\omega_0}[(\delta\dot x^0 + \delta z^0\omega_y)\cos\omega_y t - \\ &- (\delta\dot z^0 - \delta x^0\omega_y)\sin\omega_y t]\sin\omega_0 t + \frac{1}{\omega_0}\int_0^t[(\Delta n_x - r\,\Delta\dot m_y)\cos\omega_y(t-\tau) - \\ &- (\Delta n_z + 2\,\Delta m_y r\omega_y + 3\omega_0^2\Delta r)\sin\omega_y(t-\tau)]\sin\omega_0(t-\tau)\,d\tau,\end{aligned}\right\} \qquad (2.209)$$

$$
\left.\begin{aligned}
\delta y &= \delta y^0 \cos \omega_y t + \frac{\delta \dot{y}^0}{\omega_0} \sin \omega_0 t + \\
&+ \frac{1}{\omega_0} \int_0^t (\Delta n_y + \Delta \dot{m}_x r - \Delta m_z r \omega_y) \sin \omega_0 (t - \tau)\, d\tau, \\
\delta z &= (\delta x^0 \sin \omega_y t + \delta z^0 \cos \omega_y t) \cos \omega_0 t + \\
&+ \frac{1}{\omega_0} [(\delta \dot{z}^0 - \delta x^0 \omega_y) \cos \omega_y t + \\
&+ (\delta \dot{x}^0 + \delta z^0 \omega_y) \sin \omega_y t] \sin \omega_0 t + \\
&+ \frac{1}{\omega_0} \int_0^t [(\Delta n_x - \Delta \dot{m}_y r) \sin \omega_y (t - \tau) + \\
&+ (\Delta n_z + 2 \Delta m_y r \omega_y + \\
&\quad + 3\omega_0^2 \Delta r) \cos \omega_y (t - \tau)] \sin \omega_0 (t - \tau)\, d\tau.
\end{aligned}\right\} \qquad (2.209)
$$

For constant instrumental errors and $\omega_y^2 \ll \omega_0^2$, Eqs. (2.209) give the following approximate relations:

$$
\left.\begin{aligned}
\delta x &= (\delta x^0 \cos \omega_y t - \delta z^0 \sin \omega_y t) \cos \omega_0 t + \\
&+ \frac{1}{\omega_0} (\delta \dot{x}^0 \cos \omega_y t - \delta \dot{z}^0 \sin \omega_y t) \sin \omega_0 t + \\
&+ \frac{\Delta n_x}{\omega_0^2} (1 - \cos \omega_0 t \cos \omega_y t) + \\
&\quad + \left(\frac{\Delta n_z}{\omega_0^2} + 3 \Delta r\right) \cos \omega_0 t \sin \omega_y t, \\
\delta y &= \frac{\Delta n_y}{\omega_0^2} + \left(\delta y^0 - \frac{\Delta n_y}{\omega_0^2}\right) \cos \omega_0 t + \frac{\delta \dot{y}^0}{\omega_0} \sin \omega_0 t, \\
\delta z &= (\delta x^0 \sin \omega_y t + \delta z^0 \cos \omega_y t) \cos \omega_0 t + \\
&+ \frac{1}{\omega_0} (\delta \dot{x}^0 \sin \omega_y t + \delta z^0 \cos \omega_y t) \sin \omega_0 t - \\
&- \frac{\Delta n_x}{\omega_0^2} \sin \omega_y t \cos \omega_0 t + \\
&\quad + \left(\frac{\Delta n_z}{\omega_0^2} + 3 \Delta r\right) (1 - \cos \omega_0 t \cos \omega_y t).
\end{aligned}\right\} \qquad (2.210)
$$

If in (2.209) we take $\omega_y = \omega_0$, we obtain the errors δx, δy, δz in the coordinates of a satellite in circular orbit. For constant instrumental errors, the corresponding expressions take the form

$$
\left.\begin{aligned}
\delta x &= \delta x^0 + \frac{\delta \dot{x}^0}{2\omega_0} \sin 2\omega_0 t + \frac{\delta \dot{z}^0}{2\omega_0} (\cos 2\omega_0 t - 1) + \\
&\quad + \frac{\Delta n_x}{4\omega_0^2} (1 - \cos 2\omega_0 t) + \\
&\quad + \left(\frac{\Delta n_z}{4\omega_0^2} + \frac{\Delta m_y r}{2\omega_0} + \frac{3}{4} \Delta r\right) (\sin 2\omega_0 t - 2\omega_0 t), \\
\delta y &= \delta y^0 \cos \omega_0 t + \frac{\delta \dot{y}^0}{\omega_0} \sin \omega_0 t + \\
&\quad + \frac{\Delta n_y - \Delta m_z r \omega_0}{\omega_0^2} (1 - \cos \omega_0 t),
\end{aligned}\right\} \qquad (2.211)
$$

$$\left.\begin{aligned}\delta z = \delta z^0 + \frac{\delta \dot z^0}{2\omega_0}\sin 2\omega_0 t + \frac{\delta \dot x^0}{\omega_0}(1-\cos 2\omega_0 t) + \\ + \frac{\Delta n_x}{4\omega_0^2}(2\omega_0 t - \sin 2\omega_0 t) + \left(\frac{\Delta n_z}{4\omega_0^2} + \frac{\Delta m_y r}{2\omega_0} + \frac{3}{4}\Delta r\right)(1-\cos 2\omega_0 t).\end{aligned}\right\} \tag{2.211}$$

From Eqs. (2.206), matrices (1.10), (1.7) of direction cosines, and the obvious equality

$$\lambda = \left(u\cos q + \frac{V}{r}\right)t \tag{2.212}$$

we can obtain appropriate expressions for δx, δy, δz for an object moving along a parallel with constant velocity V relative to the Earth. We will not give here these (fairly complex) expressions. If, necessary, the reader will derive them without much difficulty.

2.4.5. Integration of the first group of error equations for Keplerian motion

For the case of Keplerian motion, the vector equation (2.175) projected onto the orbital axes xyz gives a fourth-order system

$$\left.\begin{aligned}\delta\ddot x + \left(\frac{\mu}{r^3} - \omega_y^2\right)\delta x + \dot\omega_y \delta z + 2\omega_y \delta\dot z = \Delta n_x - 2\Delta m_y \dot r - \Delta\dot m_y r, \\ \delta\ddot z + \left(\frac{\mu}{r^3} - \omega_y^2\right)\delta z - \dot\omega_y \delta x - 2\omega_y \delta\dot x = \Delta n_z + 2\Delta m_y r\omega_y + \frac{3u\,\Delta r}{r^3}\end{aligned}\right\} \tag{2.213}$$

and a second-order equation

$$\delta\ddot y + \frac{u}{r^3}\delta y = \Delta n_y + 2\Delta m_x \dot r + \Delta\dot m_x r - \Delta m_z r\omega_y. \tag{2.214}$$

Eqs. (2.213), (2.214), unlike Eq. (1.92) are not the variational equations of Keplerian motion; Poincaré's theorem, used in solving Eq. (1.92), is no longer applicable to the solution of Eqs. (2.213), (2.214).

However, comparison of Eqs. (1.92) and (2.213), (2.214) suggests a method of integration, as we have already found out in solving Eqs. (2.95).

Eq. (2.214) coincides with the second equation in (2.95). The solution of the latter is (2.98), and for low-eccentricity orbits and constant projections of the vectors Δn and Δm on the orbital axes its solution is (2.105). The problem thus reduces to the solution of Eqs. (2.213).

Comparison of Eqs. (2.213) with Eqs. (2.96) — the projections of the vector equation (1.92) on the orbital axes x and z — shows that the homogeneous system (2.213) has two particular solutions:

$$\left.\begin{aligned}\delta x = \frac{r}{a}, \quad \delta z = 0; \\ \delta x = 0, \quad \delta z = \frac{r}{a}.\end{aligned}\right\} \tag{2.215}$$

Its order can thus be lowered to second by the substitution of variables

$$\delta x = \frac{r}{a}\int p\,dt, \quad \delta z = \frac{r}{a}\int q\,dt. \tag{2.216}$$

The equations for p and q take the form

$$\left.\begin{aligned} \dot{p} + 2\frac{\dot{r}}{r}p + 2\omega_y q = 0, \\ \dot{q} + 2\frac{\dot{r}}{r}q - 2\omega_y p = 0. \end{aligned}\right\} \tag{2.217}$$

Multiplying the second equation in (2.217) by $j = \sqrt{-1}$, adding it to the first equation, and introducing a new (complex variable)

$$z = p + jq, \tag{2.218}$$

we end up with a first-order equation for z

$$\dot{z} + 2z\left(\frac{\dot{r}}{r} - j\omega_y\right) = 0, \tag{2.219}$$

which is immediately integrated. The general solution of this equation is

$$z = \frac{C}{r^2}(\cos 2v + i \sin 2v), \tag{2.220}$$

where v, as before, is the true anomaly, and C is a complex constant.

Changing back to δx, δz and using (2.113), we obtain two particular solutions of Eqs. (2.213)

$$\left.\begin{aligned} \delta x = \frac{r}{a}\sin 2v, \qquad \delta z = \frac{r}{a}\cos 2v; \\ \delta x = -\frac{r}{a}\cos 2v, \quad \delta z = \frac{r}{a}\sin 2v. \end{aligned}\right\} \tag{2.221}$$

Eqs. (2.221), together with Eqs. (2.215), constitute a system of four partial solutions of Eqs. (2.213). A matrix α is now formed from these particular solutions and their derivatives (using (1.162)), whose elements are

$$\left.\begin{aligned} &\alpha_{11} = \frac{r}{a}, \quad \alpha_{12} = 0, \quad \alpha_{13} = \frac{r}{a}\sin 2v, \quad \alpha_{14} = \frac{r}{a}\cos 2v, \\ &\alpha_{21} = 0, \quad \alpha_{22} = \frac{r}{a}, \quad \alpha_{23} = -\frac{r}{a}\cos 2v, \\ &\alpha_{24} = \frac{r}{a}\sin 2v, \\ &\alpha_{31} = \frac{ve}{\sqrt{1-e^2}}\sin v, \quad \alpha_{32} = 0, \\ &\alpha_{33} = \frac{ve}{\sqrt{1-e^2}}\sin v \sin 2v + \frac{2av\sqrt{1-e^2}}{r}\cos 2v, \\ &\alpha_{34} = \frac{ve}{\sqrt{1-e^2}}\sin v \cos 2v - \frac{2av\sqrt{1-e^2}}{r}\sin 2v, \\ &\alpha_{41} = 0, \quad \alpha_{42} = \frac{ve}{\sqrt{1-e^2}}\sin v, \\ &\alpha_{43} = -\frac{ve}{\sqrt{1-e^2}}\sin v \cos 2v + \frac{2av\sqrt{1-e^2}}{r}\sin 2v, \\ &\alpha_{44} = \frac{ve}{\sqrt{1-e^2}}\sin v \sin 2v + \frac{2av\sqrt{1-e^2}}{r}\cos 2v. \end{aligned}\right\} \tag{2.222}$$

The determinant of the matrix α is the Wronskian of our system of particular solutions. If the homogeneous system (2.213) is reduced to the Cauchy form,

i.e., represented as a system of first-order differential equations solved for the respective derivatives, the matrix comprising the coefficients of the right-hand sides will have zero diagonal elements, as in the case of Eqs. (1.163). The Wronskian is therefore constant. Its value for $t = 0$ is readily calculated. Seeing that $v(0) = 0$, we get from (2.222)

$$|\alpha| = 4v^2(1 - e^2) \neq 0. \tag{2.223}$$

Our system of particular solutions of homogeneous equations (2.213) is thus linearly independent. The general solution of the homogeneous system (2.213) is therefore*

$$\delta x = \sum_{i=1}^{4} C_i a_{1i}, \quad \delta z = \sum_{i=1}^{4} C_i a_{2i}. \tag{2.224}$$

* The general solution of homogeneous equations (2.213) has been derived by this method in my paper mentioned in the footnote to p. 70. A.I.Lur'e, going over the manuscript of the present book, suggested an elegant direct technique for the derivation of the general solution of the homogeneous equation (2.175), which is given below.

From (1.155) and homogeneous equation (2.175) it follows (dots denote differentiation with respect to time)

$$\ddot{\boldsymbol{r}} \times \delta\boldsymbol{r} + \delta\ddot{\boldsymbol{r}} \times \boldsymbol{r} = \frac{d}{dt}(\dot{\boldsymbol{r}} \times \delta\boldsymbol{r} + \delta\dot{\boldsymbol{r}} \times \boldsymbol{r}) = 0,$$

$$\dot{\boldsymbol{r}} \times \delta\boldsymbol{r} + \delta\dot{\boldsymbol{r}} \times \boldsymbol{r} = \boldsymbol{a}, \quad \text{where } \boldsymbol{a} \text{ is a constant vector.}$$

Hence

$$\delta\dot{r}r^2 - \delta r\boldsymbol{r}\cdot\dot{\boldsymbol{r}} = r\,\delta\dot{\boldsymbol{r}}\cdot\boldsymbol{r} + \boldsymbol{r}\times\boldsymbol{a} - \dot{r}\boldsymbol{r}\cdot\delta\boldsymbol{r}. \tag{*}$$

Now by (1.155), (2.175)

$$\boldsymbol{r}\cdot\delta\ddot{\boldsymbol{r}} - \ddot{\boldsymbol{r}}\cdot\delta\boldsymbol{r} = 0,$$

$$\boldsymbol{r}\cdot\delta\dot{\boldsymbol{r}} - \dot{\boldsymbol{r}}\cdot\delta\boldsymbol{r} = c = \text{const.}$$

Insertion in (*) gives (noting that $\boldsymbol{r}\cdot\dot{\boldsymbol{r}} = r\dot{r}$)

$$\delta\dot{r}r^2 - \delta r r\dot{r} = \delta\boldsymbol{r}\times(\boldsymbol{r}\times\dot{\boldsymbol{r}}) + c\boldsymbol{r} + \boldsymbol{r}\times\boldsymbol{a}.$$

But $\boldsymbol{r}\times\dot{\boldsymbol{r}} = r^2\dot{v}\boldsymbol{y}$. Therefore,

$$\delta\dot{\boldsymbol{r}} - \delta\boldsymbol{r}\frac{\dot{r}}{r} + \dot{v}\boldsymbol{y}\times\delta\boldsymbol{r} = (c\boldsymbol{r} + \boldsymbol{r}\times\boldsymbol{a})\frac{1}{r^2},$$

which is equivalent to the equations

$$(\delta\dot{z} + j\delta\dot{x}) - \left(\frac{\dot{r}}{r} - 2j\dot{v}\right)(\delta z + j\,\delta x) = \frac{1}{r}(c - ja_y),$$

$$\delta\dot{y} - \delta y\frac{\dot{r}}{r} = \frac{a_x}{r}.$$

Integrating these equations, we get

$$\delta z + j\,\delta x = (c_1 + jc_2)re^{-2jv} + (c - ja_y)2jr^2\dot{v}, \quad \delta y = c_3 r + a_x v/r\dot{v}.$$

The solution contains all the six arbitrary constants

$$c,\ a_x,\ a_y,\ c_1,\ c_2,\ c_3.$$

The general solution of the inhomogeneous system is obtained by variation of constants. Taking C_i to be functions of time, we obtain the equations

$$\left.\begin{aligned}
&\sum_{i=1}^{4} \dot{C}_i \alpha_{1i} = 0, \quad \sum_{i=1}^{4} \dot{C}_i \alpha_{2i} = 0, \\
&\sum_{i=1}^{4} \dot{C}_i \alpha_{3i} = \Delta n_x - 2\,\Delta m_y \dot{r} - \Delta \dot{m}_y r, \\
&\sum_{i=1}^{4} \dot{C}_i \alpha_{4i} = \Delta n_z + 2\,\Delta m_y r \omega_y + \frac{3\mu\,\Delta r}{r^3}.
\end{aligned}\right\} \tag{2.225}$$

The elements of the matrix $\beta = \alpha^{-1}$ are

$$\left.\begin{aligned}
&\beta_{11} = \frac{a}{r}, \quad \beta_{12} = \frac{e \sin v}{2(1-e^2)}, \quad \beta_{13} = 0, \\
&\beta_{14} = -\frac{r}{a} \frac{1}{2v\sqrt{1-e^2}}, \\
&\beta_{21} = -\frac{e \sin v}{2(1-e^2)}, \qquad \beta_{22} = \frac{a}{r}, \\
&\beta_{23} = \frac{r}{a} \frac{1}{2v\sqrt{1-e^2}}, \qquad \beta_{24} = 0, \\
&\beta_{31} = -\frac{e \sin v \cos 2v}{2(1-e^2)}, \qquad \beta_{32} = -\frac{e \sin v \sin 2v}{2(1-e^2)}, \\
&\beta_{33} = \frac{r}{a} \frac{\cos 2v}{2v\sqrt{1-e^2}}, \qquad \beta_{34} = \frac{r}{a} \frac{\sin 2v}{2v\sqrt{1-e^2}}, \\
&\beta_{41} = \frac{e \sin v \sin 2v}{2(1-e^2)}, \qquad \beta_{42} = -\frac{e \sin v \cos 2v}{2(1-e^2)}, \\
&\beta_{43} = -\frac{r}{a} \frac{\sin 2v}{2v\sqrt{1-e^2}}, \qquad \beta_{44} = \frac{r}{a} \frac{\cos 2v}{2v\sqrt{1-e^2}}.
\end{aligned}\right\} \tag{2.226}$$

Using the matrix β, which is by definition the inverse of α, we obtain

$$C_i(t) = \int_0^t \left[\beta_{i3}(\Delta n_x - 2\,\Delta m_y \dot{r} - \Delta \dot{m}_y r) + \beta_{i4}\left(\Delta n_z + 2\,\Delta m_y r \omega_y + \frac{3\mu\,\Delta r}{r^3}\right)\right] dt + C_i^0 \tag{2.227}$$
$$(i = 1, 2, 3, 4).$$

To write the solution of Eqs. (2.213), it now remains to insert (2.227) in (2.224), having first determined C_i^0 from the initial conditions of Eqs. (2.213). This gives

$$\left.\begin{aligned}
\delta x = \sum_{i=1}^{4} \alpha_{1i} \Bigg\{ &\int_0^t \left[\beta_{i3}(\Delta n_x - 2\,\Delta m_y \dot{r} - \Delta \dot{m}_y r) + \right. \\
&\left. + \beta_{i4}\left(\Delta n_z + 2\,\Delta m_y r \omega_y + \frac{3\mu\,\Delta r}{r^3}\right)\right] dt + C_i^0 \Bigg\}, \\
\delta z = \sum_{i=1}^{4} \alpha_{2i} \Bigg\{ &\int_0^t \left[\beta_{i3}(\Delta n_x - 2\,\Delta m_y \dot{r} - \Delta \dot{m}_y r) + \right. \\
&\left. + \beta_{i4}\left(\Delta n_z + 2\,\Delta m_y r \omega_y + \frac{3\mu\,\Delta r}{r^3}\right)\right] dt + C_i^0 \Bigg\},
\end{aligned}\right\} \tag{2.228}$$

where

$$\left.\begin{aligned}
&C_1^0 = \frac{\delta x^0}{1-e} - \frac{\delta \dot{z}^0 (1-e)}{2v\sqrt{1-e^2}}, \quad C_2^0 = \frac{\delta z^0}{1-e} + \frac{\delta \dot{x}^0 (1-e)}{2v\sqrt{1-e^2}}, \\
&C_3^0 = \frac{\delta \dot{x}^0 (1-e)}{2v\sqrt{1-e^2}}, \qquad C_4^0 = \frac{\delta \dot{z}^0 (1-e)}{2v\sqrt{1-e^2}}.
\end{aligned}\right\} \tag{2.229}$$

For low-eccentricity orbits, using (2.100), (2.102), we obtain to terms of the order e inclusive:

$$
\left.\begin{aligned}
&\alpha_{11} = 1 - e\cos\nu t, \quad \alpha_{12} = 0,\\
&\alpha_{13} = \sin 2\nu t + e\sin\nu t\,(3\cos 2\nu t - 1),\\
&\alpha_{23} = \cos 2\nu t - e\,(3\sin\nu t\sin 2\nu t + \cos\nu t),\\
&\alpha_{21} = 0, \quad \alpha_{22} = 1 - e\cos 2\nu t,\\
&\alpha_{23} = -\cos 2\nu t + e\,(3\sin\nu t\sin 2\nu t + \cos\nu t),\\
&\alpha_{31} = \sin 2\nu t + e\sin\nu t\,(3\cos 2\nu t - 1);\\
&\beta_{11} = 0, \quad \beta_{14} = -\frac{1}{2\nu}(1 - e\cos\nu t),\\
&\beta_{12} = \frac{1}{\nu}(1 - e\cos\nu t), \quad \beta_{24} = 0,\\
&\beta_{13} = \frac{1}{2\nu}[\cos 2\nu t - e\,(3\sin\nu t\sin 2\nu t + \cos\nu t)],\\
&\beta_{14} = \frac{1}{2\nu}[\sin 2\nu t + e\sin\nu t\,(3\cos 2\nu t - 1)],\\
&\beta_{32} = -\frac{1}{2\nu}[\sin 2\nu t + e\sin\nu t\,(3\cos 2\nu t - 1)],\\
&\beta_{34} = \frac{1}{2\nu}[\cos 2\nu t - e\,(3\sin\nu t\sin 2\nu t + \cos\nu t)];\\
&C_1^0 = \delta x^0(1 + e) - \frac{\delta\dot z^0}{2\nu}(1 - e),\\
&C_2^0 = \delta z^0(1 + e) + \frac{\delta\dot x^0}{2\nu}(1 - e),\\
&C_3^0 = \frac{\delta\dot z^0}{2\nu}(1 - e), \quad C_4^0 = \frac{\delta\dot z^0}{2\nu}(1 - e).
\end{aligned}\right\} \qquad (2.230)
$$

If the instrumental errors are constant and the α_{ij} and β_{ij} are inserted from (2.230) into (2.228), integration and obvious simplifications which follow from the fact that e is small give the following expressions for the errors δx, δz:

$$
\left.\begin{aligned}
&\delta x = \frac{\Delta n_x}{4\nu^2}(1 - \cos 2\nu t) +\\
&+\left(\frac{\Delta n_z}{4\nu^2} + \frac{a\,\Delta m_y}{2\nu} + \frac{3\,\Delta r}{4}\right)(\sin 2\nu t - 2\nu t) +\\
&+ \delta x^0 + \frac{\delta\dot x^0}{2\nu}\sin 2\nu t + \frac{\delta\dot z^0}{2\nu}(\cos 2\nu t - 1) +\\
&+ e\left[\frac{\Delta n_x}{2\nu^2}\left(-\frac{5}{4}\cos\nu t + 2\cos 2\nu t - \frac{3}{4}\cos 3\nu t\right) +\right.\\
&+\frac{\Delta n_z}{2\nu^2}\left(\nu t\cos\nu t + \frac{3}{4}\sin\nu t - 2\sin 2\nu t + \frac{3}{4}\sin 3\nu t\right) +\\
&+\frac{5a\,\Delta m_y}{3\nu}(-2\sin\nu t + \sin 2\nu t) +\\
&+\Delta r\left(\frac{3}{2}\nu t\cos\nu t - \frac{39}{8}\sin\nu t + \frac{9}{8}\sin 3\nu t\right) +\\
&+\delta x^0(1 - \cos\nu t) + \frac{\delta\dot z^0}{2\nu}(1 - \cos 2\nu t - 3\sin\nu t\sin 2\nu t) +\\
&\left.+\frac{\delta\dot x^0}{2\nu}(-\sin\nu t - \sin 2\nu t + 3\sin\nu t\cos 2\nu t)\right],\\
&\delta z = \frac{\Delta n_x}{4\nu^2}(2\nu t - \sin 2\nu t) +\\
&+\left(\frac{\Delta n_z}{4\nu^2} + \frac{a\,\Delta m_y}{2\nu} + \frac{3\,\Delta r}{4}\right)(1 - \cos 2\nu t) + \delta z^0 +\\
&+\frac{\delta\dot x^0}{2\nu}(1 - \cos 2\nu t) + \frac{\delta\dot z^0}{2\nu}\sin 2\nu t +\\
&+ e\left[\frac{\Delta n_x}{2\nu^2}\left(-\nu t\cos\nu t - \frac{3}{4}\sin\nu t +\right.\right.
\end{aligned}\right\} \qquad (2.231)
$$

$$\left.\begin{aligned}
&+ 2\sin 2\nu t - \frac{3}{4}\sin 3\nu t\Big) + \\
&+ \frac{\Delta n_z}{2\nu^2}\Big(-\frac{5}{4}\cos\nu t + 2\cos 2\nu t - \frac{3}{4}\cos 3\nu t\Big) + \\
&+ \frac{\Delta m_y a}{\nu}\Big(-1 + \frac{5}{3}\cos\nu t - \frac{5}{6}\cos 2\nu t + \frac{1}{6}\cos 3\nu t\Big) + \\
&+ \frac{9}{8}\frac{\Delta r}{}(\cos\nu t - \cos 3\nu t) + \delta z^0(1 - \cos\nu t) + \\
&+ \frac{\delta\dot{x}^0}{2\nu}(-1 + \cos 2\nu t + 3\sin\nu t\sin 2\nu t) + \\
&+ \frac{\delta\dot{z}^0}{2\nu}(-\sin 2\nu t - \sin\nu t + 3\sin\nu t\cos 2\nu t)\Big].
\end{aligned}\right\} \qquad (2.231)$$

For $e = 0$, i.e., a circular orbit, Eqs. (2.231) reduce ($\nu = \omega_0^*$, $r = a$) to the first and third equations in (2.211).

§2.5. DEPENDENCE OF NAVIGATION ERRORS ON INSTRUMENTAL AND SETTING INACCURACIES. COMPARISON WITH AUTONOMOUS INERTIAL SYSTEMS

2.5.1. Dependence of position and orientation errors on setting errors

In §2.2, in the derivation of the error equations of inertial systems using additional guidance information on the distance from the Earth's center (or from its surface), we established that only the first group of differential error equations, i.e., Eqs. (1.92), (1.95), and the ensuing relations change compared to autonomous inertial systems. The other equations are not affected. Thus, in our analysis of the dependence of the total position and orientation errors of the object (or the inertial platform) on instrumental errors, altimeter errors $\Delta r = \Delta h$, and setting errors in the initial conditions, we may proceed from the results of 1.3.5, where the corresponding treatment for an autonomous inertial system was carried out.

The total position errors δx_2, δy_2, δz_2 are expressed according to Eqs. (1.93), (1.94) by the relations

$$\left.\begin{aligned}
&\delta r_2 = \delta r + \delta r_1, \quad \delta r_1 = \theta \times r, \\
&\delta r_2 = \delta x_2 x + \delta y_2 y + \delta z_2 z.
\end{aligned}\right\} \qquad (2.232)$$

The projections of these equations on the moving axes xyz, with the z axis pointing along the radius vector r give, by (1.108),

$$\left.\begin{aligned}
&\delta x_2 = \delta x + \delta x_1, \quad \delta y_2 = \delta y + \delta y_1, \quad \delta z_2 = \delta z, \\
&\delta x_1 = \theta_y r, \quad \delta y_1 = -\theta_x r.
\end{aligned}\right\} \qquad (2.233)$$

Orientation errors of the co-moving axes are determined by a small rotation vector θ_1 whose projections, by Eqs. (1.118), are

$$\left.\begin{aligned}
&\theta_{1x} = -\frac{\delta y}{r}, \quad \theta_{1y} = \frac{\delta x}{r}, \\
&\theta_{1z} = -\theta_z + \beta_{l2}\frac{\partial\beta_{i1}}{\partial\eta^s}[\delta r\cdot\eta_s + r(\theta_y\beta_{s1} - \theta_x\beta_{s2})],
\end{aligned}\right\} \qquad (2.234)$$

where

$$\delta r = \delta x x + \delta y y + \delta z z.$$

In Eqs. (2.233), (2.234), the projections θ_x, θ_y, θ_z are the solutions of Eqs. (1.93) or (1.96). For autonomous inertial systems, δx, δy, δz are solutions of Eqs. (1.92) or (1.95). In our case, for δx, δy, δz in Eqs. (2.233), (2.234) we should insert the solutions of Eqs. (2.15a) or (2.16) for three-accelerometer systems and solutions of Eqs. (2.28) for two-accelerometer systems.

Thus, in order to establish the difference between the perturbed motion of autonomous systems and systems with altimeter correction, it suffices to compare the properties of solutions of Eqs. (1.95) with the properties of solutions of Eqs. (2.15a), (2.28). This comparison was partly carried out in the previous analysis of stability and integration of Eqs. (2.15a), (2.28). In view of the great importance of this problem, we will now consider it in more detail.

First let us compare the solutions of the homogeneous equations (1.95), (2.16), (2.28), i.e., the perturbed motion of an autonomous inertial system and of a system with altimeter correction in cases when setting inaccuracies (errors in the initial conditions) are the only source of perturbation.

As we have shown in 1.3.2, some of the cases when Eqs. (1.95) reduce to equations with constant coefficient lead to instability of the autonomous inertial system. The autonomous inertial system is definitely unstable in the practically important region

$$\omega_0^2 - \omega_y^2 - \omega_x^2 - \omega_z^2 > 0. \tag{2.235}$$

The roots of the characteristic equation of (1.95) in region (2.235) contain a positive root which at low velocities is close to $\omega_0\sqrt{2}$. The moduli of the errors δx, δy, δz will therefore increase exponentially, rapidly deviating from the initial setting. The numerical estimates of 1.3.5 show that an autonomous inertial system may function with an accuracy comparable with the accuracy of the initial setting only for 10—15 min from the start.

In case of Keplerian motion, the solutions of Eqs. (1.95) also point to instability. Indeed, assuming zero instrumental errors, we obtain from Eqs. (1.166)

$$\delta x = \sum_{i=1}^{4} C_i^0 A_{1i}, \quad \delta z = \sum_{i=1}^{4} C_i^0 A_{2i}, \quad \delta y = \sum_{i=1}^{3} C_{i+1}^0 B_{1i}, \tag{2.236}$$

where C_i^0 are constants determined by the inaccuracy of the initial conditions, A_{1i}, A_{2i}, B_{1i} are the elements of the matrix of particular solutions (1.167). If the elements B_{1i} are periodic functions of time, some of the elements A_{1i}, A_{2i} are proportional to time (A_{11} and A_{21}), which leads to instability. Note that A_{21} contains time as a factor in the second term with a coefficient e. For motion in a circular orbit, the solutions δx, δz of the homogeneous equations (1.95), according to (1.152), have the form

$$\left.\begin{aligned} \delta x &= \delta x^0 + \frac{\delta \dot{x}^0}{\nu}(4\sin\nu t - 3\nu t) + \\ &\quad + 6\,\delta z^0(\sin\nu t - \nu t) + \frac{2\,\delta \dot{z}^0}{\nu}(\cos\nu t - 1), \\ \delta z &= \frac{2\,\delta \dot{x}^0}{\nu}(1 - \cos\nu t) + \delta z^0(4 - 3\cos\nu t) + \frac{\delta \dot{z}^0}{\nu}\sin\nu t. \end{aligned}\right\} \tag{2.237}$$

From the expression for δx we see that the errors $\delta\dot{x}^0$, $\delta\dot{z}^0$ introduce a term which increases linearly with time. For a low-eccentricity orbit, the expression for δz also contains time-proportional terms (multiplied by e).*

The stability situation for Eqs. (2.15a), (2.28) is incomparably better. For Eqs. (2.28) in cases when they are reduced to equations with constant coefficients, the stability condition is precisely inequality (2.235). For an object which is stationary in the $O_1\xi_*\eta_*\zeta_*$ axes, the solution of the homogeneous equations (2.28), according to (2.69), are harmonic oscillations of frequency ω_0,

$$\left.\begin{aligned}\delta x &= \delta x^0 \cos\omega_0 t + \frac{\delta\dot{x}^0}{\omega_0}\sin\omega_0 t,\\ \delta y &= \delta y^0 \cos\omega_0 t + \frac{\delta\dot{y}^0}{\omega_0}\sin\omega_0 t.\end{aligned}\right\} \tag{2.238}$$

For motion with constant velocity along the arc of a great circle of a sphere of constant radius concentric with the Earth, the expression for δy in (2.238) remains as above and the equation for the error δx in the plane of motion takes the form (see the first equation in (2.72)

$$\delta x = \delta x^0 \sqrt{\omega_0^2 - \omega_y^2}\, t + \frac{\delta\dot{x}^0}{\sqrt{\omega_0^2 - \omega_y^2}}\sin\sqrt{\omega_0^2 - \omega_y^2}\, t. \tag{2.239}$$

Here the harmonic frequency is $\nu = \sqrt{\omega_0^2 - \omega_y^2}$.

From Eq. (2.239) we see that for a satellite in a circular orbit, Eq. (2.28) leads to instability, since in this case

$$\delta x = \delta x^0 + \delta\dot{x}^0 t. \tag{2.240}$$

The motion of a satellite in an arbitrary elliptical orbit is also unstable. Indeed, in this case, by (2.115),

$$\delta x = \frac{r}{a}C_1^0 + \frac{rv}{a\nu\sqrt{1-e^2}}C_2^0; \tag{2.241}$$

yet the true anomaly v is a monotonically increasing function of time. On the average it is proportional to time. If the eccentricity e of the orbit is small, we have from (2.120)

$$\delta x = \delta x^0[1 + e(1 - \cos\nu t)] + \delta\dot{x}^0\left[t + e\left(\frac{2}{\nu}\sin\nu t - t - t\cos\nu t\right)\right]. \tag{2.242}$$

In connection with Eqs. (2.28) we have noted that the coefficients of these equations are in general variable even for motion at constant distance from the Earth's center, and their solutions may thus prove to be unstable. However, at velocities substantially lower than the circular velocity, the solutions of the homogeneous equations (2.28) are bounded by estimates of the form (2.146), (2.158). It follows from these estimates that the perturbed motion of an inertial system with two accelerometers remains close to harmonic oscillations of frequency ω_0.

Further improvement of the stability of an inertial system is provided by Eqs. (2.15a), (2.16). We recall that they apply to the case of an inertial system retaining all the three accelerometers; the external information on

* See §5.4 and §5.5 in "Theory of Inertial Navigation: Autonomous Systems." 1966.

the altitude h is used only to form the magnitude of the Earth's gravitational acceleration.

In this case, assuming arbitrary motion of the object at a constant distance from the Earth's center, the inertial system is stable. This follows from the representation (2.206) of the solution of Eq. (2.15a).

When the external altimeter information is used in this way, the inertial system is also stable for Keplerian motion. To establish this point, consider Eqs. (2.228), (2.98). When the instrumental errors are zero, we have from the preceding

$$\left.\begin{aligned} \delta x = \sum_{i=1}^{4} \alpha_{1i} C_i^0, \quad \delta z = \sum_{i=1}^{4} \alpha_{2i} C_i^0, \\ \delta y = \sum_{i=1}^{2} B_{1i} C_{i+1}^0. \end{aligned}\right\} \tag{2.243}$$

But all the matrix elements α_{ij} and B_{ij} of the particular solutions, as we see from (2.226), (1.167), are periodic functions of time.

In particular, for low-eccentricity orbits, the homogeneous equations (2.16) have the solutions (see Eqs. (2.105), (2.231))

$$\left.\begin{aligned} \delta x = {} & \delta x^0 [1 + e(1 - \cos \nu t)] + \frac{\delta \dot{x}^0}{2\nu} [\sin 2\nu t + \\ & + e(-\sin \nu t - \sin 2\nu t + 3 \sin \nu t \cos 2\nu t)] + \\ & + \frac{\delta \dot{z}^0}{2\nu} [\cos 2\nu t - 1 + e(1 - \cos 2\nu t - 3 \sin \nu t \sin 2\nu t)]. \\ \delta y = {} & \delta y^0 [\cos \nu t + e(\cos \nu t - 1 - \sin^2 \nu t)] + \\ & + \frac{\delta \dot{y}^0}{\nu} [\sin \nu t + e(-\sin \nu t + \sin \nu t \cos \nu t)]. \\ \delta z = {} & \delta z^0 [1 + e(1 - \cos \nu t)] + \frac{\delta \dot{x}^0}{2\nu} [1 - \cos 2\nu t + \\ & + e(-1 + \cos 2\nu t + 3 \sin \nu t \sin 2\nu t)] + \frac{\delta \dot{z}^0}{2\nu} [\sin 2\nu t + \\ & + e(-\sin 2\nu t - \sin \nu t + 3 \sin \nu t \cos 2\nu t)]. \end{aligned}\right\} \tag{2.244}$$

For arbitrary motion, when $r(t)$ is an arbitrary function of time, the solutions of Eqs. (2.16) may prove to be unstable. Here stability and instability are entirely determined by the form of the function $r(t)$.

The homogeneous system (2.16) is equivalent to the vector equation

$$\frac{d^2 \delta r}{dt^2} + \frac{\mu}{r^3} \delta r = 0. \tag{2.245}$$

μ/r^3 is always positive. Hence, at any finite distance from the Earth's center, the solution of Eq. (2.245) is an oscillatory function. The amplitude of these oscillations may increase only in case of parametric resonance.

If the object moves near some constant $r = r^0$, so that $r = r^0 + \rho(t)$, we have for the projections of the vector δr on the axes ξ_*, η_*, ζ_*, as in (2.203),

$$\left|\frac{\delta \xi_*}{\delta \xi_*^0}\right|, \quad \left|\frac{\delta \eta_*}{\delta \eta_*^0}\right|, \quad \left|\frac{\delta \zeta_*}{\delta \zeta_*^0}\right| < \exp \frac{3\omega_0 \max |\rho|}{r^0} t. \tag{2.246}$$

The upper-bound estimates (2.246) for the divergence of the amplitudes of solutions of Eq. (2.245) from the initial values are clearly too high. They are nevertheless fairly effective. Thus, if the object moves on the Earth's

surface, we have $\max|\rho| \approx 2 \cdot 10^4$ m, $r^0 \approx 6.4 \cdot 10^6$ m, $\omega_0 \approx 1.25 \cdot 10^{-3}$ sec^{-1}. The amplitude will thus increase by a factor $e \approx 2.78$ in a time of the order of $25 \cdot 10^4$ sec, i.e., about 70 hrs. For $\max|\rho| \approx 10^5$ m, this time drops to 15 hrs. If the inertial system functions for a few hours only, the right-hand sides of inequalities (2.246) for $|\rho| < 10^5$ m are virtually unity.

2.5.2. Navigation errors due to accelerometer and gyroscope errors

Let us now consider the relations between the instrumental errors and δx, δy, δz. We will only consider the case of constant instrumental errors, for which explicit expressions for δx, δy, δz have been derived in the previous sections. First we will discuss the instrumental errors of the sensitive elements of the inertial system — accelerometers and gyroscopes; the altimeter errors $\Delta r = \Delta h$ are treated at a later stage.

Let us compare Eqs. (1.141), (2.70), (2.208), which are the respective solutions of Eqs. (1.96), (2.28), (2.16) for a stationary object. The instrumental errors Δn_x, Δn_y occupy equivalent positions in the expressions for δx and δy. The only difference is in the expression for δz. For an autonomous inertial system, i.e., in Eqs. (1.141), the error δz associated with Δn_z increases exponentially. In a three-accelerometer system with altimeter correction, as we see from the third relation in (2.208), a constant error Δn_z only leads to harmonic oscillations about a displaced state of equilibrium. The effect of Δn_z is thus the same as the effect of Δn_x, Δn_y in the expressions for δx, δy, which are identical for all the three cases. The gyroscope errors Δm_x, Δm_y, Δm_z do not enter Eqs. (1.141), (2.70), (2.208). In accordance with Eq. (2.233), the gyroscope errors enter the expressions for the total position errors δx_2, δy_2, δz_2 through the projections θ_x, θ_y making identical contribution in all the three cases.

Let us now compare Eqs. (1.150), (2.73), and (2.209), which are the solutions of Eqs. (1.96), (2.28), and (2.16) for motion with constant velocity along the great circle of a sphere of constant radius concentric with the Earth, which does not rotate with the Earth. The expressions for δy are the same in all the three cases; they differ from the second expression in (1.141) for the case of a stationary base only in that they contain the error Δm_z. It thus remains to compare Eqs. (1.150), (2.73), (2.209) for δx and δz. Retaining in the expressions for δx only terms associated with instrumental errors, we obtain from (1.150), (2.73), and (2.209) respectively:

$$\left.\begin{aligned}
\delta x &= \frac{\Delta n_x}{\omega_0^2 - \omega_y^2}\left\{1 - \frac{1}{\mu^2+\nu^2}\left[(\omega_0^2 - \omega_y^2 + \mu^2)\cos\nu t + \right.\right. \\
&\quad \left.\left. + (\omega_0^2 - \omega_y^2 - \nu^2)\operatorname{ch}\mu t\right]\right\} + \\
&\quad + \frac{2\omega_y \mu\nu\,(\Delta n_z + 2r\omega_y\,\Delta m_y)}{(\omega_0^2 - \omega_y^2)(2\omega_0^2 + \omega_y^2)(\mu^2+\nu^2)}(\mu\sin\nu t - \nu\operatorname{sh}\mu t), \\
\delta x &= \frac{\Delta n_x}{\omega_0^2 - \omega_y^2}(1 - \cos\omega_0 t), \\
\delta x &= \frac{1}{\omega_0^2 - \omega_y^2}\left[\Delta n_x\left(1 - \cos\omega_0 t\cos\omega_y t - \frac{\omega_y}{\omega_0}\sin\omega_0 t\sin\omega_y t\right) + \right. \\
&\quad \left. + (\Delta n_z + 2r\omega_y\,\Delta m_y)\left(\cos\omega_0 t\sin\omega_y t - \frac{\omega_y}{\omega_0}\sin\omega_0 t\cos\omega_y t\right)\right].
\end{aligned}\right\} \qquad (2.247)$$

Similarly from (1.150), (2.209)

$$\left.\begin{aligned}
\delta z &= \frac{\Delta n_z + 2r\omega_y \Delta m_y}{2\omega_0^2 + \omega_y^2}\left\{-1 + \frac{1}{(\omega_0^2-\omega_y^2)(\mu^2+\nu^2)}\times\right.\\
&\left.\times\left[\mu^2(\omega_0^2-\omega_y^2-\nu^2)\cos\nu t + \nu^2(\omega_0^2-\omega_y^2+\mu^2)\operatorname{ch}\mu t\right]\right\} -\\
&\qquad - \frac{2\omega_y \Delta n_x}{\mu\nu(\mu^2+\nu^2)}(\mu\sin\nu t - \nu\operatorname{sh}\mu t),\\
\delta z &= \frac{1}{\omega_0^2-\omega_y^2}\left[(\Delta n_z + 2r\omega_y \Delta m_y)\times\right.\\
&\times\left(1 - \cos\omega_0 t\cos\omega_y t - \frac{\omega_y}{\omega_0}\sin\omega_0 t\sin\omega_y t\right) -\\
&\left.- \Delta n_x\left(\cos\omega_0 t\sin\omega_y t - \frac{\omega_y}{\omega_0}\sin\omega_0 t\cos\omega_y t\right)\right].
\end{aligned}\right\} \qquad (2.248)$$

From the first equalities in (2.247), (2.248) we see (as already noted above) that in an autonomous system the errors δx, δz associated with the instrumental errors Δn_x, Δn_z, Δm_y increase exponentially with time. Of main interest are the second and third inequalities in (2.247). For a two-accelerometer system, as we see from the second equality, δx only depends on Δn_x, and

$$|\delta x| < 2|\Delta n_x|\frac{1}{\omega_0^2-\omega_y^2}. \qquad (2.249)$$

For a three-accelerometer system with altimeter correction, the error δx depends on the errors Δn_z and Δm_y, as well as on Δn_x, and we have the estimate

$$|\delta x| < \frac{1}{\omega_0^2-\omega_y^2}(2|\Delta n_x| + |\Delta n_z + 2r\omega_y \Delta m_y|). \qquad (2.250)$$

From (2.249), (2.250) we see that the maximum partial error associated with the factor Δn_x is the same in the two alternative systems with altimeter correction. Note that the second relation in (2.248) also leads to an estimate similar to (2.250):

$$|\delta x| < \frac{1}{\omega_0^2-\omega_y^2}(|\Delta n_x| + 2|\Delta n_z + 2r\omega_y \Delta m_y|). \qquad (2.251)$$

In the case of slow motion, i.e., for velocities much less than the circular velocity, we have the following approximate equalities from (2.247):

$$\left.\begin{aligned}
\delta x &= \frac{\Delta n_x}{\omega_0^2}(1-\cos\omega_0 t) - \frac{\omega_y \Delta n_z \sqrt{2}}{3\omega_0^3}\operatorname{sh}\omega_0\sqrt{2}\,t,\\
\delta x &= \frac{\Delta n_x}{\omega_0^2}(1-\cos\omega_0 t),\\
\delta x &= \frac{1}{\omega_0^2}[\Delta n_x(1-\cos\omega_0 t\cos\omega_y t) + \Delta n_z\cos\omega_0 t\sin\omega_y t].
\end{aligned}\right\} \qquad (2.252)$$

Similarly from (2.248)

$$\left.\begin{aligned}
\delta z &= \frac{\Delta n_z}{2\omega_0^2}(\operatorname{ch}\omega_0\sqrt{2}\,t - 1),\\
\delta z &= \frac{1}{\omega_0^2}[\Delta n_z(1-\cos\omega_0 t\cos\omega_y t) - \Delta n_x\cos\omega_0 t\sin\omega_y t].
\end{aligned}\right\} \qquad (2.253)$$

Comparison of the first and second relations in (2.252) shows that in case of slow motion along the arc of a fixed-orientation great circle, a two-accelerometer system should be preferred to a three-accelerometer system with altimeter correction, since for equal accelerometer errors the maximum error δx in the first system may be smaller. This conclusion also emerges from a comparison of errors in slow motion along a parallel.

At velocities close to the circular velocity, and in particular for satellites, the situation is somewhat different. Let us compare the solutions of Eqs. (1.96), (2.28), and (2.16) for Keplerian motion.

For simplicity, consider motion in a circular orbit. For an autonomous inertial system, we have from (1.153)

$$\left.\begin{aligned}
\delta x &= \frac{\Delta n_x}{\omega_0^2}\left[-\frac{3(\omega_0 t)^2}{2}+4(1-\cos\omega_0 t)\right]+ \\
&\quad +\left(\frac{2\Delta n_z}{\omega_0^2}+\frac{4a\,\Delta m_y}{\omega_0}\right)(\sin\omega_0 t-\omega_0 t), \\
\delta y &= \frac{\Delta n_y - a\omega_0\,\Delta m_z}{\omega_0^2}(1-\cos\omega_0 t), \\
\delta z &= \frac{2\Delta n_x}{\omega_0^2}(\omega_0 t-\sin\omega_0 t)+\frac{\Delta n_z}{\omega_0^2}(1-\cos\omega_0 t).
\end{aligned}\right\} \qquad (2.254)$$

For a two-accelerometer system, we have from (2.76) or from (2.105), (2.120) with zero initial conditions

$$\delta x = \frac{\Delta n_x t^2}{2}, \quad \delta y = \frac{\Delta n_y - a\omega_0\,\Delta m_z}{\omega_0^2}(1-\cos\omega_0 t). \qquad (2.255)$$

Finally, for a three-accelerometer system with altimeter correction, we have from (2.211) or from (2.105), (2.231)

$$\left.\begin{aligned}
\delta x &= \frac{\Delta n_x}{4\omega_0^2}(1-\cos 2\omega_0 t)+ \\
&\quad +\left(\frac{\Delta n_z}{4\omega_0^2}+\frac{a\,\Delta m_y}{2\omega_0}\right)(\sin 2\omega_0 t-2\omega_0 t), \\
\delta y &= \frac{\Delta n_y - a\omega_0\,\Delta m_z}{\omega_0^2}(1-\cos\omega_0 t), \\
\delta z &= \frac{\Delta n_x}{4\omega_0^2}(2\omega_0 t-\sin 2\omega_0 t)+ \\
&\quad +\left(\frac{\Delta n_z}{4\omega_0^2}+\frac{a\,\Delta m_y}{2\omega_0}\right)(1-\cos 2\omega_0 t).
\end{aligned}\right\} \qquad (2.256)$$

Comparison of (2.254), (2.255), and (2.256) shows that the three-accelerometer system with altimeter correction gives the least errors for a satellite in a circular orbit. In this case the errors δx and δz are only proportional to time, whereas in the other two alternatives the expression for δx is a parabolic function of time. This result is also applicable to elliptical orbits and to near-Keplerian motion.

2.5.3. Altimeter errors

We have so far discussed the dependence of δx, δy, δz on the errors of the sensitive elements of the inertial system, i.e., Δn_x, Δn_y, Δn_z, Δm_x, Δm_y, Δm_z.

However, if the inertial system uses external information on the distance h from the Earth's surface, additional errors are introduced due to the inaccuracy $\Delta h (= \Delta r)$ in this guidance information. Let us compare the errors δx, δy, δz associated with Δr in the two alternative systems with altimeter correction. Without this analysis, our understanding of the performance of the two systems is fundamentally incomplete.

As in the case of instrumental errors, we consider only constant Δr. If necessary, we may naturally consider the case of variable Δr, with known function $\Delta r(t)$, since the corresponding solutions of Eqs. (1.96), (2.28) and (2.16) have been derived in the previous sections for arbitrary Δr. This remark is equally applicable to the instrumental errors of the sensitive elements.

A constant error Δr in a two-accelerometer system does not lead to additional errors δx, δy for a stationary object. This is evident from (2.70). The same result is obtained for motion with constant velocity along the arc of a fixed-orientation great circle, as we see from (2.73). Incidentally, this can be established without using the solutions (2.70), (2.73). It suffices to show that for a stationary object and for motion along the arc of a great circle, Δr drops out from the right-hand sides of Eqs. (2.28).

The case of motion along a parallel in a two-accelerometer system leads to additional errors, as we see from (2.94):

$$\left.\begin{aligned} \delta x &= \frac{\Delta r \omega_y \omega_z}{\omega_0^2} \cos \omega_y t \sin \omega_z t, \\ \delta y &= -\frac{\Delta r \omega_y \omega_z}{\omega_0^2} (1 - \cos \omega_0 t \cos \omega_z t). \end{aligned}\right\} \tag{2.257}$$

For a satellite in a circular orbit, two-accelerometer systems do not acquire additional errors; this we see from (2.75). In case of elliptical motion an additional error δx arises. Thus, for a low-eccentricity orbit we have from (2.120)

$$\delta x = -2e \Delta r (\sin \nu t - \nu t). \tag{2.258}$$

In a three-accelerometer system, we have for a stationary object

$$\delta z = 3 \Delta r (1 - \cos \omega_0 t). \tag{2.259}$$

This follows from the third relation in (2.208).

For motion along the arc of a great circle, we have from (2.209)

$$\left.\begin{aligned} \delta x &= \frac{3 \Delta r \omega_0^2}{\omega_0^2 - \omega_y^2} \left(\cos \omega_0 t \sin \omega_y t - \frac{\omega_y}{\omega_0} \sin \omega_0 t \cos \omega_y t \right), \\ \delta y &= 0, \\ \delta z &= \frac{3 \Delta r \omega_0^2}{\omega_0^2 - \omega_y^2} \left(1 - \cos \omega_0 t \cos \omega_y t - \frac{\omega_y}{\omega_0} \sin \omega_0 t \sin \omega_y t \right). \end{aligned}\right\} \tag{2.260}$$

For motion along a parallel

$$\delta x = -\frac{3 \Delta r \omega_0^2}{\omega_0^2 - \omega^2} \left(-\cos \omega_0 t \sin \omega t + \frac{\omega}{\omega_0} \sin \omega_0 t \cos \omega t \right), \tag{2.261}$$

$$\left.\begin{aligned}\delta y &= -\frac{3\Delta r\omega_0^2 \sin\varphi\cos\varphi}{\omega_0^2-\omega^2}\left(1-\cos\omega_0 t\cos\omega t-\frac{\omega}{\omega_0}\sin\omega_0 t\sin\omega t\right),\\ \delta z &= \frac{3\Delta r\omega_0^2\cos^2\varphi}{\omega_0^2-\omega^2}\left(1-\cos\omega_0 t\cos\omega t-\frac{\omega}{\omega_0}\sin\omega_0 t\sin\omega t\right).\end{aligned}\right\} \tag{2.261}$$

Finally, for a satellite in circular orbit, we have from (2.211)

$$\left.\begin{aligned}\delta x &= \frac{3\Delta r}{4}(\sin 2\omega_0 t - 2\omega_0 t),\\ \delta y &= 0,\\ \delta z &= \frac{3\Delta r}{4}(1-\cos 2\omega_0 t).\end{aligned}\right\} \tag{2.262}$$

We should make one remark concerning the errors δz in a three-accelerometer system, i.e., equality (2.259), the second equality in (2.260), the third equality in (2.261), and the second equality in (2.262).

The altitude h is received from an external source of information. Given the h and the position of the object relative to the Earth, the system computes r. The error in this computation is $\Delta r\,(=\Delta h)$. Naturally the error δz in the determination of r with the accelerometer n_z, which is in a way equivalent to Δr, is larger than the latter. Thus, according to (2.259) it may reach $\delta z = 6\Delta r$. This also follows from the last relations in (2.260), (2.261). According to the second equality in (2.262), the maximum h may reach $3\Delta r/2$. Therefore, the distance from the Earth's center should be inferred from the altitude h received by the inertial system from external sources with error Δr, and not from the results of integration of the accelerometer readings n_z, which may give an error six times as large. It follows from the preceding that δx and δy are the only significant errors among (2.260)—(2.262).

Chapter 3

SCHULER-TUNED GYROPENDULUM SYSTEMS. ANALOGY WITH TWO-ACCELEROMETER INERTIAL SYSTEMS

§3.1. SCHULER PENDULUM

3.1.1. The conditions of a relative equilibrium with the pendulum axis caged to the geocentric vertical

Consider a rigid body with an axis of dynamic symmetry which is pivoted on one of the points of its axis, other than the center of mass. We are thus dealing with a symmetric physical pendulum.

Suppose the pivot of the pendulum is moving in the $O_1\xi\eta\zeta$ axes. The orientation of the pendulum symmetry axis at any time in this system is determined by the pendulum parameters, the initial conditions of pendulum motion relative to the pivot, the law of motion of the pivot, and the Earth's gravitational field.

In what follows we will be mainly concerned with appropriate choice of the pendulum parameters such that its axis remains caged to the direction to the Earth's center, i.e., to the geocentric vertical, irrespective of the motion of the pivot.

This problem was first considered by Max Schuler.* He treated the motion of a pendulum at a constant distance from the Earth's center O_1 in a plane of fixed orientation through O_1 assuming a central gravitational field. In his analysis, a special choice of parameters was found to ensure invariable attitude of the pendulum axis of symmetry along the geocentric vertical, provided that the two directions coincided initially. Generalizations of this theorem of analytical mechanics were the subject of numerous further studies, including the work of B.V. Bulgakov and A.Yu. Ishlinskii. Ishlinskii** rigorously determined the conditions to be satisfied by the pendulum parameters and the initial conditions so that the axis of dynamic symmetry invariably point along the geocentric vertical in any motion of the pivot at constant distance from the Earth's center.

Schuler's theorem provided the foundation for inertial navigation. Inertial navigation systems were first developed as mechanical devices simulating the Schuler pendulum. Classical examples are the Geckeler–Anschütz gyrocompass horizon and the two-gyro vertical.

In the final analysis, a mechanical device which fixes the direction to the Earth's center is equivalent to the two-accelerometer inertial system considered in the previous sections. Indeed, given this direction plus

* Schuler, M. Die Störung von Pendel und Kreiselapparaten durch die Beschlunigung des Fahrzeuges. — Physikalische Zeitschrift, Vol. 24, No. 16, Leipzig, 1923.

** Ishlinskii, A.Yu. Relative equilibrium of a physical pendulum with a moving pivot. — Prikladnaya Matematika i Mekhanika, Vol. 20, No. 3, 1956.

information on the distance r from the Earth's center, we can effectively solve the problem of navigation. The system only requires additional gyroscopes in order to determine the orientation of the geocentric vertical in the $O_1\xi_*\eta_*\zeta_*$ axes. For navigation in a coordinate system fixed to the Earth, the system also needs a clock so as to account for the variation in the position of the $O_1\xi_*\eta_*\zeta_*$ axes relative to the Earth.

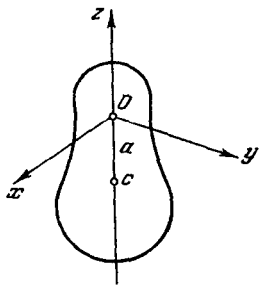

FIGURE 3.1.

The analogy between these systems, however, is much more fundamental than a mere similarity in the final results. As we will see in the following, the equations of oscillation of the Schuler pendulum and its analogs about the position of relative equilibrium precisely coincide with the first group of differential error equations of a two-accelerometer inertial system. This analogy is of particular significance since the properties of the Schuler pendulum are readily amenable to exact analysis.

Two problems have to be solved in connection with the properties of the Schuler pendulum: we should find the conditions under which the dynamic symmetry axis of the pendulum is caged to the geocentric vertical, and further derive and analyze the equations of motion of the dynamic symmetry axis about this position of relative equilibrium.

Let us consider the first of the two problems above. With the physical pendulum we associate the axes $Oxyz$ (Figure 3.1), the origin coinciding with the point of suspension and the axis z directed along the pendulum symmetry axis, away from the center of mass c. The coordinates of the center of mass are thus

$$x_c = y_c = 0, \quad z_c = -a. \tag{3.1}$$

The $Oxyz$ axes are directed along the principal axes of the ellipsoid of inertia of the pendulum. This is an ellipsoid of revolution, as the pendulum has a dynamic symmetry axis. The moments of inertia of the pendulum are thus

$$J_{xy} = J_{yz} = J_{zx} = 0, \quad J_{xx} = J_{yy} = A, \quad J_{zz} = C. \tag{3.2}$$

The equations of motion of the pendulum are best written in the $O\xi_*\eta_*\zeta_*$ axes with the origin at the point O and the axes parallel to the $O_1\xi_*\eta_*\zeta_*$ axes, maintaining fixed orientation relative to the distant stars. To write the equations of motion, we use the theorem of the angular momentum. Since the $O\xi_*\eta_*\zeta_*$ axes are in uniform rectilinear motion relative to the inertial space, the theorem of the angular momentum is written in the usual form:

$$\frac{d\boldsymbol{K}}{dt} = \boldsymbol{M}, \tag{3.3}$$

where $\boldsymbol{K}$ is the angular momentum, and $\boldsymbol{M}$ is the resultant external torque on the pendulum.

Using Eqs. (3.2), we get

$$K_x = A\omega_x, \quad K_y = A\omega_y, \quad K_z = C\omega_z. \tag{3.4}$$

where ω_x, ω_y, ω_z are the projections of the absolute angular velocity of the pendulum on the $Oxyz$ axes.

Projecting Eq. (3.3) on the x, y, z axes, we get Euler's equations

$$\left.\begin{aligned} A\frac{d\omega_x}{dt}+(C-A)\omega_y\omega_z &= M_x,\\ A\frac{d\omega_y}{dt}-(C-A)\omega_x\omega_z &= M_y,\\ C\frac{d\omega_z}{dt} &= M_z. \end{aligned}\right\} \tag{3.5}$$

In Eqs. (3.5), M_x, M_y, M_z are the x, y, z projections of the resultant torque about the point O of the forces applied to the pendulum. Since the axes $O\xi\eta\zeta$, in which Eqs. (3.3), (3.5) are written are moving axes, translatory inertial forces and Coriolis forces must be taken into consideration together with gravitation and reaction forces in calculating the torques. The torque of the reaction force is zero, since the reaction is along a line through the point O (a frictionless pivot is assumed). The Coriolis forces are also zero, since the $O\xi\eta\zeta$ system moves uniformly and rectilinearly. For this reason the inertial forces of each mass element of the pendulum are all parallel, and can be reduced to a resultant force $\boldsymbol{Q}$ applied to the center of mass c. The Earth's gravitation is assumed to be homogeneous within the volume of the pendulum. Writing m for the mass of the pendulum, we thus have for the resultant gravitational force*

$$\boldsymbol{F}=m\boldsymbol{g}; \tag{3.6}$$

this force is applied to the center of mass and acts in the direction of the Earth's gravitation at the point O. The magnitude $|\boldsymbol{g}|$ is also evaluated in this case at the point O.

Using the above remarks and Eqs. (3.1), we obtain for the torques

$$M_x=a(F_y+Q_y),\quad M_y=-a(F_x+Q_x),\quad M_z=0. \tag{3.7}$$

Now let the dynamic symmetry axis point fixedly to the Earth's center. The point of suspension O moves at constant distance r from the Earth's center, and the Earth's gravitational field is spherical. Then

$$F_x=F_y=0. \tag{3.8}$$

Further,

$$\left.\begin{aligned} Q_x=-mw_x,\quad Q_y=-mw_y,\\ Q_z=-mw_z, \end{aligned}\right\} \tag{3.9}$$

where w_x, w_y, w_z are the projections of the absolute acceleration of the point O,

$$\left.\begin{aligned} w_x&=\frac{dv_x}{dt}+\omega_y v_z-\omega_z v_y,\\ w_y&=\frac{dv_y}{dt}+\omega_z v_x-\omega_x v_z,\\ w_z&=\frac{dv_z}{dt}+\omega_x v_y-\omega_y v_x. \end{aligned}\right\} \tag{3.10}$$

Since the z axis points to the Earth's center and $r=\text{const}$, we have

$$v_x=r\omega_y,\quad v_y=-r\omega_x,\quad v_z=0. \tag{3.11}$$

* The effect of the resultant gravitational torque will be discussed later.

Inserting (3.8)—(3.11) in Eqs. (3.7) and then in Eqs. (3.5), we obtain the following equations of relative equilibrium of the physical pendulum in this case:

$$\left.\begin{aligned}(A-mar)\left(\frac{d\omega_x}{dt}-\omega_y\omega_z\right)+C\omega_y\omega_z=0,\\ (A-mar)\left(\frac{d\omega_y}{dt}+\omega_x\omega_z\right)-C\omega_x\omega_z=0,\\ C\frac{d\omega_z}{dt}=0.\end{aligned}\right\} \tag{3.12}$$

Eqs. (3.12) should be identically satisfied. For arbitrary ω_x, ω_y, ω_z this can be achieved only if

$$\frac{A}{ma}=r, \quad C=0. \tag{3.13}$$

The first equality in (3.13) is Schuler's famous condition: the reduced length of the physical pendulum should be equal to the Earth's radius. The second equality further requires that the entire mass of the pendulum be concentrated on the z axis. This requirement can be replaced by an equivalent condition. For $C \neq 0$, we have from the third equality in (3.12)

$$\omega_z=\omega_z^0=\text{const}. \tag{3.14}$$

If we now take $\omega_z^0 = 0$, the first two relations in (3.12) are satisfied for any C if the first condition in (3.13) is met. In this case the projection of the absolute angular velocity of the pendulum on the dynamic symmetry axis is zero,

$$\omega_z=\omega_z^0=0. \tag{3.15}$$

Condition (3.15) and the second condition in (3.13) can be combined into a single equality:

$$K_z=C\omega_z=0. \tag{3.16}$$

Note that (3.15) refers to the choice of the initial conditions. The other initial conditions are the following: initially the dynamic symmetry axis of the pendulum should point to the Earth's center and the initial values ω_x^0, ω_y^0 should be such that initially the rate of turning of the pendulum z axis in space is equal to the rate of turning of the radius-vector r from the Earth's center to the pivot.

Conditions (3.13), (3.16) and the particular choice of the initial conditions ensure a state of relative equilibrium in which the dynamic symmetry axis of the pendulum fixedly points to the Earth's center, for any motion of the pivot at a constant distance from the Earth's center.

As we have noted above, condition (3.16) is satisfied either if the pendulum is an infinitely thin rod or if $\omega_z = 0$. In the former case, the pendulum is a mathematical idealization and cannot be built in practice. The second requirement, however, also leads to an unfeasible Schuler pendulum. Indeed, consider a pendulum in the form of a ring 0.5 m in diameter, with its pivot distant a from the center of mass. Now, if the first condition in (3.13) is to be satisfied, a should be of the order of $0.01\,\mu$, and the pendulum is clearly unfeasible.

Our interest in the Schuler pendulum is mainly as an analog of a certain class of inertial navigation systems. In the following we will therefore assume that conditions (3.13), (3.16) are satisfied, without considering the practical or the physical (as in the case $C = 0$) feasibility of the corresponding pendulum.

The previous treatment was carried out assuming a spherical gravitational field. If we remove this restriction and also permit a variable r, the pendulum sees additional torques which can be appropriately compensated. Let

$$r = r(t), \quad v_z = \frac{dr}{dt}, \quad F_x \neq 0, \; F_y \neq 0. \tag{3.17}$$

Assuming condition (3.15) to be satisfied, we obtain instead of the first two equalities in (3.12)

$$\left.\begin{aligned} (A - mar)\frac{d\omega_x}{dt} &= a\left(F_y + 2m\omega_x \frac{dr}{dt}\right) + M_x^1, \\ (A - mar)\frac{d\omega_y}{dt} &= a\left(-F_x + 2m\omega_y \frac{dr}{dt}\right) + M_y^1. \end{aligned}\right\} \tag{3.18}$$

Eqs. (3.18) should be identically satisfied, irrespective of ω_x, ω_y. This leads to the condition

$$A - mar = 0, \tag{3.19}$$

and the compensating torques M_x^1, M_y^1 should be formed in such a way that the right-hand sides of (3.18) vanish.

In (3.19) r is variable, whereas conditions (3.18) were derived for constant r. The resulting difficulties can be bypassed in two ways. One possibility is to satisfy condition (3.19) for some $r^0 = \text{const}$, and introduce suitable additional torques so that the total compensating torques take the form

$$\left.\begin{aligned} M_x^1 &= -a\left(F_y + 2m\omega_x \frac{dr}{dt}\right) + ma(r - r^0)\frac{d\omega_x}{dt}, \\ M_y^1 &= a\left(-F_x + 2m\omega_y \frac{dr}{dt}\right) + ma(r - r^0)\frac{d\omega_y}{dt}. \end{aligned}\right\} \tag{3.20}$$

Another possibility is to replace the pendulum by a mechanical system with a constant moment of inertia A and a center of mass c (at the point $(z_c = -a)$ moving in such a way that

$$a = \frac{A}{mr(t)}, \tag{3.21}$$

or, conversely, a system with constant a and variable A in accordance with Eq. (3.19):

$$A = mar(t). \tag{3.22}$$

An appropriate mechanical system can be visualized as a pendulum with hollows in which suitable masses move in accordance with conditions (3.21), (3.22),* without displacing the dynamic symmetry axis from the axis z.

* As the relative velocities of these masses are small, the Coriolis forces are of course negligible.

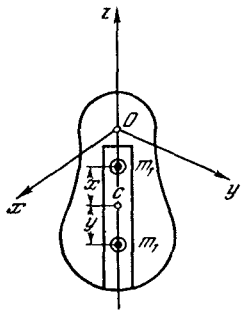

FIGURE 3.2.

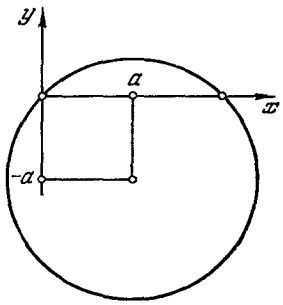

FIGURE 3.3.

Consider the following example in which Eq. (3.21) is satisfied. Take a pendulum with a cylindrical cavity along the z axis (Figure 3.2), with two equal masses m_1 moving on the two sides of the center of mass c. Let the distance of one of these masses (e.g., that lying between the pivot O and the center of mass c) from the center of mass be x, and the distance of the other y. We then obtain the following expressions for the total moment of inertia and the coordinate z_c of the center of mass:

$$\left.\begin{aligned} A &= A_0 + m_1[(a-x)^2 + (a+y)^2], \\ z_c &= \frac{a(m+2m_1) + m_1(y-x)}{2m_1+m}. \end{aligned}\right\} \tag{3.23}$$

Taking $A = A_0 = \text{const}$, we obtain from the first relation in (3.23) the equation of a circle shown in Figure 3.3. The radius of this circle is $a\sqrt{2}$, its center lies at the point $x = a$, $y = -a$. The second equation in (3.23) is the equation of a straight line

$$z_c - a = \frac{m_1}{2m_1+m}(y-x). \tag{3.24}$$

On the arc of the circle in the first quadrant, $z_c - a$ takes on the values

$$0 \geqslant z_c - a > \frac{2am_1}{2m_1+m}. \tag{3.25}$$

Thus, leaving A constant, we can satisfy condition (3.21). If condition (3.21) is satisfied, the compensating torques (3.20) take the simple form

$$M_x^1 = -a\left(F_y + 2m\omega_x\frac{dr}{dt}\right), \quad M_y^1 = a\left(-F_x + 2m\omega_y\frac{dr}{dt}\right). \tag{3.26}$$

The feasibility of condition (3.22) is also obvious. To satisfy this condition, we need only place four masses in a plane normal to the dynamic symmetry axis; these masses should lie on two mutually perpendicular lines meeting the dynamic symmetry axis, at equal distances from the axis.

In this case a is constant, and the equatorial moment of inertial A can be made variable, so as to ensure condition (3.22)

For variable A, we obtain from the vector equation (3.3)

$$\left.\begin{aligned} A\frac{d\omega_x}{dt}+\omega_x\frac{dA}{dt}+(C-A)\omega_y\omega_z=M_x,\\ A\frac{d\omega_y}{dt}+\omega_y\frac{dA}{dt}-(C-A)\omega_x\omega_z=M_y,\\ C\frac{d\omega_z}{dt}=M_z. \end{aligned}\right| \tag{3.27}$$

When condition (3.22) is satisfied,

$$\frac{dA}{dt}=ma\frac{dr}{dt}. \tag{3.28}$$

Therefore, if (3.15) or (3.16) is also satisfied, the compensating torques are given by the equalities

$$M_x'=-a\left(F_y+m\omega_x\frac{dr}{dt}\right),\quad M_y'=a\left(-F_x+m\omega_y\frac{dr}{dt}\right). \tag{3.29}$$

In (3.21), (3.22), only one of the two parameters a and A was assumed variable. If they are both variable, the compensating torques can be further simplified: the second term in parentheses can be eliminated. Indeed, let a be proportional to r. Then

$$A=mar,\quad a=kr. \tag{3.30}$$

Hence

$$\frac{dA}{dt}=2ma\frac{dr}{dt}. \tag{3.31}$$

The compensating torques now depend only on the projections F_x, F_y of the gravitational force:

$$M_x'=-aF_y,\quad M_y'=aF_x. \tag{3.32}$$

The problem of forming the compensating torques is equivalent to the problem of forming the nonspherical component of the Earth's gravitational field in the equations of inertial navigation. The forces F_x, F_y are functions of the coordinates of the pivot O relative to the Earth and the orientation of the axes xyz relative to the Earth. We can form these torques only if the pendulum together with the gyroscopes constitute a navigation system which actually measures the relevant coordinates and orientation.

Another point should be considered in this context. Can we choose the parameters of the pendulum (or of some mechanical system satisfying conditions (3.21), (3.22), (3.30)) so that the dynamic symmetry axis is caged to the gravitational force (or the force of gravity), and not to the geocentric vertical, for any motion of the pivot O? If this were so, the compensating torques (3.32) would become superfluous. This clearly cannot be achieved by varying A and a, since they are already constrained by the two conditions (3.30). In some cases, however, an approximate solution nevertheless can be obtained. If instead of eliminating the compensating torques we learn to form them as specified functions of coordinates and time, any orientation of the dynamic symmetry axis can be made a position of relative equilibrium.

In conclusion note that in order to satisfy conditions (3.19), (3.21), (3.22), (3.30) for variable r, we require external information on the distance r. The pendulum system itself cannot measure this distance internally. Clearly, r can be determined if the altitude h of the pendulum pivot above the Earth's surface is known (from altimeter readings). Since the direction of the vertical is also known, the Schuler pendulum (combined with gyroscopes) can be used to measure the coordinate of the moving pivot relative to the Earth's surface. Using Eq. (2.3) for known h, we can find r. The situation thus essentially reduces to that in an inertial system with altimeter correction.

3.1.2. Equations of small oscillations of the Schuler pendulum about the relative equilibrium

Let us now consider perturbed motion of the Schuler pendulum, i.e., oscillations about a position of relative equilibrium. We will consider the most general case, when conditions (3.30) are satisfied.

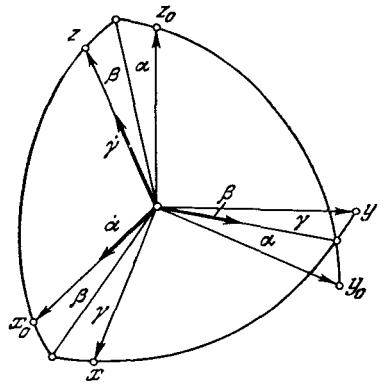

FIGURE 3.4.

We will use $x_0y_0z_0$ to designate the xyz axes fixed to the unperturbed pendulum. The perturbed position of the pendulum, or equivalently of its axes xyz, is described by the angles α, β, γ (Figure 3.4). The direction cosines between the axes x, y, z and x_0, y_0, z_0 thus form the following matrix:

$$\begin{array}{c|ccc} & x & y & z \\ x_0 & \cos\beta\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\cos\alpha\sin\gamma & \sin\beta\cos\gamma + \sin\alpha\cos\beta\sin\gamma \\ y_0 & \cos\beta\sin\gamma + \sin\alpha\sin\beta\cos\gamma & \cos\alpha\cos\gamma & \sin\beta\sin\gamma - \sin\alpha\cos\beta\cos\gamma \\ z_0 & -\cos\alpha\sin\beta & \sin\alpha & \cos\alpha\cos\beta. \end{array} \tag{3.33}$$

Let us first consider the case of small angles α and β, when matrix (3.33) takes the simpler form

$$\begin{array}{c|ccc} & x & y & z \\ x_0 & 1 & -\gamma & \beta \\ y_0 & \gamma & 1 & -\alpha \\ z_0 & -\beta & \alpha & 1. \end{array} \tag{3.34}$$

To describe the perturbed motion of the pendulum, we require only equations for the angles α, β, and γ, which can be obtained from Eqs. (3.27).

Let the projections of the absolute angular velocity of the unperturbed pendulum on the $x_0y_0z_0$ axes be ω_{x_0}, ω_{y_0}, ω_{z_0}. The left-hand sides of Eqs. (3.27) contain the projections ω_x, ω_y, ω_z of the absolute angular velocity of the perturbed pendulum on the axes x, y, z. In accordance with (3.34), we have

$$\left.\begin{aligned} \omega_x &= \omega_{x_0} + \omega_{z_0}\gamma - \omega_{z_0}\beta + \dot{\alpha}, \\ \omega_y &= \omega_{y_0} - \omega_{x_0}\gamma + \omega_{z_0}\alpha + \dot{\beta}, \\ \omega_z &= \omega_{z_0} - \omega_{y_0}\alpha + \omega_{x_0}\beta + \dot{\gamma}. \end{aligned}\right\} \tag{3.35}$$

To find the torques M_x, M_y, M_z entering the right-hand sides of Eqs. (3.27), we need the projections w_x, w_y, w_z of the acceleration of the point O on the axes x, y, z:

$$\left.\begin{aligned} w_x &= w_{x_0} - \omega_{z_0}\beta + \omega_{y_0}\gamma, \\ w_y &= w_{y_0} + \omega_{z_0}\alpha - w_{x_0}\gamma, \\ w_z &= w_{z_0}\beta - w_{y_0}\alpha + w_{z_0}. \end{aligned}\right\} \tag{3.36}$$

Here, in accordance with Eqs. (3.10), (3.11) and the second equality in (3.17),

$$\left.\begin{aligned} w_{x_0} &= r(\omega_{y_0} + \omega_{x_0}\omega_{z_0}) + 2\omega_{y_0}\dot{r} \\ w_{y_0} &= r(-\dot{\omega}_{x_0} + \omega_{y_0}\omega_{z_0}) - 2\omega_{x_0}\dot{r}, \\ w_{z_0} &= \ddot{r} - r(\omega_{x_0}^2 + \omega_{y_0}^2) \end{aligned}\right\} \tag{3.37}$$

The torques M_x, M_y, M_z of the perturbed pendulum include the torques of the translatory inertial forces on the pivot O, the gravity torques, and the additional compensating torques. Therefore, using (3.7), (3.32), we write

$$\left.\begin{aligned} M_x &= a(F_y + Q_y) - aF_{y_0}, \\ M_y &= -a(F_x + Q_x) + aF_{x_0}, \\ M_z &= 0. \end{aligned}\right\} \tag{3.38}$$

Here the inertial forces Q_x, Q_y, Q_z are given by Eqs. (3.9) with w_x, w_y, w_z substituted from (3.36).

For the projections F_x and F_y, using the table of direction cosines (3.34), we obtain expressions similar to the first two in (3.36):

$$\left.\begin{aligned} F_x &= F_{x_0} - F_{z_0}\beta + F_{y_0}\gamma, \\ F_y &= F_{y_0} + F_{z_0}\alpha - F_{x_0}\gamma. \end{aligned}\right\} \tag{3.39}$$

Combining (3.38), (3.39), we get

$$\left.\begin{aligned} M_x &= a(F_{z_0}\alpha - F_{x_0}\gamma + Q_y), \\ M_y &= -a(-F_{z_0}\beta + F_{y_0}\gamma + Q_x). \end{aligned}\right\} \tag{3.40}$$

We now insert in Eqs. (3.27) the angular velocities ω_x, ω_y, ω_z from (3.35) and the torques M_x, M_y, M_z from (3.40), (3.9), (3.36), (3.37). After obvious manipulations, dropping terms of second order in α, β, γ and their

derivatives, we obtain

$$\left.\begin{aligned}
&(A-mar)(\dot{\omega}_{x_0}-\omega_{y_0}\omega_{z_0})-C\omega_{y_0}\omega_{z_0}+\\
&+A(\omega_{y_0}\dot{\gamma}-\dot{\omega}_{y_0}\gamma-\omega_{z_0}\dot{\beta}-\dot{\omega}_{z_0}\beta+\ddot{\alpha})+\\
&+\frac{dA}{dt}(\omega_{x_0}+\omega_{y_0}\gamma-\omega_{z_0}\beta+\dot{\alpha})+\\
&+(C-A)(-\omega_{y_0}^2\alpha-\omega_{x_0}\omega_{y_0}\beta+\omega_{y_0}\dot{\gamma}-\\
&-\omega_{x_0}\omega_{z_0}\gamma+\omega_{z_0}^2\alpha+\omega_{z_0}\dot{\beta})=a(F_{z_0}\alpha-F_{x_0}\gamma)+\\
&+2am\omega_{x_0}\dot{r}-am[\ddot{r}-r(\omega_{x_0}^2+\omega_{y_0}^2)]\alpha+\\
&+am[r(\dot{\omega}_{y_0}+\omega_{x_0}\omega_{z_0})+2\omega_{y_0}\dot{r}]\gamma,\\
&(A-mar)(\dot{\omega}_{y_0}+\omega_{x_0}\omega_{z_0})-C\omega_{x_0}\omega_{z_0}+A(-\omega_{x_0}\dot{\gamma}-\dot{\omega}_{x_0}\gamma+\\
&-\omega_{z_0}\alpha+\omega_{z_0}\dot{\alpha}+\ddot{\beta})+\frac{dA}{dt}(\omega_{y_0}-\omega_{x_0}\gamma+\omega_{z_0}\alpha+\dot{\beta})-\\
&-(C-A)(-\omega_{x_0}\omega_{y_0}\alpha+\omega_{x_0}^2\beta+\omega_{x_0}\dot{\gamma}+\omega_{y_0}\omega_{z_0}\gamma-\\
&-\omega_{z_0}^2\beta+\omega_{z_0}\dot{\alpha})-a(F_{z_0}\beta-F_{y_0}\gamma)+2am\omega_{y_0}\dot{r}-\\
&-am[\ddot{r}-r(\omega_{x_0}^2-\omega_{y}^2)]\beta-\\
&+am[r(-\omega_{x_0}+\omega_{y_0}\omega_{z_0})-2\omega_x\dot{r}]\gamma,\\
&C\frac{d}{dt}(\omega_{z_0}-\omega_{y_0}\alpha+\omega_{x_0}\beta+\dot{\gamma})=0.
\end{aligned}\right\} \tag{3.41}$$

Integration of the last equation gives

$$\omega_{z_0}-\omega_{y_0}\alpha+\omega_{x_0}\beta+\dot{\gamma}=\omega_{z_0}^0-\omega_{y_0}^0\alpha^0+\omega_{x_0}^0\beta^0+\dot{\gamma}^0=\text{const}. \tag{3.42}$$

Here the superscript 0 designates, as before, the initial values of the corresponding variables.

Let the pendulum parameters satisfy conditions (3.15), (3.30). However, there is a certain inaccuracy in these equalities, so that

$$\left.\begin{aligned}
&\omega_{z_0}^0=\Delta\omega_{z_0}^0, \quad A=ma(r+\Delta r)+\Delta A,\\
&a=k(r+\Delta r)+\Delta a,
\end{aligned}\right\} \tag{3.43}$$

where Δr is the error in the distance r from the Earth's center, i.e., the error of the external source of information on r used in forming conditions (3.30).

In Eqs. (3.43) Δa and ΔA are the instrumental errors for (3.30). We will not consider other errors, e.g., those associated with nonzero products of inertia and imperfect dynamic symmetry of the pendulum. After all, the Schuler pendulum is only considered as an analog of actual inertial navigation systems, whereas effects associated with products of inertia and dynamic asymmetry are characteristic of the Schuler pendulum as such. Note, however, that any other errors in pendulum parameters can always be reduced to some equivalent errors Δa and ΔA. The latter are apparently equivalent to the fundamental instrumental errors, which were previously introduced in the analysis of error equations of inertial system.

Starting with Eqs. (3.43), we write

$$\left.\begin{aligned}
&\frac{dA}{dt}=2am\dot{r}+2ma\,\Delta\dot{r}+2mk\dot{r}\,\Delta r+m\dot{r}\,\Delta a+mr\,\Delta\dot{a}+\Delta\dot{A},\\
&A-amr=am\,\Delta r+\Delta A.
\end{aligned}\right\} \tag{3.44}$$

From the integral (3.42) and the first equality in (3.43) we further have

$$\omega_{z_0} = \Delta\omega^{\circ}_{z_0} - \omega^{0}_{y_0}\alpha^{\circ} + \omega^{0}_{x_0}\beta^{\circ} + \dot{\gamma}^{\circ} + \omega_{y_0}\alpha - \omega_{x_0}\beta - \dot{\gamma}. \tag{3.45}$$

Inserting (3.43), (3.44), (3.45) in the first two equations in (3.41), we omit the second-order products of the instrumental errors Δa, ΔA and their time derivatives with the small angles α, β, γ, $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$, $\ddot{\alpha}$, $\ddot{\beta}$. Collecting and simplifying, we obtain dividing through by ma

$$\left.\begin{aligned}
&r\ddot{\alpha} + 2\dot{r}\dot{\alpha} + r\alpha - \omega_{y_0}\omega_{y_0} r\beta - \omega^2_{x_0} r\alpha - \frac{1}{m}(F_{z_0}\alpha - F_{x_0}\gamma) = \\
&\quad = -\dot{\omega}_{x_0}\Delta r - 2\omega_{x_0}\Delta\dot{r} - C\omega^{\circ}_{x_0}(\Delta\omega_{z_0} - \omega^{\circ}_{y_0}\alpha^{\circ} + \\
&\qquad + \omega^{\circ}_{x_0}\beta^{\circ} + \dot{\gamma}^{\circ}) - \frac{\Delta A}{am}\dot{\omega}_{x_0} - \\
&\qquad - \omega_{x_0}\left(\frac{r\Delta\dot{a}}{a} + \frac{\Delta\dot{A}}{ma}\right) + \omega_{x_0}\dot{r}\,\frac{\Delta a}{a}, \\
&r\ddot{\beta} + 2\dot{r}\dot{\beta} + r\ddot{\beta} - \omega_{x_0}\omega_{x_0} r\alpha - \omega^2_{y_0} r\beta - \frac{1}{m}(F_{z_0}\beta - F_{y_0}\gamma) = \\
&\quad = -\dot{\omega}_{y_0}\Delta r - 2\omega_{y_0}\Delta\dot{r} + C\omega_{x_0}(\Delta\omega^{\circ}_{z_0} - \omega^{\circ}_{y_0}\alpha^{\circ} + \\
&\qquad + \omega^{\circ}_{x_0}\beta^{\circ} + \dot{\gamma}^{\circ}) - \frac{\Delta A}{am}\dot{\omega}_{y_0} - \\
&\qquad - \omega_{y_0}\left(\frac{r\Delta\dot{a}}{a} + \frac{\Delta\dot{A}}{ma}\right) + \omega_{y_0}\dot{r}\,\frac{\Delta a}{a}.
\end{aligned}\right\} \tag{3.46}$$

Clearly if $\Delta a = \Delta A = \Delta r = \Delta\omega^{\circ}_{z_0} = 0$, Eqs. (3.46) have the trivial solution $\alpha = \beta \equiv 0$ whenever $\alpha^0 = \beta^0 = \gamma^0 = \dot{\alpha}^0 = \dot{\beta}^0 = \dot{\gamma}^0 = 0$. This again confirms that under conditions (3.15), (3.30) the dynamic symmetry axis of the pendulum is fixedly directed to the Earth's center irrespective of the actual motion of the pivot, provided that initially the axis of the pendulum pointed in that direction and its orientation changed at the appropriate rate.

We will now expand the terms $F_{z_0}\alpha - F_{x_0}\gamma$ and $F_{z_0}\beta - F_{y_0}\gamma$ in Eqs. (3.46). To this end note that according to (1.40)

$$\boldsymbol{F} = m\,\mathrm{grad}\,V = m\left(\frac{\mu\boldsymbol{r}}{r^3} + \mathrm{grad}\,\varepsilon\right), \tag{3.47}$$

where ε is a small parameter which characterizes the nonspherical component of the Earth's gravitational field. Hence,

$$\left.\begin{aligned}
F_{x_0} &= m\,\mathrm{grad}_{x_0}\varepsilon, \quad F_{y_0} = m\,\mathrm{grad}_{y_0}\varepsilon, \\
F_{z_0} &= m\left(-\frac{\mu}{r^2} + \mathrm{grad}_{z_0}\varepsilon\right).
\end{aligned}\right\} \tag{3.48}$$

The products

$$\alpha\,\mathrm{grad}_{z_0}\varepsilon, \quad \beta\,\mathrm{grad}_{z_0}\varepsilon, \quad \gamma\,\mathrm{grad}_{x_0}\varepsilon, \quad \gamma\,\mathrm{grad}_{y_0}\varepsilon \tag{3.49}$$

are of second order of smallness, and we may therefore take

$$\left.\begin{aligned}
F_{z_0}\alpha - F_{x_0}\gamma &= -m\alpha\frac{\mu}{r^2}, \\
F_{z_0}\beta - F_{y_0}\gamma &= -m\beta\frac{\mu}{r^2}.
\end{aligned}\right\} \tag{3.50}$$

Note that the neglect of the terms (3.49) is equivalent to ignoring the variation of the nonspherical components of the Earth's gravitation in the derivation of the first group of differential error equations.

Using (3.50), Eqs. (3.46) take the form

$$\left.\begin{aligned}
&r\ddot{\alpha}+2\dot{r}\dot{\alpha}+\ddot{r}\alpha-\omega_{x_0}\omega_{y_0}r\beta-\omega_{x_0}^2r\alpha+\tfrac{\mu}{r^2}\alpha=\\
&\quad=-\dot{\omega}_{x_0}\Delta r-2\omega_{x_0}\Delta\dot{r}-C\omega_{y_0}(\Delta\omega_{z_0}^0-\omega_{y_0}^0\alpha^0+\\
&\qquad+\omega_{x_0}^0\beta^0+\dot{\gamma}^0)-\tfrac{\Delta A}{am}\dot{\omega}_{x_0}-\\
&\qquad-\omega_{x_0}\left(\tfrac{r\Delta\dot{a}}{a}+\tfrac{\Delta\dot{A}}{am}\right)+\omega_{x_0}\dot{r}\,\tfrac{\Delta a}{a},\\
&r\ddot{\beta}+2\dot{r}\dot{\beta}+\ddot{r}\beta-\omega_{x_0}\omega_{y_0}r\alpha-\omega_{y_0}^2r\beta+\tfrac{\mu}{r^2}\beta=\\
&\quad=-\dot{\omega}_{y_0}\Delta r-2\omega_{y_0}\Delta\dot{r}+C\omega_{x_0}(\Delta\omega_{z_0}^0-\omega_{y_0}^0\alpha^0+\\
&\qquad+\omega_{x_0}^0\beta^0+\dot{\gamma}^0)-\tfrac{\Delta A}{am}\dot{\omega}_{y_0}-\\
&\qquad-\omega_{y_0}\left(\tfrac{r\Delta\dot{a}}{a}+\tfrac{\Delta\dot{A}}{ma}\right)+\omega_{y_0}\dot{r}\,\tfrac{\Delta a}{a}.
\end{aligned}\right\}\qquad(3.51)$$

We have considered the case when conditions (3.15), (3.30) are satisfied. Let us now consider the case when condition (3.30) and the second condition in (3.13) are satisfied, i.e., when the pendulum is an infinitely thin rod. Here condition (3.30) is satisfied by a certain displacement of masses along the z axis; the particular mechanisms used to satisfy conditions (3.15) is of no relevance for our discussion.

Let the equality $C = 0$ be satisfied with a certain error, so that

$$C=\Delta C.\qquad(3.52)$$

From Eqs. (3.41), Eq. (3.42), the second and third equation in (3.43), and Eqs. (3.44), (3.50), (3.52), acting along the same lines as in the derivation of Eqs. (3.51) and retaining only first-order terms in α and β, we obtain the equations

$$\left.\begin{aligned}
&r\ddot{\alpha}+2\dot{r}\dot{\alpha}+\left(\ddot{r}-r\omega_{x_0}^2-r\omega_{z_0}^2+\tfrac{\mu}{r^2}\right)\alpha-\\
&\qquad-(\omega_{y_0}\omega_{x_0}+\dot{\omega}_{z_0})r\beta-2\omega_{z_0}(\dot{r}\beta+r\dot{\beta})=\\
&\quad=(\omega_{y_0}\omega_{z_0}-\dot{\omega}_{x_0})\Delta r-2\omega_{x_0}\Delta\dot{r}-\omega_{y_0}\omega_{z_0}\Delta C-\\
&\quad-\tfrac{\Delta A}{am}(\dot{\omega}_{x_0}-\omega_{y_0}\omega_{z_0})-\omega_{x_0}\left(\tfrac{r\Delta\dot{a}}{a}+\tfrac{\Delta\dot{A}}{am}-\dot{r}\,\tfrac{\Delta a}{a}\right),\\
&r\ddot{\beta}+2\dot{r}\dot{\beta}+\left(\ddot{r}-r\omega_{y_0}-r\omega_{z_0}+\tfrac{\mu}{r^2}\right)\beta-\\
&\qquad-(\omega_{x_0}\omega_{y_0}-\dot{\omega}_{z_0})r\alpha+2\omega_{y_0}(\dot{r}\alpha+r\dot{\alpha})=\\
&\quad=-(\omega_{x_0}\omega_{y_0}-\dot{\omega}_{y_0})\Delta r-2\omega_{y_0}\Delta\dot{r}+\omega_{x_0}\omega_{z_0}\Delta C-\\
&\quad-\tfrac{\Delta A}{am}(\dot{\omega}_{y_0}+\omega_{x_0}\omega_{z_0})-\omega_{y_0}\left(\tfrac{r\Delta\dot{a}}{a}+\tfrac{\Delta\dot{A}}{am}-\dot{r}\,\tfrac{\Delta a}{a}\right).
\end{aligned}\right\}\qquad(3.53)$$

The homogeneous equations (3.53), like Eqs. (3.51), have a trivial solution $\alpha=\beta\equiv 0$. For $\omega_{z_0}=0$, the homogeneous equations (3.53) are readily seen to reduce to Eqs. (3.51). Moreover, if we take $\omega_{z_0}=0$ in the right-hand sides of Eqs. (3.53), the terms with Δr, Δa, ΔA (instrumental errors of common origin in the two cases) in the right-hand sides reduce to the corresponding terms in the right-hand sides of Eqs. (3.51).

3.1.3. Equations of free oscillations of the Schuler pendulum about a position of relative equilibrium: the case of finite angles

Eqs. (3.51), (3.53) were derived assuming small angles α, β, γ, i.e., these equations only describe small oscillations of the Schuler pendulum about the relative equilibrium.

Let us now consider the equations of perturbed motion of the pendulum for finite angles α, β, γ. We will only discuss the homogeneous equations, i.e., the equations of free oscillations. This is quite sufficient, as we shall see, for the purposes of establishing the analogy between the Schuler pendulum and the two-accelerometer inertial system.

The position of the axes xyz fixed to the perturbed pendulum is determined relative to the unperturbed axes $x_0y_0z_0$ by the matrix of direction cosines (3.33). We are concerned mainly with oscillations of the axis z, and the angle γ from the start can be referred to the $x_0y_0z_0$ axes. (Incidentally, note that the angle γ does not enter the homogeneous equations (3.51), (3.53).) The matrix (3.34) is then replaced by

$$\begin{array}{c|ccc} & x & y & z \\ x_0 & \cos\beta & 0 & \sin\beta \\ y_0 & \sin\alpha\sin\beta & \cos\alpha & -\sin\alpha\cos\beta \\ z_0 & -\cos\alpha\sin\beta & \sin\alpha & \cos\alpha\cos\beta \end{array} \tag{3.54}$$

Let as before $\omega_{x_0}, \omega_{y_0}, \omega_{z_0}$ be the projections of the absolute angular velocity of the axes $x_0y_0z_0$ on the corresponding directions. In this case the axes $x_0y_0z_0$ are not longer fixed to the unperturbed pendulum, since the angle γ is referred to these axes. Therefore, we cannot take $\omega_{z_0} = 0$ in this case, but as we shall see in the following this does not constitute a significant difficulty.

The projections of the equations of motion of the pendulum on the axes x, y, z fixed to the pendulum can be written, in accordance with Eqs. (3.3), (3.7), in the form

$$\left.\begin{aligned} \dot{K}_x + \omega_y K_z - \omega_z K_y &= a(F_y + Q_y), \\ \dot{K}_y + \omega_z K_x - \omega_x K_z &= -a(F_x + Q_x), \\ \dot{K}_z &= 0. \end{aligned}\right\} \tag{3.55}$$

Here, using (3.54),

$$\left.\begin{aligned} \omega_x &= \omega_{x_0}\cos\beta + \omega_{y_0}\sin\alpha\sin\beta - \omega_{z_0}\cos\alpha\sin\beta + \dot{\alpha}\cos\beta, \\ \omega_y &= \omega_{y_0}\cos\alpha + \omega_{z_0}\sin\alpha + \dot{\beta} \\ \omega_z &= \omega_{x_0}\sin\beta - \omega_{y_0}\sin\alpha\cos\beta + \omega_{z_0}\cos\alpha\cos\beta + \dot{\alpha}\sin\beta. \end{aligned}\right\} \tag{3.56}$$

Similarly

$$\left.\begin{aligned} F_x + Q_x &= (F_{x_0} + Q_{x_0})\cos\beta + (F_{y_0} + Q_{y_0})\sin\alpha\sin\beta - \\ &\quad - (F_{z_0} + Q_{z_0})\cos\alpha\sin\beta, \\ F_y + Q_y &= (F_{y_0} + Q_{y_0})\cos\alpha + (F_{z_0} + Q_{z_0})\sin\alpha. \end{aligned}\right\} \tag{3.57}$$

Let

$$F_{x_0} = F_{y_0} = 0, \quad F_{z_0} = -\frac{m\mu}{r^2}, \tag{3.58}$$

which is equivalent to ignoring the variation of the nonspherical gravitation component.

We assume that the conditions

$$A = mar, \quad a = kr, \quad C = 0 \tag{3.59}$$

are satisfied, i.e., we consider a pendulum in the form of an infinitely thin rod whose small oscillations are described by Eqs. (3.53).

Using (3.59) we obtain

$$K_x = mar\omega_x, \quad K_y = mar\omega_y, \quad K_z = 0. \tag{3.60}$$

From (3.60), (3.56) we get

$$\left.\begin{aligned} K_x &= mar\,(\omega_{x_0}\cos\beta + \omega_{y_0}\sin\alpha\sin\beta - \omega_{z_0}\cos\alpha\sin\beta + \dot{\alpha}\cos\beta),\\ K_y &= mar\,(\omega_{y_0}\cos\alpha + \omega_{z_0}\sin\alpha + \dot{\beta}),\\ K_z &= 0. \end{aligned}\right\} \tag{3.61}$$

Similarly, from Eqs. (3.57), (3.58), (3.9), (3.37) and matrix (3.54) we have

$$\left.\begin{aligned} F_x + Q_x = -m\Big\{ &[r(\dot{\omega}_{y_0} + \omega_{x_0}\omega_{z_0}) + 2\omega_{y_0}\dot{r}]\cos\beta + \\ &+ [r(-\dot{\omega}_{x_0} + \omega_{y_0}\omega_{z_0}) - 2\omega_{x_0}\dot{r}]\sin\alpha\sin\beta - \\ &- \Big[\frac{\mu}{r^2} + \ddot{r} - r\left(\omega_{x_0}^2 + \omega_{y_0}^2\right)\Big]\cos\alpha\sin\beta\Big\},\\ F_y + Q_y = -m\Big\{ &[r(-\dot{\omega}_{x_0} + \omega_{y_0}\omega_{z_0}) - 2\omega_{x_0}\dot{r}]\cos\alpha + \\ &+ \Big[\frac{\mu}{r^2} + \ddot{r} - r\left(\omega_{x_0}^2 + \omega_{y_0}^2\right)\Big]\sin\alpha\Big\}. \end{aligned}\right\} \tag{3.62}$$

We now insert (3.61), (3.62) in (3.55). After some manipulations and simplifications, using (3.59), we get

$$\left.\begin{aligned} &r[\ddot{\beta} + \dot{\alpha}^2\sin\beta\cos\beta + 2\dot{\alpha}(\omega_{x_0}\sin\beta\cos\beta - \\ &\quad - \omega_{y_0}\sin\alpha\cos^2\beta + \omega_{z_0}\cos\alpha\cos^2\beta) + \\ &+ \dot{\omega}_{y_0}(\cos\alpha - \cos\beta) + \dot{\omega}_{z_0}\sin\alpha + \dot{\omega}_{x_0}\sin\alpha\sin\beta + \\ &+ \Big(\frac{\mu}{r^3} - \omega_{z_0}^2\cos\alpha\cos\beta - \omega_{y_0}^2\Big)\sin\beta\cos\alpha + \\ &+ \omega_{x_0}^2(\cos\beta - \cos\alpha)\sin\beta - \omega_{y_0}^2\sin^2\alpha\sin\beta\cos\beta + \\ &\quad + \omega_{x_0}\omega_{y_0}(\sin^2\beta - \cos^2\beta)\sin\alpha + \\ &+ \omega_{x_0}\omega_{z_0}(\cos^2\beta\cos\alpha - \cos\beta - \sin^2\beta\cos\alpha) + \\ &\quad + \omega_{y_0}\omega_{z_0}(2\cos\alpha\cos\beta - 1)\sin\alpha\sin\beta] + \\ &+ 2\dot{r}[\omega_{y_0}(\cos\alpha - \cos\beta) - 2\omega_{x_0}\sin\alpha\sin\beta + \\ &\quad + \omega_{z_0}\sin\alpha + \dot{\beta}] + \ddot{r}\cos\alpha\sin\beta = 0,\\ &r\Big\{\ddot{\alpha}\cos\beta - 2\dot{\alpha}\dot{\beta}\sin\beta + 2\dot{\beta}(-\omega_{x_0}\sin\beta + \\ &+ \omega_{y_0}\sin\alpha\cos\beta - \omega_{z_0}\cos\alpha\cos\beta) + \dot{\omega}_{x_0}(\cos\beta - \cos\alpha) + \\ &\quad + \dot{\omega}_{y_0}\sin\alpha\sin\beta - \dot{\omega}_{z_0}\cos\alpha\sin\beta + \\ &+ \Big[\frac{\mu}{r^3} - \omega_{x_0}^2 - \omega_{y_0}^2(1 - \cos\alpha\cos\beta) - \omega_{z_0}^2\cos\alpha\cos\beta\Big]\sin\alpha - \\ &\quad - \omega_{x_0}\omega_{y_0}\cos\alpha\sin\beta - \omega_{x_0}\omega_{z_0}\sin\alpha\sin\beta + \\ &+ \omega_{y_0}\omega_{z_0}(\sin^2\alpha\cos\beta - \cos^2\alpha\cos\beta + \cos\alpha)\Big\} + \\ &+ 2\dot{r}[\omega_{x_0}(\cos\beta - \cos\alpha) + \omega_{y_0}\sin\alpha\sin\beta - \\ &\quad - \omega_{z_0}\cos\alpha\sin\beta + \dot{\alpha}\cos\beta] + \ddot{r}\sin\alpha = 0. \end{aligned}\right\} \tag{3.63}$$

It is readily seen that Eqs. (3.63) have a trivial zero solution. If the angles α and β are small, the trigonometric functions can be expanded in a series retaining in Eqs. (3.63) only the linear terms in α, β, $\dot{\alpha}$, $\dot{\beta}$. This result is clearly seen to give the homogeneous equations (3.53).

Eqs. (3.63) are the equations of oscillation of an infinitely thin Schuler pendulum with $C = 0$. If we start with the condition $\omega_z = \omega_\zeta^\circ = 0$, and not $C = 0$, Eqs. (3.55) give expressions which coincide with those obtained from Eqs. (3.63) with $\omega_z = 0$. For small α, β, $\dot{\alpha}$, $\dot{\beta}$ they coincide to first approximation with Eqs. (3.51).

§3.2. THE TWO-GYRO VERTICAL AND THE GECKELER–ANSCHUTZ GYROCOMPASS HORIZON*

3.2.1. The two-gyro vertical

The two-gyro vertical is a gyroframe whose center of mass does not coincide with the suspension point of the gyroframe platform. If a set of right orthogonal axes $Oxyz$ is rigidly fixed to the gyroframe platform, placing the origin at the suspension point, the two gyroscopes will be aligned as shown in Figure 3.5. The two axles of the gyro cases lie in the plane xz, parallel to the axis x, and the angular momenta of the gyroscopes lie in the plane yz. The center of mass c of the gyroframe is on the axis z at a distance a from the suspension point, so that Eqs. (3.1) apply.

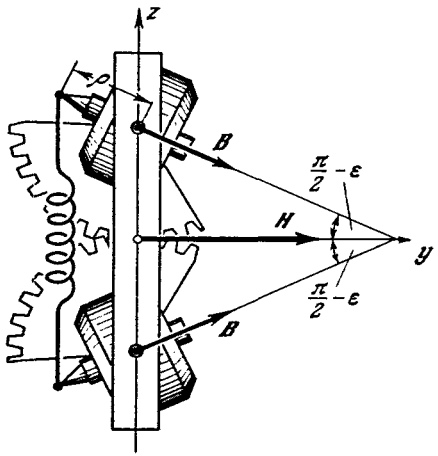

FIGURE 3.5.

The gyroscopes are geometrically linked (geared, as shown in Figure 3.5, or gimbaled), so that the tilt of each gyro about the case axle is automatically transmitted to its counterpart. The angles the two gyro angular momenta

*Ishlinskii, A.Yu. The theory of the gyrocompass horizon. — Prikladnaya Matematika i Mekhanika, Vol. 20, No. 4. 1956. Theory of the two-gyro vertical, Vol. 21, No 2. 1957.

make with the y axis are therefore equal in magnitude and of opposite direction, so that the resultant angular momentum is directed along the y axis.

Let B be the angular momentum of each gyroscope. The resultant angular momentum of the two gyroscopes is then

$$H = 2B \sin \varepsilon, \tag{3.64}$$

where ε is the angle between the angular momentum vector and the z axis.

Torques of equal magnitude and opposite direction about the case axles are applied to the gyroscopes by a special torquer. These torques are a function of the angle ε. Let N be the magnitude of these torques. In the simplest case the torque N can be set up by a spring joining the gyro cases as shown in Figure 3.5.

By the theorem of angular momentum, the equations of precession of the gyroframe are

$$-H\omega_z = M_x, \quad \frac{dH}{dt} = M_y, \quad H\omega_x = M_z, \tag{3.65}$$

where ω_x, ω_z are the projections of the absolute angular velocity of the axes xyz on the corresponding directions, M_x, M_y, M_z are the resultant torques applied to the gyroframe platform about the corresponding axes.

The system being considered has four degrees of freedom. The missing equation is obtained by considering the motion of the gyroscopes about the case axles. Here

$$-\omega_y 2B \cos \varepsilon = N, \tag{3.66}$$

where N is the torque set up by the spring joining the two cases (it is a function of the angle ε).

To complete the derivation of the equations of motion of a two-gyro vertical, it now remains to find expressions for the torques M_x, M_y, M_z. These torques, as in the case of a physical pendulum considered in the previous section, are the resultant of gravity torques and translatory inertia torques of the gyroframe. Additional artificial torques or perturbing torques resulting from the nonideal behavior of the system elements can also be taken into consideration. In accordance with (3.7), the expressions for the torques M_x, M_y, M_z can be written in the form

$$\left.\begin{aligned} M_x = a(F_y + Q_y) + M_x', \quad M_y = -a(F_x + Q_x) + M_y', \\ M_z = M_z^*, \end{aligned}\right\} \tag{3.67}$$

where $\boldsymbol{F}$ and $\boldsymbol{Q}$ are the gravitation and translatory inertia forces, M_x^*, M_y^*, M_z^* stand for all the other torques, except those produced by $\boldsymbol{F}$ and $\boldsymbol{Q}$.

Inserting (3.64) and (3.67) in (3.65) and adding Eq. (3.66) to the system, we obtain the equations of motion of the sensitive element of the two-gyro vertical:

$$\left.\begin{aligned} -\omega_z 2B \sin \varepsilon = a(F_y + Q_y) + M_x^*, \\ \frac{d}{dt}(2B \sin \varepsilon) = -a(F_x + Q_x) + M_y^*, \\ \omega_x 2B \sin \varepsilon = M_z^*, \quad -\omega_y 2B \cos \varepsilon = N. \end{aligned}\right\} \tag{3.68}$$

We will now show that for a particular choice of parameters and initial conditions, this device simulates the Schuler pendulum. Let us first find the position of relative equilibrium of the gyroframe with the z axis toward the Earth's center, irrespective of the motion of the object. The axes xyz in this state of relative equilibrium will be designated $x_0y_0z_0$. From Eqs. (3.68), using (3.37), (3.9), we then get

$$\left.\begin{aligned} -\omega_{z_0} 2B\sin\varepsilon &= a[F_{y_0} + m(r\dot{\omega}_{x_0} - r\omega_y\omega_z + 2\omega_r\dot{r})] + M_x^*, \\ \frac{d}{dt}(2B\sin\varepsilon) &= -a[F_{x_0} - m(r\dot{\omega}_{y_0} + r\omega_{x_0}\omega_{z_0} + 2\dot{r}\omega_{y_0})] + M_y^*, \\ \omega_{x_0} 2B\sin\varepsilon &= M_z^*, \quad -\omega_{y_0} 2B\cos\varepsilon = N. \end{aligned}\right\} \tag{3.69}$$

Let

$$M_z^* = 0. \tag{3.70}$$

From the third equation in (3.69), ignoring the special case $\varepsilon = 0$, we get

$$\omega_{x_0} = 0. \tag{3.71}$$

The first two equations in (3.69) thus take the simpler form

$$\left.\begin{aligned} -\omega_{z_0} 2B\sin\varepsilon &= a(F_{y_0} - mr\omega_{y_0}\omega_{z_0}) + M_x^*, \\ \frac{d}{dt}(2B\sin\varepsilon) &= -a[F_{x_0} - m(r\omega_{y_0} + 2\dot{r}\omega_{y_0})] + M_y^*. \end{aligned}\right\} \tag{3.72}$$

Let now the torques M_x^* and M_y^* be formed according to the equalities

$$M_x^* = -aF_{y_0}, \quad M_y^* = aF_{x_0}. \tag{3.73}$$

From the first equation in (3.72) we have

$$2B\sin\varepsilon = amr\omega_{y_0}, \tag{3.74}$$

and from (3.74) and the second equation in (3.72)

$$\frac{d}{dt}(ar) = 2a\dot{r}. \tag{3.75}$$

It is readily seen that condition (3.75) is satisfied if we take

$$a = kr \tag{3.76}$$

To ensure this equality, we clearly require additional information on the distance r from the Earth's center.

Turning to the last equation in (3.68), we apply (3.74) to eliminate ω_{y_0} and consequently obtain the following expression for the torque N:

$$N = -\frac{4B^2}{mar}\sin\varepsilon\cos\varepsilon. \tag{3.77}$$

Conditions (3.70), (3.73), (3.74), and (3.76) thus ensure the existence of relative equilibrium in which the axis z of the system is invariably caged to the geocentric vertical, irrespective of the particular motion of the suspension point of the platform. This explains the name two-gyro vertical.

If the Earth's gravitational field is assumed spherical and the suspension point moves at a constant distance from the Earth's center, F_{x_0}, F_{y_0} are zero, M_x^*, M_y^* can be omitted, and condition (3.76) is no longer needed. The problem thus reduces to conditions (3.70), (3.74), (3.77).

Note that if r, and hence a, are constant, the torque N can be provided by a simple spring. Indeed, turning to Figure 3.5, let ρ be the distance between the fixed end of the spring and the gyro case and k the force constant of the spring. Then

$$N = -2k\rho^2 \sin\varepsilon\cos\varepsilon, \tag{3.78}$$

assuming that the spring is relaxed for $\varepsilon = 0$. Taking further

$$k\rho^2 = \frac{4B^2}{mar}, \tag{3.79}$$

we achieve our aim.

If now r is variable, a and ρ should also vary.

We have established the requirements to be satisfied by the design parameters of the two-gyro vertical. To these we should naturally add certain requirements concerning the initial conditions of motion. One of these follows from (3.71):

$$\omega_{x_1}^0 = 0. \tag{3.80}$$

This implies that the x axis of the gyroframe should initially lie in the plane through the absolute velocity vector of the suspension point O and the Earth's center O_1.

From (3.74), (3.75) we further get

$$\sin\varepsilon^0 = \frac{a^0 m r^0 \omega_{y_0}^0}{2B}, \quad a^0 = kr^0, \quad \dot{a}^0 = k\dot{r}^0. \tag{3.81}$$

Moreover, the z axis of the gyroframe should initially point to the Earth's center.

Let us now consider the perturbed motion of the two-gyro vertical. We will analyze small oscillations of the z axis about the position of relative equilibrium assuming that equalities (3.73), (3.74), (3.76) and the initial conditions are satisfied with certain errors. There are furthermore perturbing torques associated with gimbal imperfection and imperfect balance of the platform.

The perturbed orientation of the xyz axes relative to the $x_0y_0z_0$ axes is specified by small angles α, β, γ in accordance with the matrix of direction cosines (3.34). Then ω_x, ω_y, ω_z are expressed by (3.35), Q_x, Q_y, Q_z by (3.9), (3.36), (3.37), and F_x, F_y by (3.39).

Inserting these relations in the equations of motion of the gyroframe (3.68) and introducing the perturbing torques ΔM_x, ΔM_y, ΔM_z in the right-hand sides of the equations, we get

$$\left.\begin{aligned} &-(\omega_{z_0} - \omega_{y_0}\alpha + \omega_{x_0}\beta + \dot{\gamma})\,2B\sin\varepsilon = \\ &= a(F_{z_0}\alpha - F_{x_0}\gamma + F_{y_0}) + am[r(\dot{\omega}_{x_0} - \omega_{y_0}\omega_{z_0}) + 2\omega_{x_0}\dot{r}] - \\ &\qquad - am\left[\ddot{r} - r(\omega_{x_0}^2 + \omega_{y_0}^2)\right]\alpha + \\ &\quad + am[r(\dot{\omega}_{y_0} + \omega_{x_0}\omega_{z_0}) + 2\omega_{y_0}\dot{r}]\gamma + M_x^* + \Delta M_x, \end{aligned}\right\} \tag{3.82}$$

$$\left.\begin{aligned}
&\frac{d}{dt}(2B\sin\varepsilon) = \\
&= -a(F_z\xi - F_y\gamma - F_x) + am[r(\dot\omega_y + \omega_x\omega_z) + 2\omega_y\dot r] - \\
&\qquad - am[\ddot r - r(\omega_x^2 + \omega_y^2)]\beta + \\
&\quad - am[r(-\dot\omega_x + \omega_y\omega_z) - 2\omega_x\dot r]\gamma + M_y^* + \Delta M_y, \\
&(\omega_x - \omega_z\gamma + \omega_z\beta + \dot\alpha)2B\sin\varepsilon = \Delta M_z, \\
&-(\omega_y - \omega_z\gamma - \omega_z\alpha - \dot\beta)2B\cos\varepsilon = N.
\end{aligned}\right\} \tag{3.82}$$

From the third equation in (3.82) we see that ω_{x_0} is of first order of smallness. Therefore, in the remaining equations it can be omitted whenever multiplied by another first-order quantity. Note that according to the definition (3.73) of the torques M_x^*, M_y^*, the sums $aF_y + M_x^*$ and $aF_x + M_y^*$ drop out from the right-hand sides of Eqs. (3.82). The terms $-aF_x\gamma$ and $-aF_y\gamma$ can also be neglected, as they are of second order of smallness, and for F_z we may write to the same approximation $-\frac{\mu m}{r^2}$. After these simplifications, Eqs. (3.82) take the form

$$\left.\begin{aligned}
&-(\omega_x - \omega_z\alpha + \dot\gamma)2B\sin\varepsilon = -am\frac{\mu}{r^2}\alpha + \\
&- am[r(\dot\omega_x - \omega_y\omega_z) - 2\omega_x\dot r] - \\
&- am(\ddot r - r\omega_y^2)\alpha + am(r\dot\omega_z + 2\omega_z\dot r)\gamma + \Delta M_x, \\
&\frac{d}{dt}(2B\sin\varepsilon) = -am\frac{\mu}{r^2}\beta + am[r(\dot\omega_y + \omega_x\omega_z) + \\
&+ 2\omega_z\dot r] - am(\ddot r - r\omega_y^2)\beta + am\omega_y\omega_z\gamma + \Delta M_y, \\
&(\omega_x + \omega_y\gamma - \omega_z\beta + \dot\alpha)2B\sin\varepsilon = \Delta M_z, \\
&-(\omega_y + \omega_z\alpha + \dot\beta)2B\cos\varepsilon = N.
\end{aligned}\right\} \tag{3.83}$$

Let ε_0 be the value of ε corresponding to the position of relative equilibrium; $\delta\varepsilon$ is the corresponding perturbation. Moreover, let conditions (3.74), (3.76), (3.77) be satisfied with some errors, so that

$$a = k(r + \Delta r) + \Delta a, \tag{3.84}$$

and hence

$$\left.\begin{aligned}
&2B\sin\varepsilon = -2B\cos\varepsilon_0\delta\varepsilon + \\
&\qquad + (amr + mr\,\Delta a + 2am\,\Delta r)\omega_{y_0}, \\
&\frac{N}{2B\cos\varepsilon} = -\omega_{y_0} + \frac{2B\cos\varepsilon_0\delta\varepsilon}{mar} + \frac{\Delta N}{2B\cos\varepsilon_0}.
\end{aligned}\right\} \tag{3.85}$$

In Eqs. (3.84), (3.85), Δr is the error of the external information on r. The perturbing torque ΔN can be associated, say, with initial deformation of the spring between the gyro cases.

Inserting (3.84), (3.85) in Eqs. (3.83) and retaining, as before, only first order terms, we get

$$\left.\begin{aligned}
&-r\frac{d}{dt}(\omega_y\gamma) + \frac{\omega_{z_0}2B\cos\varepsilon_0\delta\varepsilon}{ma} + \frac{\mu}{r^2}\alpha + \ddot r\alpha - \\
&- 2\dot r\omega_{y_0}\gamma = r\dot\omega_{x_0} + 2\dot r\omega_{x_0} + \Delta r\omega_{y_0}\omega_{z_0} + \frac{\Delta M_x}{am},
\end{aligned}\right\} \tag{3.86}$$

$$\left.\begin{aligned}
&-\frac{d}{dt}(2B\cos\varepsilon_0\,\delta\varepsilon)+am\omega_{y_0}\Delta r+\\
&+\omega_{y_0}(mr\,\Delta\dot a+2am\,\Delta\dot r-m\dot r\,\Delta a)=\\
&=-am\frac{\mu}{r^2}\beta+amr\omega_{x_0}\omega_{z_0}-am(\ddot r-r\omega_{y_0}^2)\beta+\\
&+amr\omega_{y_0}\omega_{z_0}\gamma+\Delta M_y,\\
&r\omega_{x_0}\omega_{y_0}+r\omega_{y_0}(\omega_{y_0}\gamma-\omega_{z_0}\beta+\dot\alpha)=\frac{\Delta M_z}{am},\\
&-(\omega_{z_0}\alpha+\dot\beta)=\frac{2B\cos\varepsilon_0\,\delta\varepsilon}{mar}+\frac{\Delta N}{2B\cos\varepsilon_0}.
\end{aligned}\right\}\qquad(3.86)$$

Eqs. (3.86) are the equations of small oscillations of a two-gyro vertical about the position of relative equilibrium. These equations allow for the principal instrumental errors and the errors Δr in the external information on the distance r.

For

$$\omega_{r_0}=0,\quad r=\text{const},\quad \Delta r=\Delta a=0,\quad \Delta M_x=\Delta M_y=\Delta M_z=\Delta N=0,\qquad(3.87)$$

Eqs. (3.86) reduce to Ishlinskii's equations.* It is readily seen that when conditions (3.87) are satisfied, Eqs. (3.86) have the trivial solution

$$\alpha=\beta=\gamma=\delta\varepsilon\equiv 0,\qquad(3.88)$$

corresponding to a position of relative equilibrium.

3.2.2. The Geckeler—Anschütz gyrocompass horizon

Let us consider another classical example of a pendulous gyroscope, which is a simulation of the Schuler pendulum. This is the Geckeler—Anschütz gyrocompass horizon.

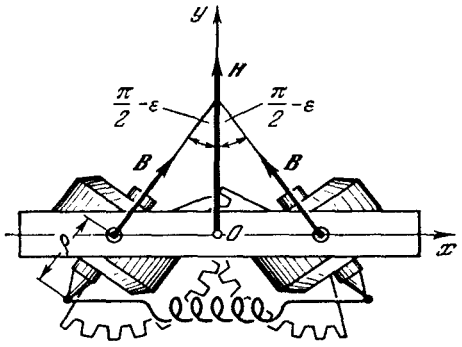

FIGURE 3.6.

The sensitive element of the gyrocompass horizon is not unlike the sensitive element of the two-gyro vertical. This is a combination of two

* See footnote on p. 117.

gyroscopes with parallel case axles and gimbal bearings fixed on a common rigid platform. The gyrocompass horizon platform can be suspended in a three-degrees-of-freedom gimbal or floated in a liquid in a spherical case. Other suspensions are also possible, but the particular suspension is of no relevance for further discussion, the main thing being that it should ensure three degrees of freedom for the sensitive element.

A set of right orthogonal axes $Oxyz$ is associated with the sensitive element of the gyrocompass. Its origin is at the suspension point of the platform, the axis y is normal to the platform, the axis z lies in the plane of the platform, parallel to the case axles. The axis x then meets the case axles and lies in the plane through the angular momenta of the two gyroscopes.

The gyro case axles are geared (Figure 3.6) or gimbaled, so that the two gyroscopes turn about the case axles through an equal angle in opposite directions. This angle is designated as before by ε. If B is the magnitude of the angular momentum of each gyroscope, the resultant angular momentum is given by

$$H = 2B\sin\varepsilon \tag{3.89}$$

and is directed along the axis y. As for a two-gyro vertical, the gyro cases are joined by a spring. The center of mass of the sensitive element lies on the negative z semiaxis at a distance a from the suspension point.

A comparison of the sensitive element of the gyrocompass horizon with that of the two-gyro vertical (Figure 3.5) reveals only the following difference: in the gyrocompass horizon the case axles lie on the x axis and are parallel to the z axis, whereas in the two-gyro vertical they lie on the z axis and are parallel to the x axis. Moreover, the center of mass of the sensitive element of the two-gyro vertical is displaced in a direction normal to the case axles, whereas in the gyrocompass horizon it is displaced along the case axles.

The precession of the gyrocompass horizon is thus entirely similar to the precession of the two-gyro vertical. Indeed, the precession equations are

$$K_x = 0, \quad K_y = H = 2B\sin\varepsilon, \quad K_z = 0. \tag{3.90}$$

The theorem of angular momentum gives for the x, y, z projections

$$\left.\begin{aligned} -\omega_z 2B\sin\varepsilon = M_x, \quad \frac{d}{dt}(2B\sin\varepsilon) = M_y, \\ \omega_x 2B\sin\varepsilon = M_z. \end{aligned}\right\} \tag{3.91}$$

which are supplemented by an equation

$$-\omega_y 2B\cos\varepsilon = N, \tag{3.92}$$

where N is the torque about the case axles.

Eqs. (3.91), (3.92) are fully identical to Eqs. (3.65), (3.66) used in our analysis of the two-gyro vertical. Further examinations of Eqs. (3.91), (3.92) therefore leads to the same relations as for the two-gyro vertical. Thus, the conditions for relative equilibrium of the sensitive element of the gyrocompass horizon are Eqs. (3.70), (3.73), (3.74), (3.76); the equations of small oscillations about the relative equilibrium are Eqs. (3.86).

As we have already noted, conditions (3.70), (3.73), (3.74), (3.76) guarantee the existence of a relative equilibrium in which the axis z of the gyrocompass horizon is directed to the Earth's center. In the position of relative equilibrium, the y axis, according to (3.71), is perpendicular to the direction of the absolute velocity of the object. The angle γ between the horizontal projection of the absolute velocity of the object and the direction to the east (i.e., the angle between the axis y and the direction to the north) is given by

$$\operatorname{tg}\gamma^* = \frac{V_N}{ru\cos\varphi + V_E}, \tag{3.93}$$

where φ is the geocentric latitude of the object, V_E and V_N are the eastward and the northward components of its horizontal velocity relative to the Earth. They are expressed in terms of the derivatives of the latitude φ and the longitude λ by the equalities:

$$V_N = r\dot{\varphi}, \quad V_E = \frac{r\dot{\lambda}}{\cos\varphi}. \tag{3.94}$$

Note that we are operating in the geocentric system of coordinates, so that the plane of the horizon is the plane perpendicular to the direction to the Earth's center.

Eq. (3.93) coincides with the standard equation for the so-called velocity error of a gyrocompass.* If the horizontal component of the velocity of an object relative to the Earth is known,

$$V = \sqrt{V_N^2 + V_E^2}, \tag{3.95}$$

γ^* can be found from the equality

$$\sin\gamma^* = \frac{V\cos\varkappa}{ru\cos\varphi}, \tag{3.96}$$

where $\varkappa$ is the so-called gyrocompass course, i.e., the angle between the axis y and the direction of the velocity vector. The relative velocity V of a sea vessel, say, is measured with a log; for aircraft moving near the Earth's surface a Doppler velocity meter is used.

Note that if V is known, the gyrocompass horizon can determine the geocentric coordinates φ and λ. Indeed, from (3.96) we have

$$4B^2\sin^2\varepsilon = ma^2r^2[(u+\dot{\lambda})^2\cos^2\varphi + \dot{\varphi}^2], \tag{3.97}$$

which together with the equality

$$V^2 = (\dot{\lambda}^2\cos^2\varphi + \dot{\varphi}^2)r^2 \tag{3.98}$$

gives a set of two differential equations for $\dot{\lambda}$ and $\dot{\varphi}$. The solution of this system for known initial conditions in fact gives λ and φ. Various arithmetic and integrating circuits can be used to build a system simulating Eqs. (3.97), (3.98), which given the values of ε and V will continuously compute the coordinates φ and λ.

* Bulgakov, B.V. Prikladnaya teoriya giroskopov (Applied Theory of Gyroscopes). — Gostekhizdat. 1955.

If V is not known, the gyrocompass horizon alone is insufficient for determining the coordinates. Indeed, although the xyz axes in a position of relative equilibrium are the co-moving axes on a sphere concentric with the Earth, we have only two projections of the absolute angular velocity of these axes:

$$\omega_x = 0, \quad \omega_z = -\frac{zB_{xyz}}{mr^2}. \tag{3.99}$$

ω_y is determined by the conditions of relative equilibrium. This is a consequence of the geometric linkage between the gyroscopes in the gyrocompass horizon and the two-gyro vertical (geared or gimbaled). Because of this linkage, the two gyroscopes are equivalent to a single gyroscope with variable angular momentum along the y axis. Yet to determine the direction to the Earth's center in space, as we have seen in the preceding, we require at least two reference directions, i.e., two geometrically independent (not linked) angular momenta.

Specifying the velocity V is in fact equivalent to specifying ω_y. It enables us to find γ' and hence to determine the orientation of the xyz axes relative to the axes rigidly fixed to the Earth. The Earth's motion (rotation) relative to the fixed-attitude axes $\xi\eta\zeta$ is known. Given V, we therefore completely specify the mutual attitude of the axes xyz and $\xi\eta\zeta$. Hence the conclusion that knowledge of V is essentially equivalent to knowledge of ω_y.

All the previous results concerning course error and Eqs. (3.97), (3.98) for the gyrocompass horizon are clearly applicable to the two-gyro vertical, since the starting equations of motion are identical.

Note, however, that despite the complete identity of the equations describing the operation of the two systems, there is a certain fundamental difference between them. Identical technological imperfections in the manufacture of the sensitive element may lead to entirely different effects in the two systems. Thus, gyroscope imbalance along the case axles introduces a torque ΔM in the equations of oscillation about the position of relative equilibrium of the two-gyro vertical (3.86), whereas for the gyrocompass horizon the same defect gives rise to the torque ΔM_y. As a result, the two-gyro vertical has a lower direction measuring accuracy and the gyrocompass horizontal a lower vertical fixing accuracy in systems with identical technical parameters.

Consider another point in connection with Eqs. (3.86). These equations determine the angles α, β, γ that the perturbed axes xyz of the sensitive gyro element fixed to the platform make with the position of relative equilibrium of the axes $x_0y_0z_0$. Eqs. (3.86) also give the deviation $\delta\varepsilon$ of the gyroscope angle from the relative equilibrium value ε_0.

Note that if the navigation accuracy is sufficient to justify determination of the course error γ', the inaccuracy of this quantity should be added to the angle γ. γ' is obtained from Eqs. (3.93), (3.96), (3.97). In this case additional information on the velocity V is used and the geographical coordinates of the object are determined while calculating γ'. To find the inaccuracy in γ', we should thus write Eqs. (3.93), (3.96), (3.97) in variational form.

Note that the gyrocompass horizon used in ships sometimes satisfies only condition (3.74), and not (3.76); r is considered constant, e.g., equal to the mean radius of the Earth. The torques M_x^*, M_z^* compensating the effect

of the nonspherical component of the Earth's gravitation are disregarded.* The errors Δr and $\Delta \dot{r}$ introduced in Eqs. (3.86) in fact allow for deviations from equality in (3.76). The consequences of the omission of the torques M_x^*, M_y^* will be discussed in the next chapter in connection with the simplified equations of inertial systems and various methods of introducing the nonspherical component of the Earth's gravitation.

§3.3. A GENERAL SCHULER-TUNED SYSTEM

3.3.1. Conditions of existence of relative equilibrium

Consider a general mechanical system mounted on a moving object in some suspension with three degrees of freedom, so that the center of mass c of the system does not coincide with the center O of the suspension. The distance between the suspension point and the center of mass of the system is a. The system may comprise various moving mechanical devices, including gyroscopes.

The origin of the axes $Oxyz$ associated with the system is placed at the suspension point, and the axis z is directed along the line between the center of mass and the suspension point, toward the center of mass. The coordinates of the center of mass are thus

$$x_c = y_c = 0, \quad z_c = -a. \tag{3.100}$$

We will now find the conditions when the z axis of the system in relative equilibrium points to the Earth's center.

Applying the theorem of the angular momentum to the motion of the system as a whole about its center of mass, we have

$$\frac{d\boldsymbol{K}}{dt} = \boldsymbol{M}, \tag{3.101}$$

where $\boldsymbol{K}$ is the angular momentum of the system, and $\boldsymbol{M}$ is the resultant external torque.

As before in calculating the resultant torque we assume the gravitational forces to reduce to a single force

$$\boldsymbol{F} = m \operatorname{grad} V, \tag{3.102}$$

applied to the center of mass in the direction of the Earth's gravitational acceleration at the point O. The inhomogeneity of the gravitational field within the volume of the system is ignored.

Adding the translatory inertia forces and an artificial torque $\boldsymbol{M}'$, we obtain from (3.101)

$$\frac{d\boldsymbol{K}}{dt} = \boldsymbol{a} \times \left(\boldsymbol{F} - m\frac{d^2\boldsymbol{r}}{dt^2}\right) + \boldsymbol{M}'. \tag{3.103}$$

Here, as before, $\boldsymbol{r}$ is the radius-vector of the point O, originating at the Earth's center O_1, $\boldsymbol{a}$ is the radius vector of the point c originating at the point O.

*Bulgakov, B.V. Prikladnaya teoriya giroskopov (Applied Theory of Gyroscopes). — Gostekhizdat. 1955.
Grammel, R. Der Kreisel, Vol. 2. — Springer-Verlag. 1950.

Let $x_0y_0z_0$ designate the axes xyz in a position of relative equilibrium. Then

$$\left.\begin{aligned} F &= F_x x_0 + F_y y_0 + F_z z_0, \\ M^* &= M_x^* x_0 + M_y^* y_0 + M_z^* z_0. \end{aligned}\right\} \tag{3.104}$$

Since in relative equilibrium a and r are collinear, we have from (3.103), (3.104)

$$\frac{dK}{dt} = -a \times m\frac{d^2r}{dt^2} + (M_x^* - aF_y)x_0 + (M_y^* + aF_x)y_0 + M_z^* z_0. \tag{3.105}$$

Let

$$M_x^* = -aF_y, \quad M_y^* = aF_x, \quad M_z^* = 0. \tag{3.106}$$

Using (3.106), we write (3.105) in the form

$$\frac{dK}{dt} = -a \times m\frac{d^2r}{dt^2}. \tag{3.107}$$

Let now

$$a = kr. \tag{3.108}$$

i.e., the distance from the center of mass to the suspension point is proportional to the distance from the Earth's center. To satisfy this condition, the system naturally requires external information on the distance r.

If condition (3.108) is satisfied, we have in relative equilibrium

$$a = -kr, \tag{3.109}$$

and thus

$$\frac{da}{dt} \times \frac{dr}{dt} = 0, \tag{3.110}$$

so that Eq. (3.107), using (3.109), can be written in the form

$$\frac{dK}{dt} = km\frac{d}{dt}\left(r \times \frac{dr}{dt}\right). \tag{3.111}$$

Integration of this equation gives

$$K - kmr \times \frac{dr}{dt} = h, \tag{3.112}$$

where h is a constant vector.

Taking $h = 0$, we write $v = \frac{dr}{dt}$ and project Eq. (3.112) on the axes x_0, y_0, z_0. This gives

$$K_x + mav_y = 0, \quad K_y - mav_x = 0, \quad K_z = 0. \tag{3.113}$$

Eqs. (3.106), (3.108), (3.113) are the necessary and sufficient conditions of existence of a relative equilibrium of a general Schuler-tuned system whose axis is directed tc the Earth's center.*

*Conditions (3.108), (3.113) were derived by an alternative technique (in application to a gyroframe) by D.M. Klimov in his paper "On the unperturbability of a gyroscopic frame." — Prikladnaya Matematika i Mekhanika, Vol. 28, No. 3, 1964.

The above conditions should be supplemented by suitable initial conditions of the system. Initially the axis z should coincide with the axis z_0. The initial rate of turning of the axis z should be equal to the rate of turning of the axis z_0 of the co-moving system (owing to the motion of the object).

Conditions (3.106), (3.108) coincide with the analogous conditions for the Schuler pendulum, two-gyro vertical, and gyrocompass horizon. The remaining conditions for these cases are readily derived from Eqs. (3.113). Thus, for a physical pendulum

$$K_{x_0} = A\omega_{x_0}, \quad K_{y_0} = A\omega_{y_0}, \quad K_{z_0} = C\omega_{z_0}.$$

The projections of the absolute velocity of the suspension point are related to the projections of the absolute angular velocity:

$$v_{x_0} = r\omega_{y_0}, \quad v_{y_0} = -r\omega_{x_0}.$$

Now from Eqs. (3.113) we have the previous conditions

$$A = mar, \quad C\omega_{z_0} = 0.$$

For the gyrocompass horizon and the two-gyro vertical, the axis x is directed along the vector v, so that $\omega_x = 0$. The first condition in (3.113) is thus identically satisfied if $H_{x_0} = 0$. The two remaining conditions give

$$(mar - J_{y_0})\omega_{y_0} - H_{y_0} = 0, \quad J_{z_0}\omega_{z_0} + H_{z_0} = 0,$$

where J_{y_0} and J_{z_0} are the moments of inertia of the gyropendulum system about the corresponding axes.

Hence, within the framework of the precession theory, when $J_{y_0} = J_{z_0} = 0$, we have

$$H_{z_0} = 0, \quad H_{y_0} = mar\omega_{y_0},$$

which coincides with the preceding conditions.

3.3.2. Perturbed motion about the relative equilibrium

Let us consider the motion of the axis z of a general Schuler-tuned system satisfying conditions (3.106), (3.108), (3.113) about the relative equilibrium in cases when the above conditions and the initial setting are observed only to some approximation. Other system errors will include the error Δr in the external information on the distance r from the Earth's center, which is used in condition (3.108)

First let us consider small oscillations of the z axis about the state of relative equilibrium.

We will again use the theorem of the angular momentum to derive the equations of motion. Taking the variations in Eq. (3.101) near the position of relative equilibrium, we find

$$\frac{d\delta K}{dt} = \delta \mathbf{M}. \tag{3.114}$$

We can now determine separately the variations δK and δM. The angular momentum K should be formed using (3.107). Thus,

$$\delta K = km\,\delta r \times \frac{dr}{dt} + kmr \times \frac{d\,\delta r}{dt} + \Delta K, \tag{3.115}$$

where ΔK is an instrumental error,

$$r = rz_0, \quad \delta r = \delta x x_0 + \delta y y_0 + \Delta r z_0. \tag{3.116}$$

To find δM, note that the variation of the torque is determined by the variation δa, the variation $\delta M'$ of the correcting torque, and the torque ΔM associated with instrumental errors. Therefore, using the expression for M in the right-hand side of Eq. (3.103), we find

$$\delta M = \delta a \times \left(F - m\frac{d^2r}{dt^2}\right) + \delta M' + \Delta M. \tag{3.117}$$

In Eq. (3.117) δa is given by

$$\delta a = -k\,\delta r - \Delta a z_0. \tag{3.118}$$

The first term in the right-hand side is associated with errors in the information on the distance r and deviations of the axis z from the exact geocentric vertical; the second term is associated with the instrumental error Δa in measuring the distance a between the suspension point and the center of mass.

Differentiating (3.115) and noting that

$$km\frac{d\,\delta r}{dt} \times \frac{dr}{dt} + km\frac{dr}{dt} \times \frac{d\,\delta r}{dt} = 0, \tag{3.119}$$

we find

$$\frac{d\,\delta K}{dt} = km\,\delta r \times \frac{d^2r}{dt^2} + kmr \times \frac{d^2\,\delta r}{dt^2} + \frac{d\,\Delta K}{dt}. \tag{3.120}$$

It now remains to insert (3.120) and (3.117) in Eq. (3.114). However, let us first somewhat simplify the expression for the torque variation (3.117). The variation $\delta M'$ of the correcting torque M' which compensates the horizontal component of the gravitational field is small and can be dropped. Now, using the first equation in (3.104), we get

$$\delta a \times F = \delta a \times (F_{x_0}x_0 + F_{y_0}y_0 + F_{z_0}z_0). \tag{3.121}$$

Dropping the product of δa by the sum of the first and second terms in parentheses (a second-order contribution) and seeing that to the same approximation

$$F_{z_0} = -\frac{\mu m}{r^2}, \tag{3.122}$$

we get

$$\delta M = \delta a \times \left(-\frac{\mu m}{r^2}z_0 + \frac{d^2r}{dt^2}\right) + \Delta M. \tag{3.123}$$

Inserting δa from (3.118), we thus get

$$\delta \boldsymbol{M} = km\,\delta \boldsymbol{r} \times \left(\frac{\mu}{r^2}\boldsymbol{z}_0 + \frac{d^2\boldsymbol{r}}{dt^2}\right) + m\,\Delta a \boldsymbol{z}_0 \times \frac{d^2\boldsymbol{r}}{dt^2} + \Delta \boldsymbol{M}. \tag{3.124}$$

We can now insert (3.124), (3.120) in Eq. (3.114). Collecting and simplifying we obtain, using the first equality in (3.116)*,

$$km\boldsymbol{r} \times \left(\frac{d^2\,\delta \boldsymbol{r}}{dt^2} + \frac{\mu\,\delta \boldsymbol{r}}{r^3}\right) = -\frac{d\,\Delta \boldsymbol{K}}{dt} + \Delta \boldsymbol{M} + m\,\Delta a \boldsymbol{z}_0 \times \frac{d^2\boldsymbol{r}}{dt^2}. \tag{3.125}$$

This is the vector equation of small oscillations about a relative equilibrium of a general Schuler-tuned system, which satisfies (with some instrumental errors) the conditions (3.106), (3.108), (3.313).

To obtain the scalar equations for δx and δy, we should insert in (3.125) the expressions for r and δr from (3.116) and project the resulting equation onto the axes x_0, y_0, using the equalities

$$\left.\begin{aligned} \frac{d\boldsymbol{z}_0}{dt}\cdot \boldsymbol{x}_0 &= \omega_{y_0}, & \frac{d^2\boldsymbol{z}_0}{dt^2}\cdot \boldsymbol{x}_0 &= \dot{\omega}_{y_0} + \omega_{x_0}\omega_{z_0},\\ \frac{d\boldsymbol{z}_0}{dt}\cdot \boldsymbol{y}_0 &= -\omega_{x_0}, & \frac{d^2\boldsymbol{z}_0}{dt^2}\cdot \boldsymbol{y}_0 &= -\dot{\omega}_{x_0} + \omega_{y_0}\omega_{z_0}, \end{aligned}\right\} \tag{3.126}$$

where ω_{x_0}, ω_{y_0}, ω_{z_0} are the projections of the absolute angular velocity of the $x_0y_0z_0$ axes on the corresponding directions.

After obvious simplifications, we thus obtain

$$\left.\begin{aligned} &\delta\ddot{x} + \left(\frac{\mu}{r^3} - \omega_{y_0}^2 - \omega_{z_0}^2\right)\delta x + \left(\omega_{x_0}\omega_{y_0} - \dot{\omega}_{z_0}\right)\delta y - \\ &- 2\omega_{z_0}\,\delta\dot{y} = -\left(\omega_{x_0}\omega_{z_0} + \dot{\omega}_{y_0}\right)\Delta r - 2\omega_{y_0}\,\Delta\dot{r} + \\ &+ \frac{1}{ma}\left(-\Delta\dot{K}_{y_0} - \omega_{z_0}\,\Delta K_{x_0} + \omega_{x_0}\,\Delta K_{z_0}\right) + \frac{\Delta M_{y_0}}{ma} + \\ &+ \frac{\Delta a}{a}\left[r\left(\dot{\omega}_{y_0} + \omega_{x_0}\omega_{z_0}\right) + 2\dot{r}\,\omega_{y_0}\right],\\ &\delta\ddot{y} + \left(\frac{\mu}{r^3} - \omega_{x_0}^2 - \omega_{z_0}^2\right)\delta y + \left(\omega_{x_0}\omega_{y_0} + \dot{\omega}_{z_0}\right)\delta x + \\ &+ 2\omega_{z_0}\,\delta\dot{x} = -\left(\omega_{y_0}\omega_{z_0} - \dot{\omega}_{x_0}\right)\Delta r + 2\omega_{x_0}\,\Delta\dot{r} + \\ &+ \frac{1}{ma}\left(\Delta\dot{K}_{x_0} + \omega_{y_0}\,\Delta K_{z_0} - \omega_{z_0}\,\Delta K_{y_0}\right) - \frac{\Delta M_{x_0}}{ma} - \\ &- \frac{\Delta a}{a}\left[r\left(\dot{\omega}_{x_0} - \omega_{y_0}\omega_{z_0}\right) + 2\dot{r}\,\omega_{x_0}\right]. \end{aligned}\right\} \tag{3.127}$$

These equations describe small oscillations of a Schuler-tuned system about a position of relative equilibrium.

Let us now derive the equations of perturbed motion for finite deviations of the axis z from relative equilibrium. We may clearly take

$$\Delta a = 0,\ \Delta K_{x_0} = \Delta K_{y_0} = \Delta K_{z_0} = 0,\ \Delta M_{x_0} = \Delta M_{y_0} = 0, \tag{3.128}$$

since the right-hand sides of Eqs. (3.127) corresponding to the instrumental errors enumerated in (3.128) retain their form for finite deviations as well.

Going over the derivation of Eqs. (3.127), we see that the homogeneous equation (3.125) is an exact equation for the perturbation δr, if we ignore the variation of the nonspherical gravitational component (as was done in the

*Andreev, V.D. Theory of Schuler-tuned systems. — Prikladnaya Matematika i Mekhanika, Vol. 29, No. 6. 1965.

derivation of this equation). Eqs. (3.127), on the other hand, are first-approximation equations. This is so because in taking the projections of the vector equation (3.125) onto the axes x_0, y_0 we used for δr its expression from (3.116)

$$\delta r = \delta x x_0 + \delta y y_0 + \Delta r z_0. \tag{3.129}$$

It is readily seen that (3.129) is valid only to first approximation. In exact equations, (3.129) should be replaced by

$$\left.\begin{aligned} &\delta r = \delta x x_0 + \delta y y_0 + \delta z z_0, \\ &(\delta x)^2 + (\delta y)^2 + (\delta z)^2 + 2r\,\delta z = 2r\,\Delta r + (\Delta r)^2. \end{aligned}\right\} \tag{3.130}$$

For small δx, δy, δz, when their squares are ignorable together with the square of Δr, the second equality in (3.130) gives

$$\delta z = \Delta r. \tag{3.131}$$

and the first equality in (3.130) thus reduces to (3.129).

Taking the projections of the vector equation (3.125) onto the axes x_0, y_0 and using (3.126), (3.128), we take δr from (3.130) and obtain a nonlinear system of scalar equations:

$$\left.\begin{aligned} &\delta\ddot{x} + \left(\frac{\mu}{r^3} - \omega_{y_0}^2 - \omega_{z_0}^2\right)\delta x + (\omega_{x_0}\omega_{y_0} - \dot{\omega}_{z_0})\,\delta y - \\ &\quad - 2\omega_{z_0}\,\delta\dot{y} + (\omega_{x_0}\omega_{z_0} + \dot{\omega}_{y_0})\,\delta z + 2\omega_{y_0}\,\delta\dot{z} = 0, \\ &\delta\ddot{y} + \left(\frac{\mu}{r^3} - \omega_{x_0}^2 - \omega_{z_0}^2\right)\delta y + (\omega_{x_0}\omega_{y_0} + \dot{\omega}_{z_0})\,\delta x + \\ &\quad + 2\omega_{z_0}\,\delta\dot{x} + (\omega_{y_0}\omega_{z_0} - \dot{\omega}_{x_0})\,\delta z - 2\omega_{x_0}\,\delta\dot{z} = 0, \\ &(\delta x)^2 + (\delta y)^2 + (\delta z)^2 + 2r\,\delta z = 2r\,\Delta r + (\Delta r)^2. \end{aligned}\right\} \tag{3.132}$$

For small δx, δy, δz, when Eq. (3.131) is valid, the nonlinear equations (3.132) reduce to the first-approximation linear equations (3.127).

§3.4. ANALOGY BETWEEN SCHULER-TUNED SYSTEMS AND TWO-ACCELEROMETER INERTIAL SYSTEMS

3.4.1. Comparison of the first group of error equations with the equations of oscillations of a gyropendulum system about relative equilibrium

The first group of error equations of an inertial system with two accelerometers oriented in the horizontal plane comprises Eqs. (2.31); the corresponding first-approximation equations are (2.28).

We will now compare these equations with Eqs. (3.132), (3.127) of the perturbed motion of a general Schuler-tuned gyropendulum system about the position of relative equilibrium with the z axis caged to the geocentric vertical. The purpose of this comparison is to establish a dynamic analogy between Schuler-tuned systems and two-accelerometer inertial systems.

We start with the first-approximation equations (2.28), (3.127). The left-hand sides of (2.28) and (3.127) are identical in form and in content. Indeed,

in both (2.28) and (3.127) δx, δy are the projections of δr on the axes x, y (x_0, y_0) of the co-moving system xyz ($x_0y_0z_0$), with the axis z (z_0) pointing along r. ω_x, ω_y, ω_z (ω_{x_0}, ω_{y_0}, ω_{z_0}) are the projections of the absolute angular velocity of these axes on the corresponding directions.

The first group of homogeneous error equations of an inertial system are thus identical in the first approximation to the equations of free oscillations of a Schuler-tuned system about relative equilibrium. It therefore appears that the two mechanical systems are dynamically similar in the absence of instrumental errors.

Let us now compare the right-hand sides of Eqs. (2.28) and (3.127) The right-hand sides contain two groups of terms: the first with the error $\Delta r\,(=\Delta h)$ in the external information concerning the distance from the Earth's center (the altitude), and the second with instrumental errors of the system elements. The terms containing Δr (and $\Delta\dot{r}$) in Eqs. (2.28) and (3.127) coincide. The other terms are different, however. This is quite natural since the terms with Δr and $\Delta\dot{r}$ are of common origin in both systems. The origin of the other terms, on the other hand, is different in the two systems. Note that the form of the right-hand sides of Eqs. (2.28) and Eqs. (3.127) is largely arbitrary. It depends on the particular instrumental errors which are taken into consideration in the derivation of the corresponding equations.

We have compared the first-approximation equations. Turning to Eqs. (2.31), (2.132) for finite deviations, we conclude that these equations also coincide. Two-accelerometer inertial systems and Schuler-tuned systems are thus dynamically similar in the large, as well as in the small.

We have so far compared the error equations (2.28), (2.31) with the equations of perturbed motion of a general Schuler-tuned systems. In §3.1 we derived Eqs. (3.53), (3.63) for the perturbed motion of the Schuler pendulum and Eqs. (3.86) of the perturbed motion of a two-gyro vertical and a Geckeler—Anschütz gyrocompass horizon. Since these devices are clearly particular cases of a general Schuler-tuned system, Eqs. (3.53), (3.63), and (3.86) should also reduce to Eqs. (2.28), (3.127) or (2.31), (2.132). Let us prove this proposition.

At a first glance, the left-hand sides of Eqs. (3.53) are not identical to the homogeneous equations (2.28), (3.127). However, substituting

$$\delta x = -r\beta, \quad \delta y = r\alpha \tag{3.133}$$

in (3.53), we get

$$\left.\begin{aligned}
&\delta\ddot{x} + \left(\frac{\mu}{r^3} - \omega_{y_0}^2 - \omega_{z_0}^2\right)\delta x + (\omega_{x_0}\omega_{y_0} - \dot{\omega}_{z_0})\,\delta y - \\
&- 2\omega_{z_0}\delta\dot{y} = -(\omega_{x_0}\omega_{z_0} + \dot{\omega}_{y_0})\,\Delta r - 2\omega_{y_0}\Delta\dot{r} + \\
&+ \Delta C\omega_{x_0}\omega_{z_0} - \frac{\Delta A}{am}(\dot{\omega}_{y_0} + \omega_{x_0}\omega_{z_0}) - \\
&- \omega_{y_0}\left(\frac{r\,\Delta\dot{a}}{a} + \frac{\Delta\dot{A}}{ma} - \frac{\dot{r}\,\Delta a}{a}\right), \\
&\delta\ddot{y} + \left(\frac{\mu}{r^3} - \omega_{z_0}^2 - \omega_{x_0}^2\right)\delta y + (\omega_{y_0}\omega_{x_0} + \dot{\omega}_{z_0})\,\delta x + \\
&+ 2\omega_{z_0}\delta\dot{x} = -(\omega_{y_0}\omega_{z_0} - \dot{\omega}_{x_0})\,\Delta r + 2\omega_{x_0}\Delta\dot{r} + \\
&+ \Delta C\omega_{y_0}\omega_{z_0} + \frac{\Delta A}{am}(\dot{\omega}_{x_0} - \omega_{y_0}\omega_{z_0}) + \\
&+ \omega_{x_0}\left(\frac{r\,\Delta\dot{a}}{a} + \frac{\Delta\dot{A}}{am} - \frac{\dot{r}\,\Delta a}{a}\right).
\end{aligned}\right\} \tag{3.134}$$

The geometrical meaning of the substitution of variables (3.133) is obvious. The left-hand sides of Eqs. (3.134) completely coincide with the left-hand sides of Eqs. (2.28), (3.127). The terms with Δr and $\Delta\dot{r}$ coincide in the right-hand sides of all the three systems. The terms with Δa and $\Delta\dot{a}$ in the right-hand sides of Eqs. (3.134) and (3.127) also exactly coincide (this can be shown using the equality $A=mar$). Remembering that $K_{x_0}=A\omega_{x_0}$, $K_{y_0}=A\omega_{y_0}$, $K_z=C\omega_{z_0}$, we can easily compare the remaining terms in the right-hand sides of Eqs. (3.134) and (3.127).

Let us now consider the equations of small oscillations of a two-gyro vertical (or a Geckeler—Anschütz gyrocompass horizon), Eqs. (3.86). At a first glance, Eqs. (3.86) and Eqs. (3.127), (2.28) have nothing in common. A more detailed analysis, however, disproves this first impression.

According to the last two equations in (3.86)

$$\left.\begin{aligned} \omega_y\gamma &= \frac{\Delta M_z}{mar\omega_{y_0}} - \omega_{x_0} + \omega_z\beta - \dot{\alpha},\\ 2B\cos\varepsilon_0\delta\varepsilon &= -mar(\omega_z\alpha+\dot{\beta}) - \frac{mar\,\Delta N}{2B\cos\varepsilon_0}. \end{aligned}\right\} \tag{3.135}$$

Inserting Eqs. (3.135) in the first two equations in (3.86), we obtain two coupled second-order differential equations for α and β. Making the substitution (3.133) in these equations, we obtain after simple manipulations

$$\left.\begin{aligned} &\delta\ddot{x} + \left(\frac{\mu}{r^3} - \omega_{y_0}^2 - \omega_{z_0}^2\right)\delta x - \dot{\omega}_{z_0}\delta y - 2\omega_{z_0}\delta\dot{y} =\\ &= \frac{1}{ma}\frac{d}{dt}\left(\frac{mar\,\Delta N}{2B\cos\varepsilon_0}\right) - \dot{\omega}_{y_0}\Delta r - 2\omega_{y_0}\Delta\dot{r} +\\ &+ \omega_{y_0}\left(\frac{\dot{r}\Delta a}{a} - \frac{r\Delta\dot{a}}{a}\right) + \omega_{z_0}\frac{\Delta M_z}{am\omega_{y_0}} + \frac{\Delta M_y}{am},\\ &\delta\ddot{y} + \left(\frac{\mu}{r^3} - \omega_{z_0}^2\right)\delta y + \dot{\omega}_{z_0}\delta x + 2\omega_{z_0}\delta\dot{x} =\\ &= -\frac{d}{dt}\left(\frac{\Delta M_z}{ma\,\omega_{y_0}}\right) - \frac{\Delta N r\omega_{z_0}}{2B\cos\varepsilon_0} -\\ &- \dot{r}\frac{\Delta M_z}{mar\omega_{y_0}} - \frac{\Delta M_x}{am} - \Delta r\omega_{y_0}\omega_{z_0}. \end{aligned}\right\} \tag{3.136}$$

Comparing the homogeneous equations (3.136) and (3.127), (2.28), we note that they coincide if we take $\omega_{x_0}=\omega_x=0$ in (3.127), (2.28). This is equivalent to projecting these equations on the axes $x_0y_0z_0$ (xyz), with the axis x_0 (x) lying in the plane through the absolute velocity vector of the point O and the Earth's center O_1. Now this is precisely the definition of the axes $x_0y_0z_0$ in Eqs. (3.136). The homogeneous system (3.136) is therefore equivalent to homogeneous equations (3.127), (2.28). The right-hand sides of these equations are also readily compared.

Eqs. (3.53), (3.86), (2.28), (3.127) discussed in the preceding are first-approximation equations. For the Schuler-pendulum we also have Eqs. (3.63) which describe free perturbed motion with finite deviations. We will now establish the equivalence of these equations with the homogeneous equations (2.31), (3.132), i.e.,

$$\left.\begin{aligned} &\delta\ddot{x} + \left(\frac{\mu}{r^3} - \omega_y^2 - \omega_z^2\right)\delta x + (\omega_x\omega_y - \dot{\omega}_z)\delta y -\\ &- 2\omega_z\delta\dot{y} + (\omega_x\omega_z + \dot{\omega}_y)\delta z + 2\omega_y\delta\dot{z} = 0,\\ &\delta\ddot{y} + \left(\frac{\mu}{r^3} - \omega_z^2 - \omega_x^2\right)\delta y + (\omega_y\omega_z + \dot{\omega}_z)\delta x +\\ &+ 2\omega_z\delta\dot{x} + (\omega_y\omega_z - \dot{\omega}_x)\delta z - 2\omega_z\delta\dot{z} = 0,\\ &(\delta x)^2 + (\delta y)^2 + (\delta z)^2 + 2r\,\delta z = 0. \end{aligned}\right\} \tag{3.137}$$

In (3.137) we make the substitution

$$\left.\begin{aligned} \delta x = -r\cos\alpha\sin\beta,\ \delta y = r\sin\alpha, \\ \delta z = r(\cos\alpha\cos\beta - 1). \end{aligned}\right\} \qquad (3.138)$$

The geometrical meaning of this substitution is again obvious.

Inserting (3.138) in the third equation in (3.137) we see that it is identically satisfied. Insertion of (3.138) in the first two equations in (3.137) and substitution of ω_x, ω_y, ω_z from (3.56) immediately gives Eqs. (3.63).

Note that we can further establish a complete analogy in the equations of the perturbed motion of a two-gyro vertical and the Geckeler—Anschütz gyrocompass horizon with Eqs. (3.137) for finite, as well as small, deviations. To obtain the equations of motion of the gyrovertical and the gyrocompass in case of finite deviations of the axes xyz from the axes $x_0y_0z_0$, we should insert in (3.68) the expressions for ω_x, ω_y, ω_z which follow from the matrix of direction cosines (3.33) (and not expressions (3.35), as in the derivation of Eqs. (3.86)). After that we eliminate γ and $2B\sin\varepsilon$; the last expression is not varied, as in the transformation from (3.83) to (3.86), but dropped altogether. Then we make the substitution (3.138), which gives Eqs. (3.137).

We have compared Eqs. (2.31) and (3.63), (3.132). These are exact equations for the perturbations δx, δy assuming a spherical gravitation field. In the derivation of the error equations (2.31) of a two-accelerometer system we ignored the variations of the nonspherical gravitation component. In the derivation of the equations of perturbed motion of a Schuler-tuned system, Eqs. (3.132), we omitted the variations of the torques compensating the effect of the nonspherical gravitation component. The terms neglected in the derivation are thus essentially the same in Eqs. (2.31) and Eqs. (3.132). It can be shown that by retaining the variations of the nonspherical component we introduce identical additional terms in these equations. The analogy between gyropendulum systems and two-accelerometer inertial systems is thus preserved in the nonspherical field also.

The situation is somewhat different with regard to the homogeneity of the gravitational field within the volume occupied by the system.

In the derivations of the equations of relative equilibrium for the Schuler pendulum and the equations of its oscillations about a relative equilibrium, we actually assumed that the effect of the Earth's gravitation on the pendulum can be reduced to a single resultant force applied to the center of mass of the pendulum. This assumption is clearly equivalent to the assumption of a homogeneous gravitation field within the volume occupied by the pendulum. In fact, however, the gravitational forces also produce a certain resultant torque about the center of mass.*

In a spherical field the resultant gravity torque appears only if the mass is distributed asymmetrically about the direction from the center of mass to the field center. The magnitude of this torque additionally depends on the geometrical size of the body. Calculations show** that for a Schuler pendulum in the form of a rigid body the torques due to field inhomogeneity may prove to be comparable with the restoring torque of the resultant force.

* This torque is considered, e.g., in lunar libration calculations. It must be taken into account in the treatment of the motion of an artificial Earth satellite about its center of mass. See, e.g., Beletskii, V.V. The motion of artificial Earth satellites about the center of mass. — In: "Iskusstvennye sputniki Zemli," No. 1. AN SSSR. 1958.

** Hoch H. Das physikalische Pendel im radialsymmetrischen Erdgewichtskraftfeld. — Zeitschrift für ang. Mathematik und Physik, Vol. 24, No. 5/6. 1944.

Introduction of the field inhomogeneity torques in the theory of the Schuler pendulum leads to a definite analogy with some gravity attitude stabilizing systems (not containing gyroscopes).* This analogy, however, is not as immediate and meaningful as the analogy with two-accelerometer inertial systems. We will not go into this problem in any more detail.

Note that Eqs. (2.28), (2.31) constitute only the first group of error equations of an inertial system. There is also the second group of error equations which characterize the gyro errors of the system.

In § 1 of this chapter we have noted that a single Schuler-tuned system is insufficient for measuring the coordinates of the object. This system only fixes the direction of the geocentric vertical. To find the coordinates, we should introduce additional gyroscopes, say, which give the fixed-orientation axes ξ,η,ζ. The angles between the z axis of the pendulum and the axes ξ,η,ζ of the gyroscopes then determine (together with r) the coordinates of the moving object. The total position errors thus include the attitude errors of the gyroscopes associated with the axes ξ,η,ζ, i.e., the second group of error equations of the inertial system. Moreover, a pendulum system without gyroscopes (e.g., the Schuler pendulum as such) does not fix any reference direction in the horizontal plane. A gyropendulum system (a gyrocompass, say), on the other hand, gives a natural reference direction. In this case, the error in the determination of the north is associated with both gyro and vertical errors, as in an inertial system with two horizontal accelerometers.

The above analysis establishes a complete dynamic analogy between a Schuler-tuned gyropendulum navigation system and a two-accelerometer inertial navigation system. Fundamentally, the two systems are equivalent. Although the Schuler-tuned system uses no accelerometers as such, the pendulum caged to the geocentric vertical essentially constitutes a two-degrees-of-freedom accelerometer which measures the horizontal accelerations. These accelerations are directly integrated by the pendulum system in the course of its motion.

3.4.2. Sufficient stability conditions for two-accelerometer inertial systems

The analogy drawn in 3.4.1 is highly important in the theory of inertial navigation. It links up two entirely different trends in the development of the theory and permits free interchange of results in both directions.

All the results of Chapter 2 obtained in the analysis of stability of Eqs. (2.28) can thus be extended to Schuler-tuned systems. On the other hand, various results in the theory of gyrocompass horizons and two-gyro verticals obtained by Bulgakov, Ishlinskii, Merkin, and Koshlyakov** become much more meaningful. For example, Ishlinskii† used a special complex-valued substitution to integrate Eq. (3.86) for the case $\mu/r^3 \gg \omega_y^2, \omega_x^2$; $r = \text{const}$. Turning to Eqs. (2.28) we note immediately that for $r = \text{const}$, $\omega_x = \omega_y = 0$, substitution of variables (2.32) with

$$\vartheta = \int_0^t \omega_{z_0}\, dt \tag{3.139}$$

* See § 3.4 of my "Theory of Inertial Navigation: Autonomous Systems," 1966.

** See references at the end of the book.

† See footnote on p. 117.

reduces these homogeneous equations to directly integrable harmonic equations $\delta\ddot{x}' + \omega_0^2 \delta x' = 0$, $\delta\ddot{y}' + \omega_0^2 \delta y' = 0$. This result sheds new light on the reducibility of the equations of the gyrocompass horizon and the two-gyro vertical* and on the relationship between Eqs. (3.86) and Geckeler's equations in the gyrocompass theory. **

On the other hand, the analogy with Schuler-tuned systems leads to the sufficient conditions for stability of an inertial system with two horizontal accelerometers in an object moving along a parallel.

The problem reduces to the investigation of nonlinear equations (2.31) or equations (3.137) (since the right-hand sides should be set equal to zero). For $r = \text{const}$, $\omega_{x_0} = 0$, $\omega_{y_0} = \text{const}$, $\omega_{z_0} = \text{const}$ (motion along a parallel), substitution (3.138) gives Eqs. (3.63), which can be written in the form

$$\left.\begin{aligned}
&\ddot{\beta} + \dot{\alpha}^2 \sin\beta\cos\beta + \\
&\quad + 2\dot{\alpha}(-\omega_{y_0}^2 \sin\alpha\cos^2\beta + \omega_{z_0}^2 \cos\alpha\cos^2\beta) + \\
&\quad + (\omega_0^2 - \omega_{z_0}^2\cos\alpha\cos\beta - \omega_{y_0}^2)\sin\alpha\cos\alpha - \\
&\quad - \omega_{y_0}^2 \sin^2\alpha\sin\beta\cos\beta + \\
&\quad + \omega_{y_0}\omega_{z_0}(2\cos\alpha\cos\beta - 1)\sin\alpha\sin\beta = 0, \\
&\ddot{\alpha}\cos\beta - 2\dot{\alpha}\dot{\beta}\sin\beta + \\
&\quad + 2\dot{\beta}(\omega_{y_0}\sin\alpha\cos\beta - \omega_{z_0}\cos\alpha\cos\beta) + \\
&\quad + \left[\frac{\mu}{r^3} - \omega_{y_0}^2(1 - \cos\alpha\cos\beta) - \omega_{z_0}^2\cos\alpha\cos\beta\right]\sin\alpha + \\
&\quad + \omega_{y_0}\omega_{z_0}(\sin^2\alpha\cos\beta - \cos^2\alpha\cos\beta + \cos\alpha) = 0.
\end{aligned}\right\} \qquad (3.140)$$

These equations are also nonlinear, but they have a first integral,† which simplifies the analysis. To find the first integral, it suffices to multiply the first equation in (3.140) by $\dot{\beta}$, the second by $\dot{\alpha}\cos\beta$, and to add them up. Integration then gives

$$\begin{aligned}
V = {} & (\dot{\alpha}\cos\beta)^2 + \dot{\beta}^2 - 2\omega_0^2\cos\alpha\cos\beta + \\
& + \omega_{z_0}^2\cos^2\alpha\cos^2\beta + \omega_{y_0}^2(\sin^2\alpha\cos^2\beta + 2\cos\alpha\cos\beta) + \\
& + 2\omega_{y_0}\omega_{z_0}(\sin\alpha\cos\beta - \sin\alpha\cos\alpha\cos\beta) = \text{const}.
\end{aligned} \qquad (3.141)$$

Let V^0 be the value of the function V for $\dot{\alpha} = \dot{\beta} = \alpha = \beta = 0$. Clearly,

$$V^0 = -2\omega_0^2 + 2\omega_{y_0}^2 + \omega_{z_0}^2. \qquad (3.142)$$

Consider the function

$$\begin{aligned}
W = V - V^0 = {} & (\dot{\alpha}\cos\beta)^2 + \dot{\beta}^2 - 2\omega_0^2(\cos\alpha\cos\beta - 1) + \\
& + \omega_{z_0}^2(\cos^2\alpha\cos^2\beta - 1) + \omega_{y_0}^2(\sin^2\alpha\cos^2\beta + 2\cos\alpha\cos\beta - 2) + \\
& + 2\omega_{y_0}\omega_{z_0}(\sin\alpha\cos\beta - \sin\alpha\cos\alpha\cos\beta).
\end{aligned} \qquad (3.143)$$

The total time derivative of the function W vanishes in virtue of (3.140):

$$\frac{dW}{dt} = 0. \qquad (3.144)$$

* Koshlyakov, V.N. Reducibility of the equations of motion of a gyrocompass horizon. — Prikladnaya Matematika i Mekhanika, Vol. 25, No. 5. 1961.

** Geckeler, W. Kreiselmechanik des Anschütz — Raumkompasses, Ingeneur Archiv, Vol. VI, No. 4, Berlin, 1935.

† Andreev, V.D. Errors in inertial navigation systems. — Izvestiya AN SSSR, tech. cyber., No. 2. 1964.

Therefore, a sufficient condition of Lyapunov stability of Eqs. (3.140) is that W be an alternating-sign function.

The series-expansion of W in powers of $\dot{\alpha}$, $\dot{\beta}$, α, β starts with the quadratic terms:

$$W = \dot{\alpha}^2 + \dot{\beta}^2 + \alpha^2(\omega_{y_0}^2 - \omega_z^2) + \beta^2(\omega_{y_0}^2 - \omega_{y_0}^2 - \omega_z^2) + \ldots \tag{3.145}$$

Applying the Sylvester test to the quadratic form in the right-hand side of Eq. (3.145), we obtain the condition of positive definiteness in the form of the inequality

$$\omega_{y_0}^2 - \omega_{y_0}^2 - \omega_z^2 > 0. \tag{3.146}$$

This inequality is thus a sufficient condition of stability of the relative equilibrium of a Schuler-tuned system and at the same time a sufficient condition of stability of a two-accelerometer system for motion along a parallel. It coincides with condition (2.49), previously derived from first-approximation equations.*

3.4.3. Damping of oscillations in a Schuler-tuned system

When condition (3.146) is satisfied, the perturbed motion of a Schuler-tuned system is Lyapunov stable. This condition, however, does not ensure asymptotic stability, and we will now consider the asymptotic stability of the relative equilibrium of the Schuler pendulum.

To ensure asymptotic stability, we introduce internal dissipation, i.e., terms of the form $h\dot{\alpha}$, $h\dot{\beta}$ in the left-hand sides of Eqs. (3.53). However, these terms cannot be formed without external guidance information. We have neither the angles α, β nor the corresponding rates of change $\dot{\alpha}$, $\dot{\beta}$, and it is therefore impossible to form (and apply to the pendulum) torques proportional to these quantities. In general, knowledge of $\dot{\alpha}$, $\dot{\beta}$ is not strictly essential for damping the pendulum oscillations. It suffices to have sign $\dot{\alpha}$ and sign $\dot{\beta}$. These functions, however, are not forthcoming either. If we use external information, we can only speak of damping torques proportional to the absolute rotation velocity of the pendulum ω_x, ω_y, ω_z, and not to the relative velocities $\dot{\alpha}$, $\dot{\beta}$. Torques of this kind appear, e.g., if viscous friction is introduced in the pendulum pivot.

Consider small oscillations of the pendulum about the relative equilibrium in the case when its pivot moves with constant velocity along the arc of the great circle on a fixed-orientation sphere concentric with the Earth.

For this motion we may take

$$\omega_x = \omega_z = 0, \quad r = \text{const}. \tag{3.147}$$

Taking $A = mar$ and adding the viscous friction torques $-k_x\omega_x$, $-k_y\omega_y$, $-k_z\omega_z$ in the right-hand sides, we obtain from Eqs. (3.41) (the case of no instrumental errors)

$$\left.\begin{aligned} \ddot{\alpha} + \tfrac{\mu}{r^3}\alpha + k_x\dot{\alpha} - k_x\omega_{y_0}\gamma + \tfrac{C}{mar}(-\omega_{y_0}^2\alpha + \omega_{y_0}\dot{\gamma}) = 0, \\ \ddot{\beta} + \left(\tfrac{\mu}{r^3} - \omega_{y_0}^2\right)\beta + k_y\dot{\beta} + k_y\omega_{y_0} = 0, \\ C\tfrac{d}{dt}(-\omega_{y_0}\alpha + \dot{\gamma}) + k_z(-\omega_{y_0}\alpha + \dot{\gamma}) = 0. \end{aligned}\right\} \tag{3.148}$$

* In application to gyrocompasses this condition was derived in Merkin, D.R. Stability of motion of a gyroframe. — Prikladnaya Matematika i Mekhanika, Vol. 25, No. 6. 1961.

If

$$C \neq 0, \tag{3.149}$$

the third equation in (3.148) is readily integrated. This gives

$$-\omega_{y_0}\alpha + \dot{\gamma} = \left(-\omega_{y_0}^0\alpha^0 + \dot{\gamma}^0\right)\exp\left(-\frac{k_z}{C}t\right), \tag{3.150}$$

and Eqs. (3.148) thus take the form

$$\left.\begin{aligned} &\ddot{\alpha} + k_x\dot{\alpha} + \omega_0^2\alpha - k_x\omega_{y_0}\gamma = -\frac{C\omega_{y_0}}{mar}\left(-\omega_{y_0}^0\alpha^0 + \dot{\gamma}^0\right)\exp\left(-\frac{k_z t}{C}\right), \\ &\ddot{\beta} + k_y\dot{\beta} + \left(\omega_0^2 - \omega_{y_0}^2\right)\beta = -k_y\omega_{y_0}, \\ &\dot{\gamma} - \omega_{y_0}\alpha = \left(-\omega_{y_0}^0\alpha^0 + \dot{\gamma}^0\right)\exp\left(-\frac{k_z}{C}t\right). \end{aligned}\right\} \tag{3.151}$$

Here, as before ω_0^2 stands for μ/r^3.

If, conversely,

$$C = 0, \tag{3.152}$$

we have from the third equation in (3.148)

$$\dot{\gamma} - \omega_{y_0}\alpha = 0. \tag{3.153}$$

Eqs. (3.148) then take the form

$$\left.\begin{aligned} &\ddot{\alpha} + k_x\dot{\alpha} + \omega_0^2\alpha - k_x\omega_{y_0}\gamma = 0, \\ &\ddot{\beta} + k_y\dot{\beta} + \left(\omega_0^2 - \omega_{y_0}^2\right)\beta = -k_y\omega_{y_0}, \\ &\dot{\gamma} - \omega_{y_0}\alpha = 0. \end{aligned}\right\} \tag{3.154}$$

The second equation in (3.151) coincides with the second equation in (3.154). Both are solved independently of the other equations. The equation

$$\ddot{\beta} + \left(\omega_0^2 - \omega_{y_0}^2\right)\beta + k_y\dot{\beta} = -k_y\omega_{y_0} \tag{3.155}$$

has a particular solution

$$\tilde{\beta} = -\frac{k_y\omega_{y_0}}{\omega_0^2 - \omega_y^2}. \tag{3.156}$$

For $\omega_0^2 - \omega_{y_0}^2 > 0$, the roots of the characteristic equation (3.155) have negative real parts, and the system is thus asymptotically stable with respect to the angle β. With time the solution of the homogeneous equation is damped, and we are left only with a constant deviation $\tilde{\beta}$.

The homogeneous systems comprising the first and third equations in (3.151), (3.154) coincide:

$$\left.\begin{aligned} &\ddot{\alpha} + k_x\dot{\alpha} + \omega_0^2\alpha - k_x\omega_{y_0}\gamma = 0, \\ &\dot{\gamma} - \omega_{y_0}\alpha = 0. \end{aligned}\right\} \tag{3.157}$$

The characteristic equation of (3.157) is

$$p^3 + k_x p^2 + \omega_0^2 p + k_x\omega_{y_0}^2 = 0. \tag{3.158}$$

Applying the Hurwitz test to this equation we find that for $\omega_0^2 - \omega_{y_0}^2 > 0$ the roots of Eq. (3.158) have negative real parts, so that the system is again asymptotically stable.

The first and third equations in (3.154) are homogeneous. The first and third equations in (3.151) have nonzero right-hand sides. However, these right-hand sides go to zero with time, and therefore the solutions of Eqs. (3.154) and (3.151) gradually draw closer approaching for $t \to \infty$ the common values

$$\alpha = 0, \quad \gamma = 0, \quad \beta = \tilde{\beta} = -\frac{k_y \omega_{y_1}}{\omega_0^2 - \omega_{y_0}^2}. \tag{3.159}$$

Damping of the Schuler pendulum thus gives rise to a velocity error $\tilde{\beta}$. This error can be eliminated only by using additional information on the velocity of the pivot. In what follows we shall see that a similar situation obtains when damping is introduced in the first group of error equations of an inertial system.

Chapter 4

SIMPLIFIED OPERATION EQUATIONS OF IDEAL AND PERTURBED TWO-ACCELEROMETER INERTIAL SYSTEMS

§4.1. GENERAL CONSIDERATIONS

The ideal operation equations of a two-accelerometer inertial system derived in Chapter 3 are a direct outcome of the fundamental equation of inertial navigation (1.3). In deriving these equations we did not make any additional simplifying assumptions apart from those employed in the derivation of the fundamental equation. Note that the results of the simplifications introduced in the derivation of Eq. (1.3) lie far beyond the practicable accuracy limits of inertial systems.*

The form of the ideal operation equations is determined by the choice of the coordinate grid for navigating the moving object and by the orientation of the sensitive axes of the gyroscopes and accelerometers. The right-hand sides of these equations are functions of the accelerometer and gyro readings, the coordinates of the object, and the rate of their change. They also include parameters which characterize the figure of the Earth, its motion and gravitational field.

Under particular circumstances, i.e., for certain groups of moving objects and specified navigation accuracy, some terms in the right-hand sides of the ideal operation equations may prove to be small. These terms naturally can be omitted from the navigation equations, resulting in a substantial simplification. Moreover, if the programmed trajectories of the object are given, an appropriate choice of the coordinate grid and the accelerometer attitude will suppress certain terms or relations in the ideal equations.

The number of individual terms and relations is not the only factor to be considered. The form of the remaining relations and terms is also of considerable significance. If it is judiciously chosen, the implementation of the navigation system is often substantially simplified.

Our problem is thus directly related to the technical implementation of the inertial navigation system; it will help to reduce the number of elements and devices used in the system, simplify their operation, and increase their reliability.

Note that this problem is closely linked with inertial system design and especially with the availability of elements and technical devices. This apparently precludes a comprehensive theoretical analysis. We can only analyze the various typical cases occurring in practice.

* See my "Theory of Inertial Navigation: Autonomous Systems." — Nauka, 1966.

In the following we will confine our treatment to systems which measure orthodromic or geographical coordinates.

An inertial system can be visualized as a gyroscopic frame with its unperturbed z axis pointing along the radius-vector r from the Earth's center to the suspension point (along the geocentric vertical) or along the normal to the level ellipsoid through the suspension point of the gyroframe (on the Earth's surface, this is the geographical vertical). The accelerometer sensitive axes n_x, n_y are assumed to coincide with the axes x and y of the gyroframe. There is no z axis accelerometer, but the system is aided by external information on the distance r from the Earth's center or, more precisely, on the altitude h.

In the following sections we will first consider the various representations and simplifications of the ideal operation equations associated with particular orientations of the axes x, y (and the accelerometers n_x, n_y) in the plane of the geocentric or the geographical horizon. After that we will investigate the simplifications associated with the small eccentricity of the Earth and the small deviation of its gravitational field from spherical. Finally, we will analyze the simplifications resulting from various constraints on the allowed trajectories of the object. These constraints include such characteristics of the object as its velocity, range, time of motion, altitude, proximity to some orthodromy surface, etc.

We have already noted in the preceding that the feasibility of various simplifications is closely related to the accuracy standards of the inertial system. Therefore, the analysis of the simplified ideal operation equations should be supplemented by an analysis of the error equations. Remember that when the ideal operation equations are changed (simplified), the error equations may also change (sometimes becoming more complex).

Note that error equations can be simplified independently of the ideal operation equations. These simplifications may stem from the peculiar features of motion. We have already used this approach in the case of plane motion, slow motion, and Keplerian motion. A number of other possibilities will be discussed in the following.

§4.2. SIMPLIFICATIONS IN AN INERTIAL SYSTEM MEASURING THE ORTHODROMIC COORDINATES

4.2.1. Exact ideal operation equations. Error equations

If the accelerometers of an inertial system lie in the plane of the geocentric horizon pointing along the co-moving orthodromic axes, the ideal operation equations are Eqs. (2.23), and the error equations (projected onto the co-moving orthodromic axes) are Eqs. (2.28), (1.96), (1.97), (1.118).

Eqs. (2.23), (2.28), (1.96), (1.97), (1.118) were derived as a particular case from the general equations of inertial navigation in curvilinear coordinates. Before trying to simplify the relevant equations, we should derive the exact ideal operation equations for the particular system being considered. This will elucidate the actual meaning of the various equations and the origin of the different terms and thus help us in further analysis.

To ensure a fairly comprehensive treatment, we will derive ideal operation equations for a somewhat more general case than Eqs. (2.23), assuming an arbitrary orientation of the accelerometers n_x and n_y in the plane of the geocentric horizon (the sensitive axes of these accelerometers remain perpendicular to one another).

We start with the fundamental relations of the kinematics of co-moving orthodromic axes and co-moving axes of arbitrary azimuthal orientation on a sphere concentric with the Earth.

Let as before the fundamental frame of reference be $O_1\xi_*\eta_*\zeta_*$ with its origin at the center of the Earth and fixed orientation. The axes fixed to the Earth are denoted as always by $O_1\xi\eta\zeta$. The axis ζ is directed along the Earth's spin axis (coinciding with its symmetry axis), and the axis ξ lies in the plane of the Greenwich meridian. If the axes $O_1\xi_*\eta_*\zeta_*$ and $O_1\xi\eta\zeta$ initially coincide (*this assumption does not involve loss of generality*), their relative orientation at any subsequent time is described by the matrix of direction cosines (1.7).

The orthodromic axes $O_1\xi'\eta'\zeta'$ which occupy some arbitrary though fixed position relative to the Earth are defined by the direction cosines β_{ij}, (1.16), relative to the axes $O_1\xi\eta\zeta$. The elements of matrix (1.16) are constant and can be expressed,* e.g., in terms of the geocentric (or geographical) coordinates of any two given points on the Earth's surface, lying in the plane $O_1\xi'\eta'$ of the axes $O_1\xi'\eta'\zeta'$.

The position of an arbitrary point O in the axes $O_1\xi'\eta'\zeta'$ is determined by its *orthodromic coordinates* (*see Figure* 1.5): *the distance* r *from the* Earth's center O_1, the angle z between the line O_1O and the plane $O_1\xi'\eta'$, and the angle S between the axis ξ' and the projection of the line O_1O on the plane $O_1\xi'\eta'$.

We now introduce the co-moving orthodromic axes $x_3y_3z_3$ (see Figure 1.5). Its origin is at the point O, the axis z_3 is directed along O_1O, the axis y_3 is normal to z_3 in the plane through the point O and the axis ζ', and the axis x_3 complements the axes y_3 and z_3 to a right orthogonal trihedron. The relative orientation of the axes $\xi'\eta'\zeta'$ and $x_3y_3z_3$ is described by the following matrix of direction cosines:

	x_3	y_3	z_3
ξ'	$-\sin S$	$-\sin z\cos S$	$\cos z\cos S$
η'	$\cos S$	$-\sin z\sin S$	$\cos z\sin S$
ζ'	0	$\cos z$	$\sin z$.

(4.1)

Together with matrices (1.16) and (1.7) this table defines the orientation of the axes $x_3y_3z_3$ in the $O_1\xi\eta\zeta$ and $O_1\xi_*\eta_*\zeta_*$ frames.

The position of the point O relative to the axes $O_1\xi\eta\zeta$ fixed to the Earth is determined (see Figure 1.4) by the geocentric coordinates r, φ, λ. These coordinates are defined in the axes $O_1\xi\eta\zeta$ in the same way as the coordinates r, z, S are defined in the axes $O_1\xi'\eta'\zeta'$. Then writing $x_2y_2z_2$ for the co-moving geocentric axes, we have the direction cosines

	x_2	y_2	z_2
ξ	$-\sin\lambda$	$-\sin\varphi\cos\lambda$	$\cos\varphi\cos\lambda$
η	$\cos\lambda$	$-\sin\varphi\sin\lambda$	$\cos\varphi\sin\lambda$
ζ	0	$\cos\varphi$	$\sin\varphi$.

(4.2)

which are analogous to direction cosines (4.1).

* See Appendix II.

Since the axes z_1 and z_2 coincide, we have from (4.1), (4.2), and (1.16)

$$\left.\begin{aligned}\cos\varphi\cos\lambda &= \beta_{11}\cos z\cos S + \beta_{12}\cos z\sin S + \beta_{13}\sin z,\\ \cos\varphi\sin\lambda &= \beta_{21}\cos z\cos S + \beta_{22}\cos z\sin S + \beta_{23}\sin z,\\ \sin\varphi &= \beta_{31}\cos z\cos S + \beta_{32}\cos z\sin S + \beta_{33}\sin z;\end{aligned}\right\} \qquad (4.3)$$

these are the relations between the geocentric and the orthodromic coordinates.

The relative orientation of the axes $x_2y_2z_2$ and $x_3y_3z_3$ is described by matrix (1.19), where the angle ψ is defined by Eqs. (1.20) which can be obtained from the matrices of direction cosines (4.1), (4.2), (1.16).

If the axes ξ', η', ζ' and ξ, η, ζ coincide, the orthodromic coordinates r, S, z reduce to the geocentric coordinates r, λ, φ. Indeed, in this case in (1.16) $\beta_{11}=\beta_{22}=\beta_{33}=1$, and the other elements are zero. From (4.3) we thus obtain

$$\begin{aligned}\cos\varphi\cos\lambda &= \cos z\cos S,\\ \cos\varphi\sin\lambda &= \cos z\sin S,\\ \sin\varphi &= \sin z.\end{aligned}$$

Moreover, by Eqs. (1.20), $\psi = 0$ and the axes $x_2y_2z_2$ and $x_3y_3z_3$ also coincide.

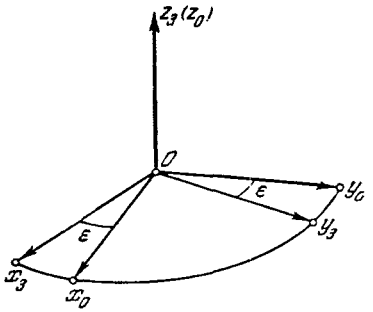

FIGURE 4.1.

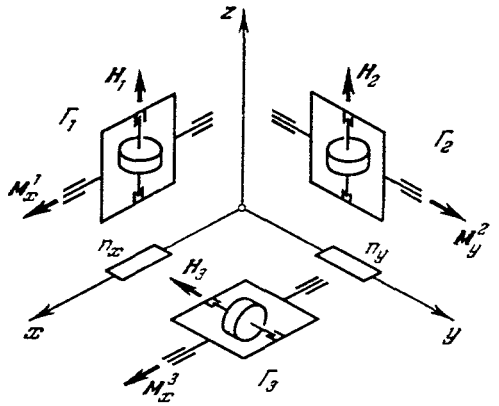

FIGURE 4.2.

Besides the geocentric and orthodromic co-moving axes $x_2y_2z_2$ and $x_3y_3z_3$ we introduce an arbitrary co-moving system $Ox_0y_0z_0$ on a sphere concentric with the Earth. The attitude of these axes (Figure 4.1) relative to the axes $x_3y_3z_3$ is specified by the following matrix of direction cosines:

	x_3	y_3	z_3
x_0	$\cos\varepsilon$	$\sin\varepsilon$	0
y_0	$-\sin\varepsilon$	$\cos\varepsilon$	0
z_0	0	0	1.

(4.4)

The two-accelerometer inertial system in this case (Figure 4.2) constitutes a platform (the axes $Oxyz$) connected to the object with a three-degrees-of-freedom gimbal (not shown in Figure 4.2). Three gyroscopes Γ_1, Γ_2, Γ_3

with angular momenta H_1, H_2, H_3 are mounted on the platform in two-degrees-of-freedom gimbals. Torquers on the gyro precession axes set up the torques M_x^1, M_y^2, M_x^3. The platform with the gyroscopes is thus a controlled spatial gyroframe. The control torques M_x^1, M_y^2, M_x^3 make the frame precess with the angular velocities

$$\omega_x = -\frac{M_y^2}{H_2}, \quad \omega_y = \frac{M_x^1}{H_1}, \quad \omega_z = \frac{M_x^3}{H_3}. \tag{4.5}$$

Two accelerometers n_x and n_y with sensitive axes along x and y axes are rigidly fixed to the gyroframe platform. If the axes xyz and $x_0y_0z_0$ initially coincide, the accelerometer readings can be used to form the control torques M_x^1, M_y^2, M_x^3, so that irrespective of the particular motion of the object we identically have

$$\omega_x \equiv \omega_{x_0}, \quad \omega_y \equiv \omega_{y_0}, \quad \omega_z \equiv \omega_{z_0}, \tag{4.6}$$

and the axes permanently coincide.

Let us derive expressions for ω_{x_0}, ω_{y_0}, ω_{z_0} in terms of the accelerometer readings. According to the fundamental equation of inertial navigation, Eq. (1.3), the accelerometer readings are

$$n_{x_0} = w_{x_0} - g_{x_0}, \quad n_{y_0} = w_{y_0} - g_{y_0}, \tag{4.7}$$

where w_{x_0}, w_{y_0} are the x_0 and y_0 projections of the absolute acceleration* of the point O where the accelerometer sensitive masses are.

Differentiation of the vector $v = \frac{dr}{dt}$ in the moving system gives Euler's equations

$$\left.\begin{aligned} w_{x_0} &= \dot{v}_{x_0} + \omega_{y_0} v_{z_0} - \omega_{z_0} v_{y_0}, \\ w_{y_0} &= \dot{v}_{y_0} + \omega_{z_0} v_{x_0} - \omega_{x_0} v_{z_0}, \\ w_{z_0} &= \dot{v}_{z_0} + \omega_{x_0} v_{y_0} - \omega_{y_0} v_{x_0}, \end{aligned}\right\} \tag{4.8}$$

where dot denotes differentiation with respect to time.

Consider some point P whose position in the axes $Ox_0y_0z_0$ is determined by the coordinates x_0^P, y_0^P, z_0^P. In accordance with the vector equality

$$v = \dot{r} + \omega \times r,$$

the projections $v_{x_0}^P$, $v_{y_0}^P$, $v_{z_0}^P$ of the absolute velocity of the point P are given by

$$\left.\begin{aligned} v_{x_0}^P &= \dot{x}_0^P + v_{x_0} + \omega_{y_0} z_0^P - \omega_{z_0} y_0^P, \\ v_{y_0}^P &= \dot{y}_0^P + v_{y_0} + \omega_{z_0} x_0^P - \omega_{x_0} z_0^P, \\ v_{z_0}^P &= \dot{z}_0^P + v_{z_0} + \omega_{x_0} y_0^P - \omega_{y_0} x_0^P. \end{aligned}\right\} \tag{4.9}$$

Here v_{x_0}, v_{y_0}, v_{z_0} are the projections of the absolute velocity of the origin of the $Ox_0y_0z_0$ axes (the point O) entering Eqs. (4.8).

Take the point P to be the Earth's center O_1. Then

$$x_0^P = 0, \quad y_0^P = 0, \quad z_0^P = -r. \tag{4.10}$$

* More precisely the acceleration in the $O_1\xi_0\eta_0\zeta_0$ frame in which Eq. (1.3) is written.

Since the point O_1 is stationary relative to the axes $O_1\xi_1\eta_1\zeta_1$ (and it is in this frame of reference that the velocities and accelerations are measured), we have

$$v^P_{x_1} = v^P_{y_1} = v^P_{z_1} = 0. \tag{4.11}$$

Inserting (4.10), (4.11) in (4.9), we find

$$v_x = r\omega_{y_1}, \quad v_{y_1} = -r\omega_x, \quad v_z = \dot{r}. \tag{4.12}$$

From (4.7), (4.8), and (4.12), we have

$$\left.\begin{aligned} v_{x_1} &= \int_0^t (n_{x_1} + \omega_{z_1} v_{y_1} - \dot{r}\omega_{y_1} + g_{x_1})\, dt + v^0_{x_1}, \\ v_{y_1} &= \int_0^t (n_{y_1} + \dot{r}\omega_{x_1} - \omega_{z_1} v_{x_1} + g_{y_1})\, dt + v^0_{y_1}, \\ \omega_{x_1} &= -\frac{v_{y_1}}{r}, \qquad \omega_{y_1} = \frac{v_{x_1}}{r}. \end{aligned}\right\} \tag{4.13}$$

Continuing the construction of the ideal operation equations, we express the projections ω_x, ω_y, ω_z of the absolute angular velocity of the axes xyz in terms of the orthodromic coordinates z, S and their derivatives $\dot{z}$, $\dot{S}$. From the definition of the axes $x_1y_1z_1$, $x_3y_3z_3$, using the matrix (4.4), we have

$$\left.\begin{aligned} \omega_x &= \omega_{x_3}\cos\varepsilon + \omega_{y_3}\sin\varepsilon, \\ \omega_y &= -\omega_{x_3}\sin\varepsilon + \omega_{y_3}\cos\varepsilon, \\ \omega_z &= \omega_{z_3} + \dot{\varepsilon}. \end{aligned}\right\} \tag{4.14}$$

where ω_{x_3}, ω_{y_3}, ω_{z_3} are the projections of the absolute angular velocity of the axes $x_3y_3z_3$ on the corresponding directions.

The axes $x_3y_3z_3$ rotate together with the axes $\xi'\eta'\zeta'$ which spin with the Earth; however, they turn additionally because the coordinates z and S change with time. Therefore,

$$\left.\begin{aligned} \omega_{x_3} &= -\dot{z} + u_{x_3}, \quad \omega_{y_3} = \dot{S}\cos z + u_{y_3}, \\ \omega_{z_3} &= \dot{S}\sin z + u_{z_3}, \end{aligned}\right\} \tag{4.15}$$

where u_{x_3}, u_{y_3}, u_{z_3} are the projections of the Earth's spin vector u on the corresponding axes. Since the vector u points along the axis ζ, we use matrices (4.1), (1.16) to find the direction cosines of the angles between this axis and the axes x_3, y_3, z_3 and write

$$\left.\begin{aligned} u_{x_3} &= u(-\beta_{31}\sin S + \beta_{32}\cos S), \\ u_{y_3} &= u(-\beta_{31}\sin z\cos S - \beta_{32}\sin z\sin S + \beta_{33}\cos z), \\ u_{z_3} &= u(\beta_{31}\cos z\cos S + \beta_{32}\cos z\sin S + \beta_{33}\sin z). \end{aligned}\right\} \tag{4.16}$$

From the second and the third equality in (4.15) we also have

$$\omega_{z_3} = (\omega_{y_3} - u_{y_3})\operatorname{tg} z + u_{z_3}. \tag{4.17}$$

Eqs. (4.14)—(4.17) enable us to express z and S in terms of ω_{x_3}, ω_{y_3}:

$$\left.\begin{aligned} z &= \int_0^t (-\omega_{x_0}\cos\varepsilon + \omega_{y_0}\sin\varepsilon + u_{x_3})\,dt + z^0, \\ S &= \int_0^t \frac{1}{\cos z}(\omega_{x_0}\sin\varepsilon + \omega_{y_0}\cos\varepsilon - u_{y_3})\,dt + S^0, \\ \varepsilon &= \int_0^t [\omega_{z_0} - (\omega_{x_0}\sin\varepsilon + \omega_{y_0}\cos\varepsilon - u_{y_3})\operatorname{tg} z + u_{z_3}]\,dt + \varepsilon^0. \end{aligned}\right\} \qquad (4.18)$$

Turning to Eqs. (4.13), (4.18), (4.16), we see that additional expressions for g_{x_0}, g_{y_0} and r are required. Using Eqs. (4.4), (1.20) and matrix (1.19) we get

$$\left.\begin{aligned} g_{x_0} &= g_{y_2}\frac{1}{\cos\varphi}[(-\beta_{31}\sin S + \beta_{32}\cos S)\cos\varepsilon + \\ &+ (-\beta_{31}\sin z\cos S - \beta_{32}\sin z\sin S + \beta_{33}\cos z)\sin\varepsilon], \\ g_{y_0} &= g_{y_2}\frac{1}{\cos\varphi}[-(-\beta_{31}\sin S + \beta_{32}\cos S)\sin\varepsilon + \\ &+ (-\beta_{31}\sin z\cos S - \beta_{32}\sin z\sin S + \beta_{33}\cos z)\cos\varepsilon]. \end{aligned}\right\} \qquad (4.19)$$

Here the projection g_{y_2} of the gravitational acceleration is a function of r and the geocentric latitude φ. It is defined by the first relation in (1.27). The third equality in (4.3) relates φ to z and S.

The distance r entering the expression for g_{y_2} and directly the right-hand sides of (4.19) is determined from the altimeter readings h and the geocentric latitude φ using Eqs. (1.13), from which the geographical latitude φ' obviously can be eliminated.

Combining Eqs. (4.13), (4.18), (4.16), (4.19), (1.13), (4.5), (4.6), the third equality in (4.3) and the first equality in (1.27), we obtain a closed system of ideal operation equations of a two-accelerometer inertial system measuring the orthodromic coordinates:

$$\left.\begin{aligned} v_{x_0} &= \int_0^t (n_{x_0} + \omega_{z_0}v_{y_0} - \dot{r}\omega_{y_0} + g_{x_0})\,dt + v_{x_0}^0, \\ v_{y_0} &= \int_0^t (n_{y_0} + \dot{r}\omega_{x_0} - \omega_{z_0}v_{x_0} + g_{y_0})\,dt + v_{y_0}^0; \\ \omega_{y_0} &= \frac{v_{x_0}}{r}, \quad \omega_{x_0} = -\frac{v_{y_0}}{r}, \\ \omega_{z_0} &= \dot{\varepsilon} + (\omega_{x_0}\sin\varepsilon + \omega_{y_0}\cos\varepsilon - u_{y_3})\operatorname{tg} z - u_{z_3}; \\ M_y^2 &= -H_2\omega_{x_0}, \quad M_x^1 = H_1\omega_{y_0}, \quad M_x^3 = H_3\omega_z; \\ z &= \int_0^t (-\omega_{x_0}\cos\varepsilon + \omega_{y_0}\sin\varepsilon + u_{x_3})\,dt + z^0, \\ S &= \int_0^t \frac{1}{\cos z}(\omega_{x_0}\sin\varepsilon + \omega_{y_0}\cos\varepsilon - u_{y_3})\,dt + S^0, \\ \varepsilon &= \int_0^t [\omega_{z_0} - (\omega_{x_0}\sin\varepsilon + \omega_{y_0}\cos\varepsilon - u_{y_3})\operatorname{tg} z + u_{z_3}]\,dt + \varepsilon^0; \\ u_{x_3} &= u(-\beta_{31}\sin S + \beta_{32}\cos S), \\ u_{y_3} &= u(-\beta_{31}\sin z\cos S - \beta_{32}\sin z\sin S + \beta_{33}\cos z), \\ u_{z_3} &= u(\beta_{31}\cos z\cos S + \beta_{32}\cos z\sin S + \beta_{33}\sin z); \end{aligned}\right\} \qquad (4.20)$$

$$
\begin{aligned}
\xi_1 &= \xi_0 - \frac{1}{\cos\varphi}\{(-\beta_{11}\sin S + \beta_{12}\cos S)\cos\varepsilon + \\
&+ (-\beta_{21}\sin z\cos S - \beta_{22}\sin z\sin S + \beta_{23}\cos z)\sin\varepsilon\}, \\
\xi_2 &= \xi_0 - \frac{1}{\cos\varphi}\{-(-\beta_{11}\sin S + \beta_{12}\cos S)\sin\varepsilon + \\
&+ (-\beta_{21}\sin z\cos S - \beta_{22}\sin z\sin S + \beta_{23}\cos z)\cos\varepsilon\}, \\
\xi_0 &= \frac{z}{2}\left(q - e^2\frac{1}{r}\right)^{\frac{1}{2}}\sin 2\varphi \times \\
&\times\left\{1 + e^2\frac{7q-9q}{14(q-e^2)}\right\}^{\frac{1}{2}}\left\{1 + \left[\frac{30q-21e^2}{14(q-e^2)} + \right.\right. \\
&\left.\left. + \frac{72q-105q}{56(q-e^2)}\sin^2\varphi\right]e^2\left(\frac{a}{r}\right)^2 + \ldots\right\}; \\
\sin\varphi &= \beta_{31}\cos z\cos S - \beta_{32}\cos z\sin S + \beta_{33}\sin z, \\
r &= \left[\frac{a(1-e^2)}{(1-e^2\sin^2\varphi')^{\frac{1}{2}}} + h\right]\frac{\sin\varphi'}{\sin\varphi}, \\
\operatorname{tg}\varphi &= \left[1 - \frac{a^2e^2}{a + h(1-e^2\sin^2\varphi')^{\frac{1}{2}}}\right]\operatorname{tg}\varphi'.
\end{aligned}
\tag{4.20}
$$

For $e = 0$ Eqs. (4.20) are clearly seen to reduce to Eqs. (2.23).

Let us now proceed with a direct derivation of the error equations. The main instrumental errors (i.e., the errors of the sensitive elements) include the accelerometer errors Δn_x, Δn_y and the gyro drift $\Delta\omega_x$, $\Delta\omega_y$, $\Delta\omega_z$, which by (4.5) are expressed in the form

$$
\Delta\omega_x = -\lambda\left(\frac{M_x^*}{H_x}\right), \quad \Delta\omega_y = \lambda\left(\frac{M_y^*}{H_y}\right), \quad \Delta\omega_z = \lambda\left(\frac{M_z^*}{H_z}\right).
\tag{4.21}
$$

Let the perturbed orientation of the axes $Oxyz$ fixed to the gyroframe be determined by small angles α, β, γ relative to the unperturbed axes:

	x	y	z
x_0	1	$-\gamma$	β
y_0	γ	1	$-\alpha$
z_0	$-\beta$	α	1.

(4.22)

The difference in the projections of the absolute angular velocity of $Oxyz$ on the perturbed and unperturbed axes is then given by

$$
\begin{aligned}
\omega_x - \omega_{x_0} &= \dot\alpha - \omega_y\gamma - \omega_z\beta, \\
\omega_y - \omega_{y_0} &= \dot\beta - \omega_x\gamma + \omega_z\alpha, \\
\omega_z - \omega_{z_0} &= \dot\gamma + \omega_x\beta - \omega_y\alpha.
\end{aligned}
\tag{4.23}
$$

Taking the variations of the sixth, seventh, and eighth equations in (4.20), we get

$$
\begin{aligned}
\delta\omega_x &= \omega_x - \omega_{x_0} - \Delta\omega_{x_0} = \dot\alpha + \omega_y\gamma - \omega_z\beta - \Delta\omega_x, \\
\delta\omega_y &= \omega_y - \omega_{y_0} - \Delta\omega_{y_0} = \dot\beta - \omega_x\gamma + \omega_z\alpha - \Delta\omega_y, \\
\delta\omega_z &= \omega_z - \omega_{z_0} - \Delta\omega_{z_0} = \dot\gamma + \omega_x\beta - \omega_y\alpha - \Delta\omega_z.
\end{aligned}
\tag{4.24}
$$

where by definition $\Delta\omega_x$, $\Delta\omega_y$, $\Delta\omega_z$ are the projections of the gyro drift velocity onto the corresponding axes.

Taking the variations of the first four equations in (4.20) (keeping the time constant), we get

$$\left.\begin{aligned}
&\delta\dot{v}_{x_0} = \delta n_x + \omega_{z_0}\delta v_{y_0} + v_{y_0}\delta\omega_{z_0} - \dot{r}\,\delta\omega_{y_0} - \omega_{y_0}\delta\dot{r} + \delta g_{x_0},\\
&\delta\dot{v}_{y_0} = \delta n_{y_0} + \dot{r}\,\delta\omega_{x_0} + \omega_{x_0}\delta\dot{r} - \omega_{z_0}\delta v_{x_0} - v_{x_0}\delta\omega_{z_0} + \delta g_{y_0},\\
&\delta v_{r_0} = r\,\delta\omega_{y_0} + \omega_{y_0}\delta r, \quad \delta v_{y_0} = -r\,\delta\omega_{x_0} - \omega_{x_0}\delta r.
\end{aligned}\right\} \tag{4.25}$$

In (4.25)

$$\left.\begin{aligned}
\delta n_x &= n_y\gamma - n_z\beta + \Delta n_{x_0},\\
\delta n_{y_0} &= -n_x\gamma + n_{z_0}\alpha + \Delta n_{y_0},
\end{aligned}\right\} \tag{4.26}$$

where n_x, n_y are determined by (4.7) and

$$n_z = w_{z_0} - g_{z_0}. \tag{4.27}$$

Eliminating all the variables, except α and β, between (4.24)—(4.26) and using (4.7), (4.27), (4.8), (4.12), we find

$$\left.\begin{aligned}
&\ddot{\alpha} + 2\frac{\dot{r}}{r}\dot{\alpha} + \left(-\frac{g_{z_0}}{r} - \omega_{x_0}^2 - \omega_{z_0}^2 + \frac{\ddot{r}}{r}\right)\alpha -\\
&-\left(\omega_{y_0}\omega_{x_0} + \dot{\omega}_{z_0} + 2\frac{\dot{r}}{r}\omega_{z_0}\right)\beta - 2\omega_{z_0}\dot{\beta} =\\
&= -\frac{1}{r}\left(\delta g_{y_0} + \Delta n_{y_0}\right) + \left(\omega_{y_0}\omega_{z_0} - \dot{\omega}_{x_0}\right)\frac{\delta r}{r} - 2\omega_{x_0}\frac{\delta\dot{r}}{r} +\\
&+ 2\,\Delta\omega_{x_0}\frac{\dot{r}}{r} + \Delta\dot{\omega}_{x_0} - \omega_{y_0}\Delta\omega_{z_0} - \omega_{z_0}\Delta\omega_{y_0},\\
&\ddot{\beta} + 2\frac{\dot{r}}{r}\dot{\beta} + \left(-\frac{g_{z_0}}{r} - \omega_{y_0}^2 - \omega_{z_0}^2 + \frac{\ddot{r}}{r}\right)\beta -\\
&-\left(\omega_{x_0}\omega_{y_0} - \dot{\omega}_{z_0} - 2\frac{\dot{r}}{r}\omega_{z_0}\right)\alpha + 2\omega_{z_0}\dot{\alpha} =\\
&= \frac{1}{r}\left(\delta g_{x_0} + \Delta n_{x_0}\right) - \left(\omega_{x_0}\omega_{z_0} + \dot{\omega}_{y_0}\right)\frac{\delta r}{r} -\\
&- 2\omega_{y_0}\frac{\delta\dot{r}}{r} + 2\,\Delta\omega_{y_0}\frac{\dot{r}}{r} + \Delta\dot{\omega}_{y_0} + \omega_{x_0}\Delta\omega_{z_0} + \omega_{z_0}\Delta\omega_{x_0}.
\end{aligned}\right\} \tag{4.28}$$

To this set we add an equation for γ which is obtained from the third equality in (4.24):

$$\dot{\gamma} = -\omega_{x_0}\beta + \omega_{y_0}\alpha + \delta\omega_{z_0} + \Delta\omega_{z_0}. \tag{4.29}$$

The initial conditions for Eqs. (4.28), (4.29) follow from the integral equations corresponding to Eqs. (4.25). These are the following:

$$\left.\begin{aligned}
&\alpha(0) = \alpha^0, \quad \beta(0) = \beta^0, \quad \gamma(0) = \gamma^0,\\
&\dot{\beta}(0) = \delta\omega_{y_0}^0 + \Delta\omega_{y_0}^0 + \omega_{x_0}^0\gamma^0 - \omega_{z_0}^0\alpha^0 - \omega_{y_0}^0\frac{\delta r^0}{r^0},\\
&\dot{\alpha}(0) = \delta\omega_{x_0}^0 + \Delta\omega_{x_0}^0 - \omega_{y_0}^0\gamma^0 + \omega_{z_0}^0\beta^0 + \omega_{x_0}^0\frac{\delta r^0}{r^0}.
\end{aligned}\right\} \tag{4.30}$$

Let us now derive the equations of errors in the coordinates z, S and in the azimuthal angle ε. Varying the ninth, tenth, and eleventh equation in (4.20), we get

$$\left.\begin{aligned}
&\delta\dot{z} = -\delta\omega_{x_0}\cos\varepsilon + \delta\omega_{y_0}\sin\varepsilon +\\
&+ (\omega_{x_0}\sin\varepsilon + \omega_{y_0}\cos\varepsilon)\,\delta\varepsilon - u\,(\beta_{31}\cos S + \beta_{32}\sin S)\,\delta S,
\end{aligned}\right\} \tag{4.31}$$

$$
\left.\begin{aligned}
&\delta\dot{S}\cos z = \delta\omega_x \sin\varepsilon + \delta\omega_y \cos\varepsilon + \\
&+ (\omega_x \cos\varepsilon - \omega_y \sin\varepsilon)\delta\varepsilon - \operatorname{tg} z\,(\omega_x \sin\varepsilon + \omega_y \cos\varepsilon)\delta z + \\
&+ u \sin z\,(-\beta_{y1}\sin S + \beta_{y2}\cos S)\delta S + \\
&\qquad + u \sec z\,(\beta_{y1}\cos S + \beta_{y2}\sin S)\delta z, \\
&\delta\dot{\varepsilon}\cos z = \delta\omega_x \cos z - \sin z\,(\delta\omega_x \sin\varepsilon + \delta\omega_y \cos\varepsilon) - \\
&- \sin z\,(\omega_x \cos\varepsilon - \omega_y \sin\varepsilon)\delta\varepsilon + \\
&+ u\,(\beta_{x1}\sin S - \beta_{x2}\cos S)\delta S - [\sec z\,(\omega_x \sin\varepsilon + \\
&\qquad + \omega_y \cos\varepsilon) + u \operatorname{tg} z\,(\beta_{x1}\cos S + \beta_{x2}\sin S)]\,\delta z.
\end{aligned}\right\} \tag{4.31}
$$

Here $\delta\omega_x$, $\delta\omega_y$, $\delta\omega_z$ are determined by (4.24). The initial conditions for Eqs. (4.31) are obvious.

Eliminating $\delta\omega_x$, $\delta\omega_y$, $\delta\omega_z$ from Eqs. (4.28), (4.29), (4.31) with the aid of (4.24), we obtain six equations in six unknowns, α, β, γ, $\delta\varepsilon$, δz, δS. These equations give the coordinate errors δz, δS, the azimuth error $\delta\varepsilon$, and the orientation errors of the inertial platform, i.e., the angles α, β, γ. Eqs. (4.28), (4.29), (4.31) thus constitute the complete set of error equations of the inertial navigation system.

At a first glance these equations are substantially different from Eqs. (2.28), (1.96), (1.97), (1.116), which are naturally extended to our particular system as well. However, a more detailed inspection of Eqs. (4.28), (4.29), (4.31) shows that they can be reduced to Eqs. (2.28), (1.96), (1.97), (1.116).

Let us start with Eqs. (4.28), where we make the substitution

$$\delta x = r\beta, \quad \delta y = -r\alpha. \tag{4.32}$$

The variations δg_x, δg_y of the nonspherical gravitation component are ignored. To the same approximation, we may take

$$g_z = -\frac{\mu}{r^2} \tag{4.33}$$

and seeing that to first approximation

$$\delta r = \Delta r = \Delta h, \tag{4.34}$$

we get

$$
\left.\begin{aligned}
&\delta\ddot{x} + \left(\frac{\mu}{r^3} - \omega_y^2 - \omega_z^2\right)\delta x + (\omega_x\omega_y - \dot{\omega}_z)\delta y - \\
&- 2\omega_z\,\delta\dot{y} = \Delta n_x - (\omega_x\omega_z + \dot{\omega}_y)\,\Delta r - 2\omega_z\,\Delta\dot{r} + \\
&\qquad + 2\,\Delta\omega_z\,\dot{r} + \Delta\dot{\omega}_z\,r - r\omega_x\,\Delta\omega_z + r\omega_z\,\Delta\omega_x, \\
&\delta\ddot{y} + \left(\frac{\mu}{r^3} - \omega_x^2 - \omega_z^2\right)\delta y + (\omega_x\omega_y + \dot{\omega}_z)\,\delta x + \\
&+ 2\omega_z\,\delta\dot{x} = \Delta n_y - (\omega_y\omega_z - \dot{\omega}_x)\,\Delta r + 2\omega_x\,\Delta\dot{r} - \\
&\qquad - 2\,\Delta\omega_x\dot{r} - \Delta\dot{\omega}_x r - r\omega_z\,\Delta\omega_z - r\omega_z\,\Delta\omega_y \\
&\qquad\qquad (\Delta r = \Delta h).
\end{aligned}\right\} \tag{4.35}
$$

Note that here the right-hand sides contain $-\Delta\omega_x$, $-\Delta\omega_y$, $-\Delta\omega_z$, whereas the right-hand sides of Eqs. (2.28) contain Δm_x, Δm_y, Δm_z. This difference between the two sets of equations is readily understood. The point is that in the derivation of Eqs. (4.35), $\Delta\omega_x$, $\Delta\omega_y$, $\Delta\omega_z$ were introduced as the positive gyrodrift components. Δm_x, Δm_y, Δm_z in Eqs. (2.28), on the other hand,

correspond to negative errors. Indeed, the vector Δm was introduced as the instrumental error of the gyroscopic absolute velocity meter. If we treat the controlled gyroframe as an angular velocity meter, utilizing the magnitude of the control torques, a positive gyro drift results in a measurement of angular velocity components which are less than the actual values precisely by the amount of the drift; this clearly corresponds to negative measurement errors for positive gyro drift. Thus,

$$\Delta m_{x_0} = -\Delta\omega_{x_0},\quad \Delta m_{y_0} = -\Delta\omega_{y_0},\quad \Delta m_{z_0} = -\Delta\omega_{z_0} \tag{4.36}$$

and Eqs. (4.35) and (2.28) are therefore identical.

Let us now turn to Eqs. (4.31). We introduce new variables*

$$\left.\begin{aligned} \alpha_2 &= -\delta z\cos\varepsilon + \delta S\cos z\sin\varepsilon,\\ \beta_2 &= \delta z\sin\varepsilon + \delta S\cos z\cos\varepsilon,\\ \gamma_2 &= \delta\varepsilon + \delta S\sin z. \end{aligned}\right\} \tag{4.37}$$

Eqs. (4.31) can now be written in the form

$$\left.\begin{aligned} \dot\alpha_2 + \omega_{y_0}\gamma_2 - \omega_{z_0}\beta_2 &= \delta\omega_{x_0},\\ \dot\beta_2 + \omega_{z_0}\alpha_2 - \omega_{x_0}\gamma_2 &= \delta\omega_{y_0},\\ \dot\gamma_2 + \omega_{x_0}\beta_2 - \omega_{y_0}\alpha_2 &= \delta\omega_{z_0}. \end{aligned}\right\} \tag{4.38}$$

It is readily seen that the variables α_2, β_2, γ_2 are the total angular errors in the determination of the coordinates and the azimuth. Introducing new variables

$$\theta_{x_0} = \alpha_2 - \alpha,\quad \theta_{y_0} = \beta_2 - \beta,\quad \theta_{z_0} = \gamma_2 - \gamma, \tag{4.39}$$

we obtain from (4.38), (4.24)

$$\left.\begin{aligned} \dot\theta_{x_0} + \omega_{y_0}\theta_{z_0} - \omega_{z_0}\theta_{y_0} &= -\Delta\omega_{x_0},\\ \dot\theta_{y_0} + \omega_{z_0}\theta_{x_0} - \omega_{x_0}\theta_{z_0} &= -\Delta\omega_{y_0},\\ \dot\theta_{z_0} + \omega_{x_0}\theta_{y_0} - \omega_{y_0}\theta_{x_0} &= -\Delta\omega_{z_0}. \end{aligned}\right\} \tag{4.40}$$

Using Eqs. (4.26), we arrive at the second group of error equations (1.96). Let now

$$\delta x_2 = r\beta_2,\qquad \delta y_2 = -r\alpha_2. \tag{4.41}$$

From (4.39), (4.32) we then have

$$\left.\begin{aligned} \delta x_1 &= r\theta_{y_0},\quad \delta y_1 = -r\theta_{x_0},\\ \delta x_2 &= \delta x + \delta x_1,\quad \delta y_2 = \delta y + \delta y_1, \end{aligned}\right\} \tag{4.42}$$

which coincide with Eqs. (1.97) if we remember that in our case we should take $x = y = 0$ in (1.97) and drop the relation for δz_2.

It now remains to show that Eqs. (1.118) follow from (4.28), (4.29), (4.31). Seeing that by definition

$$\alpha = \theta_{1x_0},\quad \beta = \theta_{1y_0}, \tag{4.43}$$

and turning to Eqs. (4.32), we obtain the first two relations in (1.118). The

* Andreev, V.D. Theory of inertial systems for autonomous measurement of the coordinates of a moving object. — Prikladnaya Matematika i Mekhanika, Vol. 28, No. 1. 1964.

third relation is similarly obtained. In particular, when $\varepsilon = 0$, we have from the third equality in (4.37) and the third equality in (4.39)

$$\gamma = \theta_{1z_0} = -\theta_{z_0} + \frac{\Delta x_2}{r} \operatorname{tg} z, \tag{4.44}$$

which coincides with Eq. (1.120), a particular case of the third relation in (1.118).

4.2.2. Some particular cases of azimuthal orientation of accelerometers

Eqs. (4.20) are the ideal operation equations of a two-accelerometer inertial system measuring the orthodromic coordinates. The accelerometers lie in the plane of the geocentric horizon, but their attitude (azimuthal orientation) in this plane is quite arbitrary. This follows from the fact that the angle ε, and hence ω_{z_0} and M_1^1, are all arbitrary. Let us consider some particular values of the angle ε and the resulting simplifications in the ideal operation equations.

If

$$\varepsilon = 0, \tag{4.45}$$

the accelerometers point along the axes of the co-moving orthodromic system. Eqs. (4.20) reduce to Eqs. (2.23), and if the axes $\xi\eta\zeta$ and $\xi'\eta'\zeta'$ coincide they further reduce to Eqs. (2.25) for the geocentric coordinates.

Another interesting case is when the angle ε is such that

$$\omega_{z_0} = 0, \quad M_1^1 = 0 \tag{4.46}$$

In this case the projection of the absolute angular velocity of the inertial platform on the geocentric vertical is zero, the result being a so-called "azimuthally free" system.* These systems do not require a torquer for the case axle of the gyroscope Γ_3 (see Figure 4.2).

If (4.46) is satisfied, the fifth and the eighth relation drop out from Eqs. (4.20) and terms with ω_{z_0} are omitted from the integrands of the first, second, and eleventh relation.

Note that instead of (4.46) we may use an alternative condition

$$\omega_{z_0} = \omega_{z_0}^0 = \text{const}, \quad M_x^1 = \text{const},$$

which sometimes (given an appropriate choice of $\omega_{z_0}^0$) proves more useful.

Let now the angle ε be chosen so that

$$v_{x_0} = v_{y_0}. \tag{4.47}$$

From the first two equations in (4.20) we thus get

$$\dot{\omega}_{z_0} = \frac{n_{y_0} - n_{x_0} + g_{y_0} - g_{x_0}}{2v_{x_0}}. \tag{4.48}$$

* See, e.g., Fridlender, G.O. Inertsial'nye sistemy navigatsii (Inertial Navigation Systems).—Fizmatgiz. 1961.

Let us consider how Eqs. (4.20) change. Of the first four equations only two are left:

$$\left.\begin{aligned} v_{x_0} &= \int_0^t (n_{x_0} + \omega_{z_0} v_{x_0} - \dot{r}\,\omega_{y_0} + g_{x_0})\,dt + v^0_{x_0}, \\ \omega_{y_0} &= \frac{v_{x_0}}{r}. \end{aligned}\right\} \tag{4.49}$$

All the equations containing ω_{x_0} change, since now

$$\omega_{x_0} = -\omega_{y_0}. \tag{4.50}$$

Thus the sixth, ninth, tenth, and eleventh equation in (4.20) give

$$\left.\begin{aligned} M^2_y &= H_1\omega_{y_0}, \\ z &= \int_0^t [\omega_{y_0}(\cos\varepsilon + \sin\varepsilon) + u_{x_3}]\,dt + z^0, \\ S &= \int_0^t \frac{1}{\cos z}[\omega_{y_0}(\cos\varepsilon - \sin\varepsilon) - u_{y_3}]\,dt + S^0, \\ \varepsilon &= \int_0^t \{\omega_{z_0} - [\omega_{y_0}(\cos\varepsilon - \sin\varepsilon) - u_{y_3}]\operatorname{tg} z + u_{z_3}\}\,dt + \varepsilon^0. \end{aligned}\right\} \tag{4.51}$$

The other relations in (4.20) do not change.

Note that condition (4.47) can be replaced by a more general one,

$$v_{x_0} = kv_{y_0}, \tag{4.52}$$

where k is an arbitrary constant coefficient, or even by a condition

$$v_{x_0} = f(v_{y_0},\ t). \tag{4.53}$$

The corresponding ideal operation equations are readily derived.

Let us consider the relative merits and shortcomings of the azimuthal accelerometer orientations based on conditions (4.45)—(4.47).

Using condition (4.46) we eliminate the torquer of the gyroscope Γ_3 and simplify the integrands ($\omega_{z_0} = 0$) in the first two equations in (4.20). This condition, however, involves an additional integration in accordance with the eleventh equation in (4.20), which is unnecessary when condition (4.45) is used. Condition (4.46) leads to more complicated integrands than condition (4.45) in the equations for z and S, i.e., in the ninth and the tenth equation in (4.20). Finally the condition $\omega_{z_0} = 0$ leads to more complex expressions for g_{x_0} and g_{y_0}. On the whole, condition (4.46) is not clearly advantageous compared to condition (4.45), when $\varepsilon = 0$.

Let us now consider condition (4.47). It involves the same number of integrations as condition (4.45). When condition (4.45) is used, we integrate to determine v_{x_0}, v_{y_0}, and with condition (4.47) integration is applied to obtain v_{x_0} and ε. The expressions for S and z are more involved for condition (4.47). The Γ_3 torquer is retained. Also note that when condition (4.47) is used, ω_{z_0} must be formed in accordance with Eq. (4.48). v_{x_0} is variable. The accelerometer readings before integration thus have to be divided by a

variable quantity. This constitutes an additional fundamental shortcoming of any system using condition (4.47). Thus, condition (4.47), like (4.46), does not lead to any simplification in system design. The most advantageous is apparently condition (4.45), when $\varepsilon = 0$ and the accelerometers point along the axes x_2, y_2 of the co-moving orthodromic frame $x y_2 z_2$. In what follows we will consider only this alternative. The corresponding ideal operation equations are Eqs. (2.23).

In conclusion note that the ideal operation equations remain exact when either of conditions (4.45)—(4.47) is used. The error equations are therefore the same in these three cases. These are Eqs. (4.35), (4.40), (4.42), (1.118) that we have previously analyzed.

4.2.3. Simplifications which follow from the nearly spherical figure of the Earth

Consider the ideal operation equations (4.20) for $\varepsilon = 0$, i.e., Eqs. (2.23). We will try to simplify these equations seeing that the Earth is nearly spherical and its gravitational field is almost central, i.e., in the final analysis we seek the simplifications associated with the small eccentricity of the Clairaut ellipsoid.

These simplifications primarily apply to the expressions for g_{x_2}, g_{y_2} and to the last two equalities in (2.23) which relate the geographical coordinates h, φ' to the geocentric coordinates r, φ.

The projection g_{x_2} of the Earth's gravitational acceleration on the axis y_2 of the co-moving geocentric frame, which lies in the plane of the meridian, is given by the first relation in (1.27). To first approximation, retaining only terms of the order e^2, we obtain

$$g_{x_2} \approx \frac{g_e}{2}(q - e^2)\left(\frac{a}{r}\right)^4 \sin 2\varphi. \tag{4.54}$$

In accordance with the first expression in (1.27), the error in (4.54) is estimated as

$$\Delta g_{x_2} \approx \frac{g_e e^2}{28}\left(\frac{a}{r}\right)^4 \sin 2\varphi \left\{7e^2 - 30q + [30q - 21e^2 + 7(7e^2 - 10q)\sin^2\varphi]\left(\frac{a}{r}\right)^2\right\}. \tag{4.55}$$

If g_{x_2}, g_{z_2} are formed using the approximate value (4.54), and not the exact g_{x_2}, we have from the eighth and the ninth relation in (2.23)

$$\left.\begin{aligned} g_x &= g_e(q - e^2)\left(\frac{a}{r}\right)^4 \sin\varphi(-\beta_{31}\sin S + \beta_{32}\cos S), \\ g_z &= g_e(q - e^2)\left(\frac{a}{r}\right)^4 \sin\varphi(-\beta_{31}\sin z\cos S - \beta_{32}\sin z\sin S + \beta_{33}\cos z). \end{aligned}\right\} \tag{4.56}$$

By (4.55), the errors Δg_x, Δg_y are

$$\left.\begin{aligned} \Delta g_{x_2} \approx \frac{g_e e^2}{14}\left(\frac{a}{r}\right)^4 \sin\varphi \Big\{ & 7e^2 - 30q + \\ & + [30q - 21e^2 + 7(7e^2 - 10q)\sin^2\varphi]\left(\frac{a}{r}\right)^2\Big\} \times \\ & \times(-\beta_{31}\sin S + \beta_{32}\cos S). \end{aligned}\right\} \tag{4.57}$$

$$\left.\begin{aligned}\Delta g_{y_0} \approx \frac{g_e e^2}{14}\left(\frac{a}{r}\right)^4 \sin\varphi' \Big\{ & 7e^2 - 30q + \\ & + [30q - 21e^2 + 7(7e^2 - 10q)\sin^2\varphi']\left(\frac{a}{r}\right)^2\Big\}\times \\ & \times(-\beta_{31}\sin z\cos S - \beta_{32}\sin z\sin S + \beta_{33}\cos z).\end{aligned}\right\} \tag{4.57}$$

The numerical values of the constants in Eqs. (4.56), (4.57) are the following:*

$$\left.\begin{aligned} g_e &= 978.049\ \text{cm/sec}^2, & e^2 &= 0.0067386, \\ q &= 0.00346775, & a &= 6\,378\,245\ \text{m}. \end{aligned}\right\} \tag{4.58}$$

Using these values we can readily obtain upper-bound estimates for the errors (4.57):

$$|\Delta g_{x_0}|,\ |\Delta g_{y_0}| \lesssim 10^{-2}\ \text{cm/sec}^2 \approx 10^{-5} g_e. \tag{4.59}$$

In (4.54), (4.56) we thus take

$$g_e(q - e^2) = 3.16\ \text{cm/sec}^2, \tag{4.60}$$

so that these expressions give g_{x_0}, g_{y_0} to within 0.01 cm/sec^2.

We will now simplify the last two equations in (2.23), which give the distance r. Eliminating φ, we obtain

$$r = \left[\left(\frac{a}{\sqrt{1 - e^2\sin^2\varphi'}} + h\right)^2\cos^2\varphi' + \left(\frac{a(1-e^2)}{\sqrt{1 - e^2\sin^2\varphi'}} + h\right)^2\sin^2\varphi'\right]^{1/2}. \tag{4.61}$$

Expanding the right-hand side in powers of e, we obtain to terms of the order e^4

$$r = r|_{e=0} + \frac{\partial r}{\partial e^2}\Big|_{e=0} e^2 + \frac{1}{2}\frac{\partial^2 r}{\partial e^4}\Big|_{e=0} e^4. \tag{4.62}$$

From (4.61) we get

$$\left.\begin{aligned} & r|_{e=0} = a + h, \\ & \frac{\partial r}{\partial e^2}\Big|_{e=0} = -\frac{a}{2}\sin^2\varphi', \\ & \frac{\partial r^2}{\partial e^4}\Big|_{e=0} = \frac{a\sin^2\varphi'}{4(a+h)}[a(4 - 5\sin^2\varphi') - h\sin^2\varphi']. \end{aligned}\right\} \tag{4.63}$$

Inserting the derivatives from (4.63) in (4.62) we obtain

$$r = a + h - \frac{ae^2}{2}\sin^2\varphi' + \frac{ae^4\sin^2\varphi'}{8(a+h)}[a(4 - 5\sin^2\varphi') - h\sin^2\varphi']. \tag{4.64}$$

Retaining only the term with e^2, we have

$$r = a + h - \frac{ae^2}{2}\sin^2\varphi'. \tag{4.65}$$

This is an expression for r in terms of h and φ'. The geographical latitude φ' enters Eqs. (2.23) only when r is being determined. If r is

* The values of e^2 and q are given for the parameters of Krasovskii's ellipsoid; see, e.g., Graur, A.V. Matematicheskaya kartografiya (Mathematical Cartography). — Leningrad University, 1956. The value of g_e is from Grushinskii, N.P. Teoriya figury Zemli (Theory of the Earth's Figure). — Fizmatgiz, 1963.

determined to within terms of the order e^2, the geographical latitude φ can be eliminated from these equations. Indeed, according to (1.15), the difference $\varphi' - \varphi$ is of the order e^2. Therefore in (4.65) the geographical latitude φ' can be replaced by the geocentric latitude φ. Hence, to terms of the order e^2 inclusive, we have

$$r = a\left(1 - \frac{e^2}{2}\sin^2\varphi\right) + h \tag{4.66}$$

To estimate the error Δr in this expression, we proceed as follows. From the last but one relation in (2.23) we have to terms of the order e^2

$$\sin\varphi = \sin\varphi'\left[1 + \frac{ae^2}{2a+h}\left(-1 + \frac{3}{2}\sin^2\varphi'\right)\right]. \tag{4.67}$$

To the same approximation

$$\sin^2\varphi - \sin^2\varphi' = \sin^2\varphi'\frac{ae^2}{2a+h}\left(-2 + 3\sin^2\varphi'\right). \tag{4.68}$$

Now (4.64) can be written in the form

$$r = a + h - \frac{ae^2}{2}\sin^2\varphi + \frac{ae^2}{2}\left(\sin^2\varphi - \sin^2\varphi'\right) + \\ + \frac{ae^2\sin^2\varphi'}{2(2a+h)}\left[a\left(4 - 5\sin^2\varphi'\right) - h\sin^2\varphi'\right] \tag{4.69}$$

Comparing (4.69) with (4.66) and using (4.68), we see that the error Δr in (4.66) is expressed in the form

$$\Delta r \approx \frac{ae^2\sin^2\varphi'}{2}\left(-1 - \frac{3}{4}\sin^2\varphi' + \frac{h}{2a+h}\right). \tag{4.70}$$

Hence,

$$|\Delta r| \leq \frac{5}{8}ae^2 \approx 180 \text{ m}, \tag{4.71}$$

this estimate being independent of h.

Using (4.66), (4.57), we write the ideal operation equations of a two-accelerometer system, Eqs. (2.23), in the simplified form

$$\left.\begin{aligned}
v_x &= \int_0^t \left(n_x + v_{y_0}\omega_{z_0} - \dot{r}\omega_{y_0} + g_{x_0}\right)dt + v_{x^0}^0, \\
v_{y_0} &= \int_0^t \left(n_{y_0} - v_{x_0}\omega_z + \dot{r}\omega_{x_0} + g_y\right)dt + v_{y_0}^0; \\
\omega_x &= -\frac{v_{y_0}}{r}, \quad \omega_{y_0} = \frac{v_{x_0}}{r}; \\
S &= \int_0^t \frac{1}{\cos z}\left[\omega_{y_0} + u\left(\beta_{31}\sin z\cos S + \beta_{32}\sin z\sin S - \right.\right. \\
&\qquad \left.\left. - \beta_{33}\cos z\right)\right]dt + S^0, \\
z &= \int_0^t \left[-\omega_{x_0} + u\left(-\beta_{31}\sin S + \beta_{32}\cos S\right)\right]dt + z^0; \\
\omega_{z_0} &= \omega_{y_0}\operatorname{tg} z + \frac{u}{\cos z}\left(\beta_{31}\cos S + \beta_{32}\sin S\right);
\end{aligned}\right\} \tag{4.72}$$

$$\left.\begin{aligned}
&g_{x_0} = g_e(q - e^2)\left(\frac{a}{r}\right)^4 \sin\varphi\,(-\beta_{31}\sin S + \beta_{32}\cos S),\\
&g_{y_0} = g_e(q - e^2)\left(\frac{a}{r}\right)^4 \sin\varphi\,(-\beta_{31}\sin z\cos S - \\
&\qquad - \beta_{32}\sin z\sin S + \beta_{33}\cos z);\\
&\sin\varphi = \beta_{31}\cos z\cos S + \beta_{32}\cos z\sin S + \beta_{33}\sin z;\\
&M_y^2 = -H_2\omega_{x_0},\quad M_x^1 = H_1\omega_{y_0},\quad M_x^3 = -H_3\omega_{z_0};\\
&r = h + a\left(1 - \frac{e^2}{2}\sin^2\varphi\right).
\end{aligned}\right\} \qquad (4.72)$$

Let us establish how these simplifications affect the error equations (4.35). The left-hand sides of these equations clearly do not change, since Eqs. (4.72) include all the terms which enter the exact equations (2.23). The simplifications introduced in Eqs. (4.72) only change the coefficients in the left-hand sides of the error equations (4.35) by terms of the first order of smallness. This is of no consequence, since on being multiplied by δx, δy, $\delta\dot{x}$, $\delta\dot{y}$ these terms give a second-order contribution. To first approximation, the left-hand sides of the first group of error equations thus do not change.

The right-hand sides of the error equations do change, however. To the intrinsic instrumental errors Δn_{x_0}, Δn_{y_0}, $\Delta\omega_{x_0}$, $\Delta\omega_{y_0}$, $\Delta\omega_{z_0}$ certain equivalent instrumental errors are added. These equivalent errors are readily determined by retracing the previous derivation of the error equations (4.35).

The right-hand sides of the error equations (4.35) will additionally acquire equivalent accelerometer errors $\Delta\tilde{n}_{x_0}$, $\Delta\tilde{n}_{y_0}$ associated with the simplification introduced in the expressions for g_{x_0}, g_{y_0}, so that

$$\Delta\tilde{n}_{x_0} = \Delta g_{x_0},\qquad \Delta\tilde{n}_{y_0} = \Delta g_{y_0}, \qquad (4.73)$$

where Δg_{x_0} and Δg_{y_0} are defined by (4.57). Estimates (4.59) show that the simplifications adopted in the expressions for g_{x_0}, g_{y_0} are equivalent to accelerometer errors of the order of $10^{-5}\,g_e$. Such errors are often ignorable. Their effect is definitely negligible at velocities substantially less than the circular orbit velocity. This follows from the analysis of the error equations in two-accelerometer systems, carried out in Chapter 2. The errors associates with deviations (4.73) for motion with any velocity can be readily estimated using the expressions of Chapter 2.

The right-hand sides of the error equations (4.35) contain terms with $\Delta r = \Delta h$, where Δh is the altimeter error. By adopting the simplified expression (4.66) for r, we add the error (4.70) to the altimeter error Δh. This equivalent altimeter error does not exceed 180 m according to (4.71).

Note that at a first glance it may seem that the error Δr in the calculation of r should lead to certain equivalent gyro drift errors $\Delta\omega_{x_0}$, $\Delta\omega_{y_0}$, $\Delta\omega_{z_0}$ and thus change the right-hand sides of the second group of error equations (4.40). This impression is based on an examination of the third and fourth relations in (2.23) and (4.72). Indeed, they give

$$\left.\begin{aligned}
&\omega_{x_0} + \Delta\tilde{\omega}_{x_0} = -\frac{v_{y_0}}{r + \Delta r} = \omega_{x_0} - \omega_{x_0}\frac{\Delta r}{r},\\
&\omega_{y_0} + \Delta\tilde{\omega}_{y_0} = \frac{v_{x_0}}{r + \Delta r} = \omega_{y_0} - \omega_{y_0}\frac{\Delta r}{r},\\
&\omega_{z_0} + \Delta\tilde{\omega}_{z_0} = (\omega_{y_0} + \Delta\tilde{\omega}_{y_0})\operatorname{tg} z + \\
&+ \frac{u}{\cos z}(\beta_{31}\cos S + \beta_{32}\sin S) = \omega_{z_0} + \Delta\tilde{\omega}_{y_0}\operatorname{tg} z.
\end{aligned}\right\} \qquad (4.74)$$

and the error Δr thus involves the following equivalent errors:

$$\left.\begin{aligned} \Delta\tilde{\omega}_{x_2} &= -\omega_{x_2}\frac{\Delta r}{r}, \quad \Delta\tilde{\omega}_{y_2} = -\omega_{y_2}\frac{\Delta r}{r}, \\ \Delta\tilde{\omega}_{z_2} &= -\omega_{y_2}\frac{\Delta r}{r}\operatorname{tg} z. \end{aligned}\right\} \tag{4.75}$$

The latter clearly result in additional torques being applied to the inertial platform, which are equivalent to gyro drift.

Examination of the entire set of variational equations corresponding to Eqs. (2.23) shows that this is not so. Indeed, the errors $\Delta\tilde{\omega}_{x_2}$, $\Delta\tilde{\omega}_{y_2}$ specified by Eqs. (4.75) enter the first two equations in (4.24) together with the gyro drift components, and hence they are carried over to Eqs. (4.28) and (4.35). However, $\Delta\tilde{\omega}_{x_2}$, $\Delta\tilde{\omega}_{y_2}$, unlike the gyro drift, also enter the right-hand sides of the first two equations in (4.31), and therefore drop out on passing to Eqs. (4.40), since in the process we take differences (4.39) of Eqs. (4.24) and (4.31). $\Delta\tilde{\omega}_{x_2}$, $\Delta\tilde{\omega}_{y_2}$ therefore do not enter the right-hand sides of the first two equations of the second group (4.40) and thus do not produce equivalent gyro errors.

$\Delta\tilde{\omega}_{z_2}$ does not enter the right-hand side of the third equation in (4.24), since here $\Delta\omega_{z_2}$ is by definition an instrumental error independent of $\Delta\omega_{x_2}$, $\Delta\omega_{y_2}$, whereas $\Delta\tilde{\omega}_{z_2}$ according to the second and third equations in (4.75) is equal to $\Delta\tilde{\omega}_{y_2}\operatorname{tg} z$. It is readily seen that $\Delta\tilde{\omega}_{z_2}$ enters Eq. (4.29), since it is contained in the term $\delta\omega_{y_2}\operatorname{tg} z$. In forming the third equation in (4.40), however, $\Delta\tilde{\omega}_{z_2}$ drops out. It does not enter the right-hand side of this equation and does not produce equivalent gyro drift either.

If using (4.75) we replace $\Delta\omega_{x_2}$, $\Delta\omega_{y_2}$, $\Delta\omega_{z_2}$ in the right-hand sides of (4.35) by $\Delta\tilde{\omega}_{x_2}$, $\Delta\tilde{\omega}_{y_2}$, $\Delta\tilde{\omega}_{z_2}$, we get

$$\left.\begin{aligned} &2\dot{r}\,\Delta\tilde{\omega}_{y_2} + r\,\Delta\dot{\tilde{\omega}}_{y_2} + r\omega_{x_2}\Delta\tilde{\omega}_{z_2} + r\omega_{z_2}\Delta\tilde{\omega}_{x_2} = \\ &\qquad = -\dot{r}\omega_{y_2}\frac{\Delta r}{r} - \dot{\omega}_{y_2}\Delta r - \omega_{x_2}\omega_{z_2}\Delta r - \\ &\qquad\qquad - \omega_{y_2}\Delta\dot{r} - \omega_{x_2}\omega_{y_2}\Delta r\operatorname{tg} z, \\ &-2\dot{r}\,\Delta\tilde{\omega}_{x_2} - r\,\Delta\dot{\tilde{\omega}}_{x_2} + r\omega_{y_2}\Delta\tilde{\omega}_{z_2} + r\omega_{z_2}\Delta\tilde{\omega}_{y_2} = \\ &= \dot{r}\omega_{x_2}\frac{\Delta r}{r} + \dot{\omega}_{y_2}\Delta r - \omega_{z_2}\omega_{y_2}\Delta r + \omega_{x_2}\Delta\dot{r} - \\ &\qquad\qquad - \omega_{y_2}^2\Delta r\operatorname{tg} z. \end{aligned}\right\} \tag{4.76}$$

However, the right-hand sides of (4.35) include Δr not only through $\Delta\tilde{\omega}_{x_2}$, $\Delta\tilde{\omega}_{y_2}$, $\Delta\tilde{\omega}_{z_2}$ but also through the variations of the terms $v_{y_2}\omega_{z_2} - \dot{r}\omega_{y_2}$ and $-v_{r_2}\omega_{z_2} + \dot{r}\omega_{x_2}$ in the integrands in the first two equations of (2.23) or (4.72). These variations are given by

$$\left.\begin{aligned} &-\omega_{y_2}\Delta\dot{r} + r\omega_{x_2}\omega_{y_2}\Delta r\operatorname{tg} z + \dot{r}\omega_{y_2}\frac{\Delta r}{r}, \\ &\omega_{x_2}\Delta\dot{r} + \omega_{y_2}^2\Delta r\operatorname{tg} z - \dot{r}\omega_{x_2}\frac{\Delta r}{r}. \end{aligned}\right\} \tag{4.77}$$

Adding (4.77) to the right-hand sides of (4.76), we obtain

$$\left.\begin{aligned} &-2\omega_{y_2}\Delta\dot{r} - (\omega_{x_2}\omega_{z_2} + \dot{\omega}_{y_2})\Delta r, \\ &2\omega_{x_2}\Delta\dot{r} - (\omega_{y_2}\omega_{z_2} - \dot{\omega}_{x_2})\Delta r. \end{aligned}\right\} \tag{4.78}$$

These expressions actually coincide with the right-hand sides of Eqs. (4.35) containing Δr.

The error Δr associated with the approximate calculation of r is equivalent to the altimeter error. They both enter the right-hand sides of the first group of error equations and in their turn are equivalent to some accelerometer errors. The right-hand sides of the second group of error equations are not affected, so that no equivalent gyro errors are involved. We have seen in the preceding that this is associated with the simultaneous appearance of $\Delta\tilde{\omega}_{x_0}$, $\Delta\tilde{\omega}_{y_0}$, $\Delta\tilde{\omega}_{z_0}$ in the expressions for the control torques and in the expressions for the coordinates z and S. This fact is attributable to the common origin of the errors, namely the deviation Δr.

If the errors $\Delta\tilde{\omega}_{x_0}$, $\Delta\tilde{\omega}_{y_0}$, $\Delta\tilde{\omega}_z$ associated with some other factor were to appear only in the equations for the coordinates, or alternatively only in the equations of the control torques, this would definitely change the right-hand sides of the second group of equations and hence result in errors equivalent to gyro drift. An appropriate example will be considered in the following.

4.2.4. Simplifications associated with the closeness of the trajectory to the Earth's surface and the orthodromy plane

The simplifications of the ideal operation equations considered in the previous subsection were independent of the particular trajectory of the object: they were entirely attributable to the small flattening of the Earth. It is, however, natural to expect that knowledge of the particular properties of motion of the object using the inertial navigation system will substantially simplify the system operation and design in some cases. In what follows we consider the simplifications associated with the particular geometrical form of the trajectory (without imposing any restrictions on the velocity of motion, however). We will be dealing with trajectories close to the Earth's surface and to the orthodromy plane $z = 0$, i.e., we are considering close motion for small values of h and z. Typical altitudes are of the order of 50—100 km, which cover present-day and future aircraft flight altitudes. The values of z are taken to be of the order of 100—500 km.

Turning to Eqs. (4.72) we note that h enters explicitly only the last equation. It is through this relation that it further enters the first four equations and the eighth and ninth equation for g_{x_0}, g_{y_0}.

The last equality in (4.72) clearly does not require any further simplification for small h.

The expressions for ω_{x_0}, ω_{y_0} for small h can be written in approximate form

$$\left.\begin{aligned}\omega_{x_0} &\approx -\frac{v_{y_0}}{a}\left(1+\frac{e^2}{2}\sin^2\varphi-\frac{h}{a}\right),\\ \omega_{y_0} &\approx \frac{v_{x_0}}{a}\left(1+\frac{e^2}{2}\sin^2\varphi-\frac{h}{a}\right).\end{aligned}\right\} \tag{4.79}$$

The error in the determination of the projections of the absolute angular velocity of the axes $x_0y_0z_0$ from these relations are equivalent to some additional error in r (apart from (4.70))

$$\Delta r' = a\left(\frac{h}{a}e^2\sin^2\varphi+\frac{e^4}{4}\sin^4\varphi-\frac{h^2}{a^2}\right). \tag{4.80}$$

For $h \approx 50$ km,

$$\Delta r' \leq 500 \text{ m}, \tag{4.81}$$

and for $h \approx 100$ km,

$$\Delta r' \leq 1.5 \text{ km}. \tag{4.82}$$

The additional error (4.80) enters only the third and the fourth equations in (4.72). In the integrands of the first two equations r is taken from the last expression in (4.72). The equivalent accelerometer errors for $\Delta r'$ are therefore

$$\left.\begin{aligned} \Delta\tilde{n}_{x_2} &= -(\omega_x\omega_z + \dot{\omega}_{y_2})\Delta r' - \omega_{y_2}\Delta\dot{r}', \\ \Delta\tilde{n}_{y_2} &= -(\omega_y\omega_z - \dot{\omega}_{x_2})\Delta r' + \omega_{x_2}\Delta\dot{r}', \end{aligned}\right\} \tag{4.83}$$

whereas for the error Δr we had, by (4.78),

$$\left.\begin{aligned} \Delta\tilde{n}_x &= -(\omega_x\omega_z + \dot{\omega}_y)\Delta r - 2\omega_z\Delta\dot{r}, \\ \Delta\tilde{n}_y &= -(\omega_y\omega_z - \dot{\omega}_x)\Delta r + 2\omega_x\Delta\dot{r}. \end{aligned}\right\} \tag{4.84}$$

Note that Eqs. (4.79) can also be written in the form

$$\left.\begin{aligned} \omega_x &= -\frac{v_{y_2}}{a} - \frac{v_y}{a}\left(\frac{e^2}{2}\sin^2\varphi - \frac{h}{a}\right), \\ \omega_y &= \frac{v_{x_2}}{a} + \frac{v_x}{a}\left(\frac{e^2}{2}\sin^2\varphi - \frac{h}{a}\right). \end{aligned}\right\} \tag{4.85}$$

Here the first terms in the right-hand sides, which make the principal contribution to the sums, are obtained from v_{x_2}, v_{y_2} which are multiplied by a constant factor $1/a$. In other words, they differ from v_{x_2}, v_{y_2} in scale only. The second terms are small compared to the first, as they are formed using $\sin^2\varphi$ and h. Therefore the function $\sin^2\varphi$ may be taken with a fairly large error. The other operations required for the formation of the second term in the right-hand sides of Eqs. (4.85) can also be performed relatively crudely. The only exact operation here is addition. Representations of the form (4.83) are often used in systems employing analog computers.

Consider the eighth and the ninth equation in (4.72). They contain the ratio a^4/r^4. Series expanding this ratio in powers of e and h/a, we obtain using the last expression in (4.72)

$$\left(\frac{a}{r}\right)^4 = 1 - 4\frac{h}{a} + 2e^2\sin^2\varphi + 10\frac{h^2}{a^2} - 10\frac{he^2}{a}\sin^2\varphi + \frac{5e^4}{4}\sin^4\varphi. \tag{4.86}$$

For altitudes of the order of 100 km, we need only take the first two terms:

$$\left(\frac{a}{r}\right)^4 = 1 - 4\frac{h}{a}, \tag{4.87}$$

which give

$$\left.\begin{aligned} g_{x_2} &= g_e(q-e^2)\left(1-4\frac{h}{a}\right)\sin\varphi\,(-\beta_{31}\sin S + \beta_{32}\cos S), \\ g_{y_2} &= g_e(q-e^2)\left(1-\frac{4h}{a}\right)\sin\varphi\,(-\beta_{31}\sin z\cos S - \\ &\qquad -\beta_{32}\sin z\sin S + \beta_{33}\cos z). \end{aligned}\right\} \tag{4.88}$$

Comparison of these expressions with the corresponding equations in (4.72) leads to the following equivalent accelerometer errors:

$$\left.\begin{aligned}\Delta\tilde{n}_x &\approx g_e(q-e^2)\left(2e^2\sin^2\varphi+10\frac{h^2}{a^2}\right)\times\\ &\qquad\times\sin\varphi(-\beta_{31}\sin S+\beta_{32}\cos S),\\ \Delta\tilde{n}_{y_0} &\approx g_e(q-e^2)\left(2e^2\sin^2\varphi+10\frac{h^2}{a^2}\right)\times\\ &\times\sin\varphi(-\beta_{31}\sin z\cos S-\beta_{32}\sin z\sin S+\beta_{33}\cos z),\end{aligned}\right\}\tag{4.89}$$

which are added to the errors (4.57). From (4.89) for $h = 100$ km we have approximately

$$|\Delta\tilde{n}_{x_0}|,\quad |\Delta\tilde{n}_{y_0}|\leqslant 5\cdot 10^{-5}g_e.\tag{4.90}$$

For altitudes of about 50 km, we can further drop the term $4h/a$ in (4.88), which gives

$$|\Delta\tilde{n}_{x_0}|,\quad |\Delta\tilde{n}_{y_0}|\leqslant 10^{-4}g_e.\tag{4.91}$$

It follows from the preceding that at altitudes of 50—100 km the functions ω_{x_0}, ω_{y_0}, g_{x_0}, g_{y_0} can be formed using fairly crude information on h, and in some cases the altitude may even be replaced by a constant average value. The altimeter is then required only in forming the vertical velocity $\dot{h}$, which enters the integrand in the first two equations in (4.72) (according to (4.69), $\dot{r}=\dot{h}+\ldots$). Therefore for small vertical velocities an autonomous system without any altimeter correction may prove to be sufficiently accurate.

Let us now consider the simplifications resulting from the fact that the trajectory is close to the orthodromy plane $z = 0$. This case applies, e.g., when the object is automatically guided by an inertial system which maintains with fair precision the programmed trajectory lying in the orthodromy plane $z = 0$. Putting $z = 0$ in (4.72) and using (4.85), (4.88), we obtain the ideal operation equations

$$\left.\begin{aligned}v_{x_0} &= \int_0^t (n_{x_0}+v_{y_0}\omega_{z_0}-r\omega_{y_0}+g_{x_0})\,dt+v_{x_0}^0,\\ v_{y_0} &= \int_0^t (n_{y_0}+\dot{r}\omega_{x_0}-v_{x_0}\omega_{z_0}+g_{y_0})\,dt+v_{y_0}^0;\\ \omega_{x_0} &= -\frac{v_{y_0}}{a}\left(1+\frac{e^2}{2}\sin^2\varphi-\frac{h}{a}\right),\\ \omega_{y_0} &= \frac{v_{x_0}}{a}\left(1+\frac{e^2}{2}\sin^2\varphi-\frac{h}{a}\right);\\ \sin\varphi &= \beta_{31}\cos S+\beta_{32}\sin S;\\ S &= \int_0^t(\omega_{y_0}-u\beta_{33})\,dt+S^0,\\ z &= \int_0^t[-\omega_{x_0}+u(-\beta_{31}\sin S+\beta_{32}\cos S)]\,dt+z^0;\\ \omega_{z_0} &= u(\beta_{31}\cos S+\beta_{32}\sin S);\\ M_x^1 &= H_1\omega_{y_0},\quad M_y^2=-H_2\omega_{x_0},\quad M_x^3=-H_3\omega_{z_0};\\ r &= h+a\left(1-\frac{e^2}{2}\sin^2\varphi\right);\\ g_{x_0} &= g_e(q-e^2)(-\beta_{31}\sin S+\beta_{32}\cos S)\sin\varphi,\\ g_{y_0} &= g_e(q-e^2)\beta_{33}\sin\varphi.\end{aligned}\right\}\tag{4.92}$$

To estimate the corresponding errors, let us consider the error equations. The left-hand sides of the error equations (4.35), (4.40) clearly do not change. The right-hand sides acquire equivalent instrumental errors $\Delta\tilde{n}_x^1$, $\Delta\tilde{n}_y^1$, $\Delta\tilde{\omega}_x^1$, $\Delta\tilde{\omega}_y^1$, $\Delta\tilde{\omega}_z^1$. They can be found by comparing the right-hand sides of Eqs. (4.92) and (4.72). Let us calculate those equivalent instrumental errors which are functions of z and are added to the previous errors.

We will first consider the equivalent accelerometer errors. They arise primarily because we calculate $\sin\varphi$ from a simplified expression, and the error in the sine leads to errors in r and g_{x_1}, g_{y_1}. Another source of the equivalent accelerometer errors is the simplified expression for ω_z and for the factors containing z in the equalities for g_{x_1}, g_{y_1}.

Comparison of the expressions for $\sin\varphi$ in Eqs. (4.72) and (4.92) gives

$$\Delta\sin\varphi = (\beta_{31}\cos S + \beta_{32}\sin S)(1-\cos z) - \beta_{33}\sin z. \tag{4.93}$$

Let z be so small that we may take

$$\sin z = \operatorname{tg} z = z, \quad \cos z = 1 - \frac{z^2}{2}. \tag{4.94}$$

Then retaining only the leading terms in (4.93), we get

$$\Delta\sin\varphi = -\beta_{33}z. \tag{4.95}$$

Using the expression for r from Eqs. (4.92) we thus get

$$\Delta r = ae^2(\beta_{31}\cos S + \beta_{32}\sin S)\beta_{33}z. \tag{4.96}$$

Similarly

$$\left.\begin{aligned} \Delta g_{x_1} &= -g_e(q-e^2)(-\beta_{31}\sin S - \beta_{32}\cos S)\beta_{33}z, \\ \Delta g_{y_1} &= -g_e(q-e^2)\left[\beta_{33}^2 - (\beta_{31}\cos S + \beta_{32}\sin S)^2\right]z, \\ \Delta\omega_z &= -\omega_{y_1}z, \end{aligned}\right\} \tag{4.97}$$

Examining the right-hand sides of the error equations (4.35), we thus obtain the equivalent accelerometer errors

$$\left.\begin{aligned} \Delta\tilde{n}_{x_1} &= -(\omega_{y_1}\omega_z + \dot{\omega}_{y_1})\Delta r - 2\dot{\omega}_{y_1}\Delta\dot{r} + \Delta g_{x_1} + r\omega_x\Delta\omega_{z_1}, \\ \Delta\tilde{n}_{y_1} &= -(\omega_{y_1}\omega_z - \dot{\omega}_{x_1})\Delta r + 2\omega_{x_1}\Delta\dot{r} + \Delta g_{y_1} + r\omega_{y_1}\Delta\omega_{z_1}. \end{aligned}\right\} \tag{4.98}$$

Here Δr, Δg_{x_1}, Δg_{y_1}, $\Delta\omega_{z_1}$ are given by Eqs. (4.96), (4.97), and the expressions for ω_{x_1}, ω_{y_1}, ω_z are obtained from (4.14)—(4.16) where we should take $\varepsilon = 0$, $z = 0$. These expressions have the form

$$\left.\begin{aligned} \omega_{x_1} &= -\dot{z} + u(-\beta_{31}\sin S + \beta_{32}\cos S), \\ \omega_{y_1} &= \dot{S} + u\beta_{33}, \\ \omega_z &= u(\beta_{31}\cos S + \beta_{32}\sin S), \end{aligned}\right\} \tag{4.99}$$

which correspond to the seventh, sixth and eighth equality in (4.92).

We will now derive numerical estimates of the errors $\Delta\tilde{n}_{x_1}$, $\Delta\tilde{n}_{y_1}$ expressed by Eqs. (4.98).

Let

$$\left.\begin{aligned} z = z^0 = \text{const} > 0, \quad r\dot{S} = V^0, \\ r = a + h\,(\approx a) = \text{const}, \end{aligned}\right\} \tag{4.100}$$

i.e., let the object move parallel to the orthodromy plane (at some distance z^0 from it) with constant velocity V^0 at a fixed distance from the Earth's center. Then from (4.95)—(4.99) we get

$$\left.\begin{aligned} \Delta\tilde{n}_{x_0} = z^0(\beta_{31}\sin S + \beta_{32}\cos S)\,[-u^2ae^2(\beta_{31}\cos S + \\ + \beta_{32}\sin S)^2\beta_{33} - 2\left(\frac{V^0}{a} + u\beta_{33}\right)V^0e^2\beta_{33} - \\ - u(V^0 + au\beta_{33}) - g_e(q - e^2)\beta_{33}], \\ \Delta\tilde{n}_{y_0} = z^0\{-aue^2(\beta_{31}\cos S + \beta_{32}\sin S)^2\beta_{33}(2V^0 + \\ + au\beta_{33}) + 2ue^2(-\beta_{31}\sin S + \beta_{32}\cos S)^2V^0\beta_{33} - \\ - a\left(\frac{V^0}{a} + u\beta_{33}\right)^2 - g_e(q - e^2)\times \\ \times[\beta_{33}^2 - (\beta_{31}\cos S + \beta_{32}\sin S)^2]\}. \end{aligned}\right\} \tag{4.101}$$

Since $e^2 \ll 1$, we obtain the approximate equalities*

$$\left.\begin{aligned} \Delta\tilde{n}_{x_0} = -z^0(-\beta_{31}\sin S + \beta_{32}\cos S)\,[(V^0 + au\beta_{33})u + g_e(q - e^2)\beta_{33}], \\ \Delta\tilde{n}_{y_0} = -z^0\left\{2a\left(\frac{V^0}{a} + u\beta_{33}\right)^2 + \right. \\ \left. + g_e(q - e^2)[\beta_{33}^2 - (\beta_{31}\cos S + \beta_{32}\sin S)^2]\right\}. \end{aligned}\right\} \tag{4.102}$$

If

$$V^{0^2} \gg a^2u^2,$$

which is applicable at velocities greater than 1000 m/sec, Eqs. (4.102) can be further simplified:

$$\left.\begin{aligned} \Delta\tilde{n}_{x_0} = -z^0(-\beta_{31}\sin S + \beta_{32}\cos S)\,uV^0, \\ \Delta\tilde{n}_{y_0} = -z^0V^0\left(\frac{V^0}{a} + 2u\beta_{33}\right). \end{aligned}\right\} \tag{4.103}$$

Seeing that

$$|-\beta_{31}\sin S + \beta_{32}\cos S| \leqslant 1, \quad |\beta_{33}| \leqslant 1,$$

we find

$$\left.\begin{aligned} \Delta\tilde{n}_{x_0} \leqslant z^0uV^0, \\ \Delta\tilde{n}_{y_0} \leqslant z^0V^0\left(\frac{V^0}{a} + 2u\right). \end{aligned}\right\} \tag{4.104}$$

For $V^0 = 1000$ m/sec and $z^0 = 15'$ ($rz^0 \approx 25$ km), we have

$$\left.\begin{aligned} \Delta\tilde{n}_{x_0} \leqslant 0.3 \text{ cm/sec}^2 \approx 3\cdot 10^{-4}g_e, \\ \Delta\tilde{n}_{y_0} \leqslant 1.25 \text{ cm/sec}^2 \approx 1.25\cdot 10^{-3}g_e. \end{aligned}\right\} \tag{4.105}$$

Let us now determine the equivalent instrumental errors $\Delta\tilde{\omega}_{x_0}$, $\Delta\tilde{\omega}_{y_0}$, $\Delta\tilde{\omega}_{z_0}$ (i.e., the equivalent gyro drift components) entering the right-hand sides of

* $g_e(q - e^2)$ is of the order au^2, and the terms containing this factor have therefore been retained here.

the second-group error equations (4.40). Since the expression for z in Eqs. (4.92) is the same as in Eqs. (4.72), we have

$$\Delta\tilde{\omega}_{x_2} = 0. \tag{4.106}$$

Comparison of the equations for S and ω_{z_2} in (4.72) and (4.92) on the other hand gives

$$\left.\begin{aligned} \Delta\tilde{\omega}_{y_2} &= u(\beta_{31}\cos S + \beta_{32}\sin S)z, \\ \Delta\tilde{\omega}_{z_2} &= -(\dot{S} + u\beta_{33})z. \end{aligned}\right\} \tag{4.107}$$

Hence the estimates

$$|\Delta\tilde{\omega}_{y_2}| < uz^0, \quad |\Delta\tilde{\omega}_{z_2}| < \left(\frac{V^0}{a} + u\right)z^0. \tag{4.108}$$

For $V^0 = 1000$ m/sec, $z^0 = 15'$, we have

$$|\Delta\tilde{\omega}_{y_2}| \leqslant 0.06 \text{ deg/hr}, \quad |\Delta\tilde{\omega}_{z_2}| \leqslant 0.2 \text{ deg/hr}. \tag{4.109}$$

Estimates (4.104), (4.108) show that the ideal operation equations (4.72) can be simplified to Eqs. (4.92) only if the deviations of the object from the orthodromy plane are indeed very small. Thus, already for deviations of about 25 km (at velocities $V^0 \approx 1000$ m/sec), the corresponding errors, as we see from the numerical estimates (4.105), (4.109), are fairly large and may prove to be unacceptable in high-accuracy systems. Indeed, as we have seen in Chapter 2, the constant errors $\Delta\tilde{n}_{x_2}$, $\Delta\tilde{n}_{y_2}$ lead to the following position errors

$$\left.\begin{aligned} \delta x &= \frac{\Delta\tilde{n}_{x_2}}{\omega_0^2}(1 - \cos\omega_0 t), \\ \delta y &= \frac{\Delta\tilde{n}_{y_2}}{\omega_0^2}(1 - \cos\omega_0 t), \end{aligned}\right\} \tag{4.109a}$$

and the constant error $\Delta\tilde{\omega}_{y_2}$ produces the coordinate error

$$\delta x_1 = r\,\Delta\tilde{\omega}_{y_2} t. \tag{4.110}$$

Estimates (4.104), (4.108) therefore correspond to the following total position errors:

$$\left.\begin{aligned} |\delta x_2| &\leqslant |\delta x| + |\delta x_1| \leqslant \left(\frac{2uV^0}{\omega_0^2} + urt\right)z^0, \\ |\delta y_2| &\leqslant \frac{2V^0}{\omega_0^2}\left(\frac{V^0}{a} + u\right)z^0. \end{aligned}\right\} \tag{4.111}$$

Hence, using relations (4.105), (4.109), we obtain the following numerical estimates for error buildup in 1 hour of system operation (the time factor is significant only for δx_2):

$$|\delta x_2| \leqslant 10 \text{ km}, \quad |\delta y_2| \leqslant 15 \text{ km}, \tag{4.112}$$

On passing from Eqs. (4.72) to Eqs. (4.92) we took $\sin z = 0$, $\cos z = 1$. A substantial improvement is accomplished if we take the next approximation,

$$\sin z = z, \quad \cos z = 1. \tag{4.113}$$

The fifth, sixth, eighth, and the last two equations in (4.92) then take the form

$$\left.\begin{aligned}
&\sin\varphi = \beta_{31}\cos S + \beta_{32}\sin S + \beta_{33}z,\\
&S = \int_0^t \{\omega_{y_0} + u[(\beta_{31}\cos S + \beta_{32}\sin S)\,z + \beta_{33}]\}\,dt + S^0,\\
&\omega_{z_0} = \omega_{y_0} z + u(\beta_{31}\cos S + \beta_{32}\sin S),\\
&g_{x_0} = g_e(q - e^2)\sin\varphi(-\beta_{31}\sin S + \beta_{32}\cos S),\\
&g_{y_0} = g_e(q - e^2)\sin\varphi[-(\beta_{31}\cos S + \beta_{32}\sin S)\,z + \beta_{33}].
\end{aligned}\right\} \tag{4.114}$$

Eqs. (4.95), (4.96) are replaced by

$$\left.\begin{aligned}
\Delta\sin\varphi &= \frac{z^2}{2}(\beta_{31}\cos S + \beta_{32}\sin S),\\
\Delta r &= -ae^2\frac{z^2}{2}(\beta_{31}\cos S + \beta_{32}\sin S)^2.
\end{aligned}\right\} \tag{4.115}$$

Eqs. (4.97) take the form

$$\left.\begin{aligned}
\Delta g_{x_0} &= g_e(q - e^2)\frac{z^2}{2}(\beta_{31}\cos S + \beta_{32}\sin S)(-\beta_{31}\sin S + \beta_{32}\cos S),\\
\Delta g_{y_0} &= g_e(q - e^2)\frac{z^2}{2}(-\beta_{31}\sin S + \beta_{32}\cos S)\beta_{33},\\
\Delta\omega_{z_0} &= -u\frac{z^2}{2}(\beta_{31}\cos S + \beta_{32}\sin S).
\end{aligned}\right\} \tag{4.116}$$

The corresponding position error estimates can be derived without any difficulty. Note that we have previously established that the errors are mainly associated with $\Delta\omega_{y_0}$ and $\Delta\omega_{z_0}$. Here the situation is basically the same. The equivalent accelerometer errors and gyro drift, corresponding to $\Delta\omega_{y_0}$, $\Delta\omega_{z_0}$, are found to be

$$\left.\begin{aligned}
\Delta\tilde{n}_{x_0} &= -u^2\frac{z^2}{2}(\beta_{31}\cos S + \beta_{32}\sin S)(-\beta_{31}\sin S + \beta_{32}\cos S),\\
\Delta\tilde{n}_{y_0} &= u\left(\frac{V^0}{a} + u\beta_{33}\right)\frac{z^2}{2}(\beta_{31}\cos S + \beta_{32}\sin S),\\
\Delta\tilde{\omega}_{y_0} &= \left(\frac{V^0}{a} + u\beta_{33}\right)\frac{z^2}{2},\\
\Delta\tilde{\omega}_{z_0} &= -u\frac{z^2}{2}(\beta_{31}\cos S + \beta_{32}\sin S).
\end{aligned}\right\} \tag{4.117}$$

From (4.117) we thus have

$$\left.\begin{aligned}
&|\Delta\tilde{n}_{x_0}| \leq au^2\frac{z^2}{2}, \quad |\Delta\tilde{n}_{y_0}| \leq au\left(\frac{V^0}{a} + u\right)\frac{z^2}{2},\\
&|\Delta\tilde{\omega}_{y_0}| \leq \left(\frac{V^0}{a} + u\right)\frac{z^2}{2}, \quad |\Delta\tilde{\omega}_{z_0}| \leq u\frac{z^2}{2}.
\end{aligned}\right\} \tag{4.118}$$

For $V^0 = 2000$ km/sec, $z = 2°$ ($rz \approx 220$ km), we get

$$\left.\begin{aligned}
&\Delta\tilde{n}_{x_0} \leq 0.015\ \text{cm/sec}^2 \approx 1.5\cdot 10^{-5} g_e,\\
&\Delta\tilde{n}_{y_0} \leq 0.075\ \text{cm/sec}^2 \approx 7.5\cdot 10^{-5} g_e,\\
&\Delta\tilde{\omega}_{y_0} \leq 0.04\ \text{deg/hr}, \quad \Delta\tilde{\omega}_{z_0} \leq 0.008\,\text{deg/hr},
\end{aligned}\right\} \tag{4.119}$$

Comparison with estimates (4.105), (4.109) shows that Eqs. (4.114) are much more effective than the corresponding equations in (4.92).

Eqs. (4.114) were derived by inserting from (4.113) in the corresponding equations in (4.72), whereas (4.92) were derived putting $\sin z = 0$, $\cos z = 1$ in

all the relations at the same time. There is another alternative however: in the expression for $\sin\varphi$ we may put $\sin z = 0$, $\cos z = 1$, and in the expressions for S and ω_z we may take $\sin z = z$, $\cos z = 1$. For velocities of up to 2000 m/sec and deviations of up to 200 km from the orthodromy ($z \approx 2°$) this gives estimates close to (4.119).

In conclusion note that the integrands in the first two relations in (4.92) contain $\dot{r}\omega_y$ and $\dot{r}\omega_x$. For velocities small compared to the circular velocity, r can be replaced with h in these quantities. This will lead to additional equivalent accelerometer errors

$$\left.\begin{aligned} \Delta\tilde{n}_x &= ae^2\dot{q}\omega_z \sin q \cos q, \\ \Delta\tilde{n}_y &= -ae^2\dot{q}\omega_z \sin q \cos q, \end{aligned}\right\} \tag{4.120}$$

which are estimated without any difficulty, since clearly

$$|\dot{q}| \leqslant \max\{|\omega_x|, \ |\omega_y|\}.$$

4.2.5. Different methods of accounting for the horizontal component of the Earth's gravitation

We have so far considered simplified ideal operation equations derived from approximate representations of individual relations and expressions entering these equations. Further simplification can be accomplished if we drop some terms or form them approximately as a function of time using the parameters of the programmed trajectory. The latter approach is permissible if the object has a definite programmed trajectory to which it keeps with satisfactory accuracy.

Let us consider one of such simplifications. The integrand in the first two equations in (2.23) contains the x_0 and y_0 projections of the Earth's gravitation, i.e.,

$$\left.\begin{aligned} &\frac{g_y(r, q)}{\cos q}(-\beta_{31}\sin S + \beta_{32}\cos S), \\ &\frac{g_y(r, q)}{\cos q}(-\beta_{31}\sin z\cos S - \beta_{32}\sin z\sin S + \beta_{33}\cos z). \end{aligned}\right\} \tag{4.121}$$

these functions are formed using the coordinates S and q which are measured by the inertial system. Let us elucidate what errors are involved if these projections, characterizing the nonspherical gravitation component, are omitted.

The omission of these horizontal components of the Earth's gravitation from the first two equations in (4.121) clearly introduces the following equivalent accelerometer errors in the right-hand sides of the error equations (4.35):

$$\left.\begin{aligned} \Delta\tilde{n}_x &= -g_e(q - e^2)\sin q(-\beta_{31}\sin S + \beta_{32}\cos S), \\ \Delta\tilde{n}_y &= -g_e(q - e^2)\sin q(-\beta_{31}\sin z\cos S - \\ &\qquad -\beta_{32}\sin z\sin S + \beta_{33}\cos z). \end{aligned}\right\} \tag{4.122}$$

We will confine the discussion to an object moving with constant velocity V^0 at a constant distance r^0 from the Earth's center in the orthodromy plane

$z = 0$. Taking the angle S to be initially zero (without loss of generality), we obtain at any time

$$S = \frac{V^0 t}{r^0} = \dot{S}^0 t. \tag{4.123}$$

Since for motion in the orthodromy plane $z = 0$

$$\sin\varphi = \beta_{31}\cos S + \beta_{32}\sin S,$$

Eqs. (4.122) for $\Delta\tilde{n}_{x_0}$, $\Delta\tilde{n}_{y_0}$ take the form

$$\left.\begin{aligned} \Delta n_{x_0} &= -g_e(q-e^2)\left(\frac{\beta_{32}^2-\beta_{31}^2}{2}\sin 2S + \beta_{31}\beta_{32}\cos 2S\right), \\ \Delta n_{y_0} &= -g_e(q-e^2)\beta_{31}(\beta_{33}\cos S + \beta_{32}\sin S), \end{aligned}\right\} \tag{4.124}$$

where S is expressed by (4.123).

Suppose that there are no intrinsic instrumental errors and the right-hand sides of the error equations (4.35) contain only $\Delta\tilde{n}_{x_0}$, $\Delta\tilde{n}_{y_0}$. The error equations thus take the form

$$\left.\begin{aligned} \delta\ddot{x} + (\omega_0^2 - \omega_y^2 - \omega_z^2)\delta x + (\omega_x\omega_y - \dot{\omega}_z)\delta y - 2\omega_z\delta\dot{y} &= \Delta\tilde{n}_{x_0}, \\ \delta\ddot{y} + (\omega_0^2 - \omega_z^2 - \omega_x^2)\delta y + (\omega_y\omega_x + \dot{\omega}_z)\delta x + 2\omega_z\delta\dot{x} &= \Delta\tilde{n}_{y_0}, \end{aligned}\right\} \tag{4.125}$$

where $\omega_0^2 = \mu/r^{0^3}$.

Let $\dot{S}^0$ be small so that

$$(\dot{S}^0 + u)^2 \ll \omega_0^2. \tag{4.126}$$

We have seen in §2.3 that in this case Eqs. (4.125) can be replaced by the simplified equations

$$\left.\begin{aligned} \delta\ddot{x} + \omega_0^2\delta x &= \Delta\tilde{n}_{x_0}, \\ \delta\ddot{y} + \omega_0^2\delta y &= \Delta\tilde{n}_{y_0}. \end{aligned}\right\} \tag{4.127}$$

The solutions of these equations coincide to high accuracy with the solutions of Eqs. (4.125) at least during several periods of frequency ω_0.

Taking zero initial conditions for Eqs. (4.127), we obtain their solutions in the form

$$\left.\begin{aligned} \delta x &= \frac{1}{\omega_0}\int_0^t \Delta\tilde{n}_{x_0}(\tau)\sin\omega_0(t-\tau)\,d\tau, \\ \delta y &= \frac{1}{\omega_0}\int_0^t \Delta\tilde{n}_{y_0}(\tau)\sin\omega_0(t-\tau)\,d\tau. \end{aligned}\right\} \tag{4.128}$$

Inserting for $\Delta\tilde{n}_{x_0}$, $\Delta\tilde{n}_{y_0}$ their expressions from (4.124) and integrating (using (4.123)), we get

$$\left.\begin{aligned} \delta x &= -\frac{g_e(q-e^2)}{\omega_0^2 - 4\dot{S}^{0^2}}\left[\frac{\beta_{32}^2-\beta_{31}^2}{2}(\sin 2S - 2\frac{\dot{S}^0}{\omega_0}\sin\omega_0 t + \beta_{31}\beta_{32}(\cos 2S - \cos\omega_0 t)\right], \\ \delta y &= -\frac{g_e(q-e^2)}{\omega_0^2 - \dot{S}^{0^2}}\left[\beta_{31}\beta_{33}(\cos S - \cos\omega_0 t) + \beta_{31}\beta_{32}\left(\sin S - \frac{\dot{S}^0}{\omega_0}\sin\omega_0 t\right)\right]. \end{aligned}\right\} \tag{4.129}$$

Expanding these expressions in powers of $\dot{S}^0/\omega_0$ and retaining terms of the order $(\dot{S}^0/\omega_0)^2$, we get

$$\left.\begin{aligned}
\delta x = &-\frac{g_e(q-e^2)}{\omega_0^2}\left\{\frac{\beta_{32}^2-\beta_{31}^2}{2}\sin 2S + \right.\\
&+\beta_{31}\beta_{32}\cos 2S - \beta_{31}\beta_{32}\cos\omega_0 t - \\
&-\frac{\dot{S}^0}{\omega_0}(\beta_{32}^2-\beta_{31}^2)\sin\omega_0 t + \frac{2\dot{S}^{0\,2}}{\omega_0^2}[2\beta_{31}\beta_{32}(\cos\omega_0 t - \\
&\left.-\cos 2S) - (\beta_{32}^2-\beta_{31}^2)\sin 2S]\right\}.\\
\delta y = &-\frac{g_e(q-e^2)}{\omega_0^2}\{\beta_{31}\beta_{33}\cos S + \beta_{31}\beta_{32}\sin S + \\
&+\beta_{31}\beta_{33}\cos\omega_0 t + \beta_{31}\beta_{32}\frac{\dot{S}^0}{\omega_0}\sin\omega_0 t + \\
&+\frac{\dot{S}^{0\,2}}{\omega_0^2}[\beta_{31}\beta_{33}(\cos\omega_0 t - \cos S) - \beta_{31}\beta_{32}\sin S]\}.
\end{aligned}\right\} \qquad (4.130)$$

Comparison of Eqs. (4.130) and (4.124) gives δx, δy in the form

$$\left.\begin{aligned}
\delta x = &\frac{\Delta\tilde{n}_{x_0}}{\omega_0^2} - \frac{\Delta\tilde{n}_{x_0}^0}{\omega_0^2}\cos\omega_0 t - \frac{\Delta\tilde{n}_{x_0}^0}{\omega_0^3}\sin\omega_0 t - \\
&-\frac{g_e(q-e^2)}{\omega_0^2}\frac{2\dot{S}^{0\,2}}{\omega_0^2}[2\beta_{31}\beta_{32}(\cos\omega_0 t - \cos 2S) - \\
&\left.-(\beta_{32}^2-\beta_{31}^2)\sin 2S],\right.\\
\delta y = &\frac{\Delta\tilde{n}_{y_0}}{\omega_0^2} - \frac{\Delta\tilde{n}_{y_0}^0}{\omega_0^2}\cos\omega_0 t - \frac{\Delta\tilde{n}_{y_0}^0}{\omega_0^3}\sin\omega_0 t + \\
&+\frac{g_e(q-e^2)}{\omega_0^2}[\beta_{31}\beta_{33}(\cos\omega_0 t - \cos S) - \beta_{31}\beta_{32}\sin S].
\end{aligned}\right\} \qquad (4.131)$$

According to (4.43), (4.32), the errors δx, δy correspond to a deviation of the axis z of the inertial platform from the direction to the Earth's center by the angles

$$\theta_{1x_0} = -\frac{\delta y}{r^0}, \quad \theta_{1y_0} = \frac{\delta x}{r^0}. \qquad (4.132)$$

On the other hand,

$$\theta'_{1x_0} = -\frac{\Delta\tilde{n}_{y_0}}{r\omega_0^2}, \quad \theta'_{1y_0} = \frac{\Delta\tilde{n}_{x_0}}{r\omega_0^2} \qquad (4.133)$$

are in fact the angles which characterize the deviation of the gravitational acceleration vector from the direction to the Earth's center.

From (4.130)—(4.133) it follows that if initially the platform z axis deviates from the direction to the Earth's center by

$$\theta_{1x_0}^0 = -\theta_{1x_0}^{0'}, \quad \theta_{1y_0}^0 = -\theta_{1y_0}^{0'}, \qquad (4.134)$$

i.e., it points along the gravitational acceleration vector, and it has an additional angular velocity such that

$$\dot{\theta}_{1x_0}^0 = -\dot{\theta}_{1x_0}^{0'}, \quad \dot{\theta}_{1y}^0 = -\dot{\theta}_{1y_0}^{0'} \qquad (4.135)$$

the z axis of the platform will remain caged to the gravitational acceleration vector to within

$$\left.\begin{aligned}\Delta\theta_{1x_0} &= -\frac{g_e(q-e^2)}{r^0\omega_0^2}\frac{\dot{S}^{0^2}}{\omega_0^2}a,\\ \Delta\theta_{1y_0} &= -\frac{g_e(q-e^2)}{r^0\omega_0^2}\frac{2\dot{S}^{0^2}}{\omega_0^2}b.\end{aligned}\right\} \tag{4.136}$$

Here

$$\left.\begin{aligned}a &= \beta_{31}\beta_{33}(\cos\omega_0 t-\cos S)-\beta_{31}\beta_{32}\sin S,\\ b &= 2\beta_{31}\beta_{32}(\cos\omega_0 t-\cos 2S)-(\beta_{32}^2-\beta_{31}^2)\sin 2S.\end{aligned}\right\} \tag{4.137}$$

We choose $\beta_{ij}=n_{ij}$ in accordance with Eqs. (II.4) in Appendix II. Then

$$\beta_{31}=\sin\varphi_1,\quad \beta_{32}=\cos\varphi_1\cos\psi_0,\quad \beta_{32}=\cos\varphi_1\sin\psi_0,$$

where φ_1 is the latitude of the starting point, and ψ_0 is the angle between the orthodromy plane $z=0$ and the northward direction at the starting point.

Inserting these expressions for β_{ij} in Eqs. (4.137) for a and b, we get

$$\left.\begin{aligned}a &= \tfrac{1}{2}\sin 2\varphi_1[\sin\psi_0\cos\omega_0 t-\sin(\psi_0+S)],\\ b &= \sin 2\varphi_1\cos\psi_0(\cos\omega_0 t-\cos 2S)-(\cos^2\varphi_1\cos^2\psi_0-\sin^2\varphi_1)\sin 2S.\end{aligned}\right\} \tag{4.138}$$

From these equations we get

$$\max a=1,\quad \max b=2. \tag{4.139}$$

The latter value is obtained for $\varphi_1=\frac{\pi}{4}$ and $\psi_0=0$. In virtue of (4.139), we have the estimates

$$\left.\begin{aligned}|\Delta\theta_{1x_0}| &\leqslant (e^2-q)\frac{\dot{S}^{0^2}}{\omega_0^2},\\ |\Delta\theta_{1y_0}| &\leqslant (e^2-q)\frac{4\dot{S}^{0^2}}{\omega_0^2}.\end{aligned}\right\} \tag{4.140}$$

For sea vessels, when the velocity relative to the Earth is less than 50 knots (≈ 30 m/sec), the right-hand sides of Eqs. (4.140) are vanishingly small. Turning to Eqs. (4.132), (4.131) we see that insertion of these vanishingly small velocities in the expressions for θ_{1x_0}, θ_{1y_0} makes even the terms linear in $\dot{S}^0/\omega_0$ exceedingly small. Indeed,

$$\left.\begin{aligned}\frac{\Delta\tilde{n}^0_{x_0}}{r^0\omega_0^3} &= (e^2-q)(\beta_{32}^2-\beta_{31}^2)\frac{\dot{S}^0}{\omega_0},\\ \frac{\Delta\tilde{n}^0_{y_0}}{r^0\omega_0^3} &= (e^2-q)\beta_{31}\beta_{32}\frac{\dot{S}^0}{\omega_0},\end{aligned}\right\} \tag{4.141}$$

and for $\dot{S}^0$ corresponding to a velocity of 50 knots the right-hand sides of Eqs. (4.141) are at most 0.04′ and 0.02′, respectively. Thus, for θ_{1x_0} and θ_{1y_0} we obtain the approximate expressions

$$\theta_{1x_0}=-\frac{\Delta\tilde{n}_{y_0}}{r^0\omega_0^2}+\frac{\Delta\tilde{n}^0_{y_0}}{r^0\omega_0^2}\cos\omega_0 t,\qquad \theta_{1y_0}=\frac{\Delta\tilde{n}_{x_0}}{r^0\omega_0^2}-\frac{\Delta\tilde{n}^0_{x_0}}{r^0\omega_0^2}\cos\omega_0 t. \tag{4.142}$$

Eqs. (4.142), in particular, provide an answer to the question presented at the end of §3.2, namely what will be the results if the horizontal gravitation component is not compensated in the gyrocompass horizon. Comparing Eqs. (3.106) for the compensating torques M_x^*, M_y^* with Eqs. (4.124) for $\tilde{n}_x$, $\tilde{n}_y$, we readily see that Eqs. (4.142) give the deviation of the z axis of the gyrocompass horizon from the geocentric vertical. Hence it follows that if initially the z axis of the gyrocompass horizon is pointed along the Earth's gravitation vector (i.e., along the gravitational vertical), it will remain caged to the gravitational vertical with high accuracy.

For subsonic craft, when $V^0 < 300$ m/sec, the right-hand sides of inequalities (4.140) are negligibly small, and the right-hand sides of (4.141) are respectively equal to 0'.13 and 0'.07.

For flight speeds $V^0 \approx 1000$ m/sec, we have from inequalities (4.140)

$$|\Delta\theta_{1x}| < 5 \cdot 10^{-5} \approx 0.'17, \quad |\Delta\theta_{1y}| < 2 \cdot 10^{-4} \approx 0.'7. \tag{4.143}$$

The corresponding estimates in (4.141) are 0'.34 and 0'.17.

Thus, if the terms characterizing the horizontal gravitation component are dropped from the first two equations in (4.72), then in case of low velocities, e.g., up to 300 m/sec, the following terms should be additionally introduced in the right-hand sides of the fifth and the sixth equations in (4.72):

$$\Delta S = -\frac{\Delta\tilde{n}_y}{r^0\omega_0^2}, \quad \Delta z = -\frac{\Delta\tilde{n}_{x_0}}{r^0\omega_0^2}. \tag{4.144}$$

For velocities of the order of 1000 m/sec, in the three equations in (4.72) used in forming the gyroframe control torques, the projections ω_{x_0}, ω_{y_0}, ω_{z_0} should be replaced with $\omega_{x_0} + \Delta\omega_{x_0}$, $\omega_{y_0} + \Delta\omega_{y_0}$, $\omega_{z_0} + \Delta\omega_{z_0}$, where

$$\left.\begin{aligned} \Delta\omega_{x_0} &= \frac{\Delta\tilde{n}_y^0}{r^0\omega_0^2}, \quad \Delta\omega_{y_0} = -\frac{\Delta\tilde{n}_{x_0}^0}{r^0\omega_0^2}, \\ \Delta\omega_{z_0} &= \Delta\omega_y \operatorname{tg} z. \end{aligned}\right\} \tag{4.145}$$

At any velocity, the gyroframe z axis should be initially directed along the gravitational vertical. It is therefore inadvisable to try and change Eqs. (4.72) by eliminating g_{x_0}, g_{y_0} from the first two equations and adding relations (4.144), (4.145) to the system. Indeed, the amount of computational work will not be reduced, whereas to orient the platform z axis along the gravitational vertical is no easier than along the geocentric vertical.

A different approach is possible, however: instead of dropping the terms g_{x_0}, g_{y_0} characterizing the horizontal gravitation components in the integrands in the first two equations in (4.72), it would be better to omit the projections of the gravitational components on the axes x_0, y_0. In this case adding expressions similar to (4.144), (4.145), the platform z axis can be initially oriented along the gravitational vertical. The platform axis will then retain this orientation (with slight errors) during the entire course of motion.

The x_0 and y_0 projections of the centrifugal force of the Earth's spin acting on a unit mass at the point O are given by

$$\left.\begin{aligned} f_{x_0} &= -ru^2 \sin\varphi\,(-\beta_{31}\sin S + \beta_{32}\cos S), \\ f_{y_0} &= -ru^2 \sin\varphi\,(-\beta_{31}\sin z\cos S - \\ &\qquad -\beta_{32}\sin z\sin S + \beta_{33}\cos z). \end{aligned}\right\} \tag{4.146}$$

The integrands of the first two equations in (4.72) can be written in the form

$$\left.\begin{aligned} n_{x_0} + v_{y_0}\omega_{z_0} - \dot{r}\omega_{y_0} + g_{x_0} = \\ = n_{x_0} + v_{y_0}\omega_{z_0} - \dot{r}\omega_{y_0} - f_{x_0} - \\ - g_e e^2 \sin\varphi\,(-\beta_{31}\sin S + \beta_{32}\cos S), \\ n_{y_0} + \dot{r}\omega_{x_0} - v_{x_0}\omega_{z_0} + g_{y_0} = \\ = n_{y_0} + \dot{r}\omega_{x_0} - v_{x_0}\omega_{z_0} - f_{y_0} - \\ - g_e e^2 \sin\varphi\,(-\beta_{31}\sin z\cos S - \\ - \beta_{32}\sin z\sin S + \beta_{33}\cos z). \end{aligned}\right\} \qquad (4.147)$$

The first two equations in (4.92) now take the form

$$\left.\begin{aligned} v_{x_0} = \int_0^t [n_{x_0} + v_{y_0}\omega_{z_0} - \dot{r}\omega_{y_0} - f_{x_0} - \\ - g_e e^2 \sin\varphi\,(-\beta_{31}\sin S + \beta_{32}\cos S)]\,dt + v^0_{x_0}, \\ v_{y_0} = \int_0^t [n_{y_0} + \dot{r}\omega_{x_0} - v_{x_0}\omega_{z_0} - f_{y_0} - \\ - g_e e^2 \sin\varphi\,(-\beta_{31}\sin z\cos S - \\ - \beta_{32}\sin z\sin S + \beta_{33}\cos z)]\,dt + v^0_{y_0}. \end{aligned}\right\} \qquad (4.148)$$

The last term in the integrand can be omitted. Using Eqs. (4.146) we get

$$\left.\begin{aligned} v_{x_0} = \int_0^t [n_{x_0} + v_{y_0}\omega_{z_0} - \dot{r}\omega_{y_0} + \\ + ru^2 \sin\varphi\,(-\beta_{31}\sin S + \beta_{32}\cos S)]\,dt + v^0_{x_0}, \\ v_{y_0} = \int_0^t [n_{y_0} + \dot{r}\omega_{x_0} - v_{x_0}\omega_{z_0} + \\ + ru^2 \sin\varphi\,(-\beta_{31}\sin z\cos S - \\ - \beta_{32}\sin z\sin S + \beta_{33}\cos z)]\,dt + v^0_{y_0}. \end{aligned}\right\} \qquad (4.149)$$

On the right in the error equations (4.35) the following equivalent accelerometer errors now appear:

$$\left.\begin{aligned} \Delta\tilde{n}_{x_0} = g_e e^2 \sin\varphi\,(-\beta_{31}\sin S + \beta_{32}\cos S), \\ \Delta\tilde{n}_{y_0} = g_e e^2 \sin\varphi\,(-\beta_{31}\sin z\cos S - \beta_{32}\sin z\sin S + \beta_{33}\cos z). \end{aligned}\right\} \qquad (4.150)$$

Inserting the expression for $\sin\varphi$ in terms of the trigonometric functions of the angles S, z and constants β_{ij}, we obtain for $z = 0$

$$\left.\begin{aligned} \Delta\tilde{n}_{x_0} = g_e e^2\left(\frac{\beta_{32}^2 - \beta_{31}^2}{2}\sin 2S + \beta_{31}\beta_{32}\cos 2S\right), \\ \Delta\tilde{n}_{y_0} = g_e e^2(\beta_{31}\beta_{33}\cos S + \beta_{31}\beta_{32}\sin S). \end{aligned}\right\} \qquad (4.151)$$

The right-hand sides of Eqs. (4.151) differ from Eqs. (4.124) only in a constant factor (just like Eqs. (4.150) differ from Eqs. (4.122)). The solution of Eqs. (4.125) is therefore (4.131) and all the previous conclusions derived from this solution remain in force. In particular, the estimates (4.140) and the equalities (4.141) are applicable, the only difference being that e^2 is substituted for $(e^2 - q)$. Since $e^2 \approx 2(e^2 - q)$ the above numerical estimates are doubled.

Thus, for small velocities of the object relative to the Earth, the first two equations in (4.72) can be replaced by Eqs. (4.149), adding on the right in the fifth and the sixth equation according to (4.144) and (4.150) the error terms

$$\left.\begin{aligned}\Delta S &= -ae^2\sin\varphi\,(-\beta_{31}\sin z\cos S - \\ &\qquad -\beta_{32}\sin z\cos S+\beta_{33}\cos z),\\ \Delta z &= -\frac{ae^2\sin\varphi}{\cos z}(-\beta_{31}\sin S+\beta_{32}\cos S),\end{aligned}\right\} \tag{4.152}$$

and incrementing the components ω_{x_0}, ω_{y_0}, ω_{z_0} in the corresponding equations in (4.72), according to (4.145), by

$$\Delta\omega_{x_0}=\frac{\Delta\tilde{n}_y^0}{r^0\omega_0^2},\quad \Delta\omega_{y_0}=\frac{-\Delta\tilde{n}_x^0}{r^0\omega_0^2},\quad \Delta\omega_{z_0}=\Delta\omega_{y_0}\operatorname{tg} z, \tag{4.153}$$

where $\Delta\tilde{n}_x$, $\Delta\tilde{n}_y$ are determined by Eqs. (4.150), and initially the platform z axis is pointed along the gravitational vertical, without changing the azimuthal orientation of the platform. In accordance with the above estimates, the position errors for sea vessels and subsonic craft moving close to the Earth are negligible, and the platform z axis is steadily caged (with minor errors) to the gravitational vertical. Neglecting the terms (4.153), we introduce an orientation error of some 0'.25 and a position error of about 0.5 km.

For velocities of the order of 1000 m/sec, the position errors allowing for corrections (4.145) are $\approx$ 2.5 km, and the orientation errors are $\approx$ 1'.4. These errors are quite substantial. It should be remembered, however, that only the upper-bound estimates have been derived, and the actual errors are much smaller.

4.2.6. Other simplifications

Let us consider one further possibility of simplifying the ideal operation equations (4.72). The integrands in the simplified equations (4.149) contain the sums

$$v_y\omega_{z_0}-\dot{r}\omega_{y_0}-f_{x_0},\quad \dot{r}\omega_{x_0}-v_x\omega_{z_0}-f_{y_0}. \tag{4.154}$$

They can be written in the form

$$\left.\begin{aligned}v_{x_0}\omega_{z_0}-\dot{r}\omega_{y_0}-f_{x_0} &= -r\omega_{x_0}\dot{S}\sin z+r\dot{z}u\sin\varphi-\dot{r}\omega_{y_0},\\ \dot{r}\omega_{x_0}-v_x\omega_z-f_y &= \dot{r}\omega_{x_0}-r\dot{S}(\omega_{y_0}+u\sin\varphi\cos z).\end{aligned}\right\} \tag{4.155}$$

If for r we use the approximate expression (4.65) and assume small h, $\dot{S}$, and $\dot{z}$, Eqs. (4.155) can be simplified as follows:

$$\left.\begin{aligned}v_{x_0}\omega_{z_0}-\dot{r}\omega_{y_0}-f_{x_0} &= -\omega_{y_0}\dot{h}-a\dot{S}\omega_{x_0}\sin z+a\dot{z}u\sin\varphi,\\ \dot{r}\omega_{x_0}-v_{x_0}\omega_z-f_y &= \omega_{x_0}\dot{h}-a\dot{S}(\omega_{y_0}+u\sin\varphi\cos z).\end{aligned}\right\} \tag{4.156}$$

The resulting errors are clearly equivalent to some accelerometer errors and can be easily found by subtracting the right-hand sides of Eqs. (4.155) from the right-hand sides of Eqs. (4.156). Estimates of these errors show that they are negligible for subsonic velocities near the Earth's

surface. In some cases these errors can be ignored at velocities of 1000 m/sec as well. It all depends on the accuracy standards of the inertial system.

The right-hand sides of Eqs. (4.155), (4.156) depend on the velocity relative to the Earth in such a way that they vanish when the object is stationary relative to the Earth. For small velocities of the object relative to the Earth, the right-hand sides of Eqs. (4.155), (4.156) are also small (by virtue of continuity). Thus, for a sea vessel moving on the Earth's surface near the plane $z = 0$ we find (seeing that its velocity is much less than the peripheral velocities of the points on the equator due to the Earth's spin)

$$v_{x_0}\omega_{z_0} - \dot{r}\omega_{y_0} - f_{x_0} \approx au\dot{z}\sin\varphi,$$
$$\dot{r}\omega_{x_0} - v_{x_0}\omega_{z_0} - f_{y_0} \approx -au\dot{S}(\beta_{33} + \sin\varphi).$$

For speeds of some 50 knots, when $|a\dot{z}|, |a\dot{S}| < 30$ m/sec, we have

$$\left.\begin{aligned} |v_{x_0}\omega_{z_0} - \dot{r}\omega_{y_0} - f_{x_0}| &< u|a\dot{z}| \approx 0.22 \text{ cm/sec}^2, \\ |\dot{r}\omega_{x_0} - v_{x_0}\omega_{z_0} - f_{y_0}| &< 2u|a\dot{S}| \approx 0.44 \text{ cm/sec}^2. \end{aligned}\right\} \qquad (4.157)$$

If the ship's position has to be fixed to within 1—2 nautical miles (≈ 1.8—3.6 km), the terms (4.156) in the integrands of (4.149) can be ignored. Eqs. (4.149) then take the form

$$v_{x_0} = \int_0^t n_{x_0}\,dt + v_{x_0}^0, \quad v_{y_0} = \int_0^t n_{y_0}\,dt + v_{y_0}^0. \qquad (4.158)$$

If the object's position has to be determined to within better than 1 mile, or if much higher speeds are considered (e.g., of the same order of magnitude as the peripheral velocity on the equator), the terms (4.156) can no longer be neglected in Eqs. (4.149)

Under certain conditions a different approach is possible: (4.156) or (4.155) can be formed using approximate values of the relevant variables and parameters. Here we are faced by two alternatives: the information on the relevant variables which enter (4.155), (4.156) can be supplied to the inertial system from external sources, or alternatively these variables are formed as a function of time using information on the programmed trajectory of the object. The latter alternative is naturally practicable only if the object follows a preprogrammed trajectory.

The simplified expressions (4.158), in which the terms (4.156) have been omitted, and the use of external information or programmed trajectory data will substantially change the first group of error equations (4.35) of the inertial system.

If the terms (4.156) are entirely omitted from the ideal operation equations, i.e., when Eqs. (4.158) are adopted, Eqs. (4.35) acquire on the right equivalent accelerometer errors $\Delta\tilde{n}_{x_0}$, $\Delta\tilde{n}_{y_0}$, which are equal to the right-hand sides of Eqs. (4.156) taken with minus sign. If, however, the expressions (4.156) are formed approximately, the equivalent accelerometer errors are equal to differences between the approximate and the true values of these quantities. The dependence of the right-hand sides of Eqs. (4.35) on the intrinsic instrumental errors also changes.

The left-hand sides of Eqs. (4.35) also change in both cases. Here the terms $v_y\omega_z - \dot{r}\omega_y$, $\dot{r}\omega_x - v_x\omega_z$ in the first two equations in (4.20) are not formed as a function of the coordinates and velocities measured by the inertial system. Therefore, terms contributed by the variations of these expressions drop out from the left-hand sides of Eqs. (4.35)

Neglecting, as always, the variation of the nonspherical component of the gravitational field, we obtain instead of Eqs. (4.25)

$$\left.\begin{aligned} \delta\dot{v}_x &= \delta n_x, \\ \delta\dot{v}_y &= \delta n_y. \end{aligned}\right\} \tag{4.159}$$

Inserting the expressions for δn_x, δn_y, δv_{x_0}, δv_{y_0} from (4.26), (4.24), (4.7), (4.8), (4.27), (4.12), substituting the variables from (4.32), and replacing γ with θ_{1z} and g_z with $-\mu/r^3$, we obtain the equations

$$\left.\begin{aligned} &\delta\ddot{x} - \frac{\dot{r}}{r}\delta\dot{x} + \left(\frac{\mu}{r^3} - \omega_{x_0}^2 - \omega_{y_0}^2 + \frac{\dot{r}^2}{r^2}\right)\delta x - \\ &- \omega_z\delta\dot{y} - \dot{\omega}_z\delta y - r\omega_y\dot{\theta}_{1z} + (\dot{r}\omega_y - r\omega_y\omega_z)\theta_{1z} = \\ &\quad = \Delta n_x - \dot{r}\Delta\omega_y - r\Delta\dot{\omega}_y - \dot{\omega}_y\Delta r + \omega_y\Delta\dot{r}, \\ &\delta\ddot{y} - \frac{\dot{r}}{r}\delta\dot{y} + \left(\frac{\mu}{r^3} - \omega_{x_0}^2 - \omega_{y_0}^2 + \frac{\dot{r}^2}{r^2}\right)\delta y + \\ &+ \omega_z\delta\dot{x} + \dot{\omega}_z\delta x - r\omega_y\dot{\theta}_{1z} + (\dot{r}\omega_y + r\omega_x\omega_z)\theta_{1z} = \\ &\quad = \Delta n_y - \dot{r}\Delta\omega_y - r\Delta\dot{\omega}_x + \dot{\omega}_x\Delta r + \omega_x\Delta\dot{r}. \end{aligned}\right\} \tag{4.160}$$

Together with Eqs. (4.40)—(4.44) they constitute a closed system of error equations for this case.

Comparison of Eqs. (4.160) with Eqs. (4.35) in the general case reveals fundamental differences. For $\dot{r} = \omega_x = \omega_y = \omega_z = 0$, i.e., for a stationary object, the two sets of equations coincide. This is understandable, since for a stationary object the simplified equations (4.158) are exact.

A general analysis of Eqs. (4.160) can be carried out by the methods previously applied to Eqs. (4.35). The following point should be kept in mind, however.

Eqs. (4.160) contain the projections θ_{1z} and $\dot{\theta}_{1z}$, which in virtue of Eqs. (4.44), (4.42) depend both on the variables δx, δy and on the variables θ_{x_0}, θ_{y_0}, θ_{z_0}. Eqs. (4.160) are thus coupled with Eqs. (4.40). This coupling can be eliminated by inserting for θ_{1z} in (4.160) its expression from (4.44) and expressing δx_2 in terms of δx and θ_y from (4.42).

Further simplification of the ideal operation equations, which involves omission of the terms (4.156) or their formation as a function of time, is meaningful only for small ω_{x_0}, ω_{y_0}, ω_{z_0}, of the order of magnitude of the Earth's spin. The smallness of ω_{x_0}, ω_{y_0}, ω_{z_0} also can be utilized in this case for simplifying the error equations (4.160). The error equations (4.160) can also be simplified seeing that z is generally small.

§4.3. SIMPLIFICATIONS IN AN INERTIAL SYSTEM WHICH MEASURES THE GEOGRAPHICAL COORDINATES

4.3.1. Exact ideal operation equations. Error equations

Consider an inertial system which measures the geographical coordinates h, φ', λ. The two accelerometers in this system lie in the unperturbed plane of the geographical horizon; one of them lies in the meridional plane through the current point of the object and the other in the plane of the corresponding parallel. The ideal operation equations of a two-accelerometer system of this kind using a gyrostable platform are Eqs. (2.26). The corresponding error equations, as we have noted in 2.2.2, are Eqs. (2.28), (1.96), (1.97), (1.118).

These equations were derived in Chapter 2 from the general equations of inertial navigation in curvilinear coordinates. As in the preceding section, we will now derive the ideal operation equations (and the corresponding error equations) for the particular system being considered. We will proceed along the same lines as in the derivation of the analogous equations for orthodromic coordinates.

Let, as before, $O_1\xi\eta\zeta$ be the axes fixed to the Earth (the axis ζ pointing along the Earth's spin axis), and $Ox_0y_0z_0$ the co-moving geographical axes. The relative orientation of these axes is given by the following matrix of direction cosines:

$$\begin{array}{c|ccc} & x_0 & y_0 & z_0 \\ \xi & -\sin\lambda & -\sin\varphi'\cos\lambda & \cos\varphi'\cos\lambda \\ \eta & \cos\lambda & -\sin\varphi'\sin\lambda & \cos\varphi'\sin\lambda \\ \zeta & 0 & \cos\varphi' & \sin\varphi'. \end{array} \tag{4.161}$$

We will now find the projections ω_{x_0}, ω_{y_0}, ω_{z_0} of the absolute angular velocity $\boldsymbol{\omega}$ of the axes $Ox_0y_0z_0$ on the corresponding directions. The axes $Ox_0y_0z_0$ rotate in relation to the axes $O_1\xi\eta\zeta$ with angular velocity $\boldsymbol{\omega} - \boldsymbol{u}$. The derivatives $d\boldsymbol{x}_0/dt$, $d\boldsymbol{y}_0/dt$, $d\boldsymbol{z}_0/dt$ of the unit vectors $\boldsymbol{x}_0$, $\boldsymbol{y}_0$, $\boldsymbol{z}_0$ in the axes $O_1\xi\eta\zeta$ are therefore written in the form

$$\left.\begin{aligned} \frac{d\boldsymbol{x}_0}{dt} = (\boldsymbol{\omega} - \boldsymbol{u}) \times \boldsymbol{x}_0, \quad \frac{d\boldsymbol{y}_0}{dt} = (\boldsymbol{\omega} - \boldsymbol{u}) \times \boldsymbol{y}_0, \\ \frac{d\boldsymbol{z}_0}{dt} = (\boldsymbol{\omega} - \boldsymbol{u}) \times \boldsymbol{z}_0. \end{aligned}\right\} \tag{4.162}$$

Dot-multiplying the left- and the right-hand sides of the second equation in (4.162) by $\boldsymbol{z}_0$ and the third equation by $\boldsymbol{y}_0$, we get

$$\omega_{x_0} = u_{x_0} + \frac{d\boldsymbol{y}_0}{dt} \cdot \boldsymbol{z}_0 = u_{x_0} - \frac{d\boldsymbol{z}_0}{dt} \cdot \boldsymbol{y}_0. \tag{4.163}$$

Similarly, multiplying the third equality in (4.162) by $\boldsymbol{x}_0$ or the first by $\boldsymbol{z}_0$, we get

$$\omega_{y_0} = u_{y_0} + \frac{d\boldsymbol{z}_0}{dt} \cdot \boldsymbol{x}_0 = u_{y_0} - \frac{d\boldsymbol{y}_0}{dt} \cdot \boldsymbol{z}_0. \tag{4.164}$$

Also

$$\omega_{z_0} = u_z + \frac{dx_0}{dt} \cdot y, \quad \omega_{y_0} = u_z - \frac{dy_0}{dt} \cdot x, \tag{4.165}$$

For further calculations we will use the matrix of direction cosines (4.161). According to this matrix, we have

$$\left.\begin{aligned} u_z &= 0,\ z_0 = \xi\cos\varphi'\cos\lambda + \eta\cos\varphi'\sin\lambda + \zeta\sin\varphi', \\ \frac{dy_0}{dt} &= -\xi\frac{d}{dt}(\sin\varphi'\cos\lambda) - \eta\frac{d}{dt}(\sin\varphi'\sin\lambda) + \zeta\frac{d}{dt}(\cos\varphi'), \end{aligned}\right\} \tag{4.166}$$

where ξ, η, ζ are the unit vectors along the axes ξ, η, ζ (the derivative dy_0/dt is taken in the axes $O_1\xi\eta\zeta$, i.e., assuming $d\xi/dt = d\eta/dt = d\zeta/dt = 0$).

Inserting (4.166) in (4.163), we get

$$\omega_{x_0} = -\dot{\varphi}'. \tag{4.167}$$

Similarly from (4.161) and Eqs. (4.164), (4.165) we get

$$\omega_{y_0} = (\dot{\lambda} + u)\cos\varphi', \quad \omega_{z_0} = (\dot{\lambda} + u)\sin\varphi' = \omega_{y_0}\operatorname{tg}\varphi'. \tag{4.168}$$

Now we proceed to determine the projections v_{x_0}, v_{y_0}, v_{z_0} of the absolute velocity v of the origin of the axes $Ox_0y_0z_0$ on the corresponding unit vectors. To this end we use Eqs. (4.9).

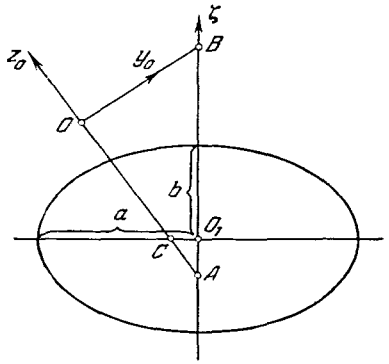

FIGURE 4.3.

Let the point P be at the Earth's center O_1. The coordinates $x_0^{O_1}$, $y_0^{O_1}$, $z_0^{O_1}$ of the point O_1 in the axes $Ox_0y_0z_0$ are then

$$\left.\begin{aligned} x_0^{O_1} &= 0, \quad y_0^{O_1} = \frac{ae^2\sin\varphi'\cos\varphi'}{(1 - e^2\sin^2\varphi')^{1/2}}, \\ z_0^{O_1} &= -h - a(1 - e^2\sin^2\varphi')^{1/2}. \end{aligned}\right\} \tag{4.169}$$

Eqs. (4.169) for $y_0^{O_1}$ and $z_0^{O_1}$ are obtained by passing the meridional section of the geoid through the axes y_0, z_0 (Figure 4.3) and examining the similar right triangles O_1AC and OAB (using Eqs. (1.8a).

Inserting (4.169) in (4.9) and seeing that O_1 is stationary relative to the axes $O_1\xi\eta\zeta$,

$$v_{x_0}^{O_1} = v_{y_0}^{O_1} = v_{z_0}^{O_1} = 0, \tag{4.170}$$

we obtain the equations

$$\left.\begin{aligned} v_{x_0} - \omega_{y_0}\left[h + a(1 - e^2\sin^2\varphi')^{1/2}\right] - \\ - \omega_z \frac{ae^2\sin\varphi'\cos\varphi'}{(1 - e^2\sin^2\varphi')^{1/2}} = 0, \\ \frac{ae^2\dot{\varphi}'[\cos^2\varphi' - \sin^2\varphi'(1 - e^2\sin^2\varphi')]}{(1 - e^2\sin^2\varphi')^{1/2}} + \\ + v_{y_0} + \omega_{x_0}\left[h + a(1 - e^2\sin^2\varphi')^{1/2}\right] = 0, \\ - \dot{h} - \frac{ae^2\sin\varphi'\cos\varphi'\dot{\varphi}'}{(1 - e^2\sin^2\varphi')^{1/2}} + v_{z_0} + \\ + \omega_z \frac{ae^2\sin\varphi'\cos\varphi'}{(1 - e^2\sin^2\varphi')^{1/2}} = 0. \end{aligned}\right\} \tag{4.171}$$

From (4.167), (4.168) $\dot{\varphi}' = -\omega_x$, $\omega_{z_0} = \omega_{y_0}\operatorname{tg}\varphi'$. Eqs. (4.171) thus take the form

$$\left.\begin{aligned} v_{x_0} - \omega_{y_0}\left[h + \frac{a}{(1 - e^2\sin^2\varphi')^{1/2}}\right] = 0, \\ v_{y_0} + \omega_{x_0}\left[h + \frac{a(1 - e^2)}{(1 - e^2\sin^2\varphi')^{1/2}}\right] = 0, \\ - \dot{h} + v_z = 0. \end{aligned}\right\} \tag{4.172}$$

Hence follow expressions analogous to (4.12):

$$v_{x_0} = r_2\omega_{y_0}, \quad v_{y_0} = -r_3\omega_{x_0}, \quad v_{z_0} = \dot{h}, \tag{4.173}$$

where

$$\left.\begin{aligned} r_2 = \frac{a}{(1 - e^2\sin^2\varphi')^{1/2}} + h, \\ r_3 = \frac{a(1 - e^2)}{(1 - e^2\sin^2\varphi')^{3/2}} + h \end{aligned}\right\} \tag{4.174}$$

are the radii of curvature of the principal normal sections of the surface $h = \text{const}$ at the point O: r_3 is the radius of curvature of the meridional section (through the axis y_0) and r_2 is the radius of curvature of the perpendicular section (through the axis x_0).

According to the fundamental equation of inertial navigation, the readings n_{x_0}, n_{y_0} of the accelerometers along the axes x_0, y_0 are expressed in the form

$$n_{x_0} = w_{x_0}, \quad n_{y_0} = w_{y_0} - g_{y_0}, \tag{4.175}$$

where by (4.8) and (4.173)

$$\left.\begin{aligned} w_{x_0} = \dot{v}_{x_0} + \dot{h}\omega_{y_0} - \omega_{z_0}v_{y_0}, \\ w_{y_0} = \dot{v}_{y_0} + \omega_{z_0}v_{x_0} - \dot{h}\omega_{x_0}. \end{aligned}\right\} \tag{4.176}$$

We have thus derived all the starting relations needed for the construction of the ideal operation equations. Solving Eqs. (4.175), (4.176) for v_{x_0}, v_{y_0} we obtain the first two equations in (2.26). From (4.173) and the second relation

in (4.168) we obtain the third, fourth, and fifth equation in (2.26). From (4.167) and the first equality in (4.168), we obtain the sixth and the seventh equation in (2.26). The eighth and the ninth equation in (2.26) coincide with (4.174). To obtain a closed system, we should further add the last six equations in (2.26) to the set.

Let us now proceed with a direct derivation of the error equations. This will be done by taking the variations of the ideal operation equations (2.26). As in the previous section, we will consider only the main instrumental errors, namely the accelerometer errors Δn_x, Δn_y, the gyroplatform drift $\Delta\omega_x$, $\Delta\omega_y$, $\Delta\omega_z$, and the altimeter error Δh.

From the first two equations in (2.26), we have

$$\left.\begin{aligned}\delta\dot v_x &= \delta n_x + \delta v_y\omega_z + v_y\delta\omega_z - \dot h\,\delta\omega_y - \Delta\dot h\omega_y,\\ \delta\dot v_y &= \delta n_y + \Delta\dot h\omega_x + \dot h\,\delta\omega_x - \delta v_x\omega_z - v_x\,\delta\omega_z + \delta g_{y_0},\end{aligned}\right\} \tag{4.177}$$

where the variation δg_{y_0} should be calculated using the last three equalities in (2.26) and the first relation in (1.27).

From the third, fourth, eighth, and ninth relations in (2.26), we get

$$\left.\begin{aligned}\delta v_x &= r_2\delta\omega_z + \omega_z\,\delta r_2,\\ \delta v_z &= -r_2\delta\omega_x - \omega_x\delta r_2,\\ \delta r_2 &= \frac{ae^2\sin\varphi'\cos\varphi'\,\delta\varphi'}{(1-e^2\sin^2\varphi')^{1/2}} + \Delta h,\\ \delta r_3 &= \frac{3ae\,(1-e^2)\sin\varphi'\cos\varphi'\,\delta\varphi}{(1-e^2\sin^2\varphi')^{5/2}} + \Delta h.\end{aligned}\right\} \tag{4.178}$$

From the sixth and the seventh equality in (2.26) we obtain the variational equations

$$\delta\dot\varphi' = -\delta\omega_{x_0},\quad \delta[(\dot\lambda + u)\cos\varphi'] = \delta\omega_{y_0}. \tag{4.179}$$

Finally, from the fifth, tenth, eleventh, and twelfth relations in (2.26) we get

$$\left.\begin{aligned}\delta\omega_{x_0} &= \dot\alpha + \omega_y\gamma - \omega_z\beta - \Delta\omega_x,\\ \delta\omega_{y_0} &= \dot\beta - \omega_x\gamma + \omega_z\alpha - \Delta\omega_y,\\ \delta\omega_{z_0} &= \dot\gamma + \omega_x\beta - \omega_y\alpha - \Delta\omega_z,\\ \delta\omega_{z_0} &= \delta\omega_{y_0}\operatorname{tg}\varphi' + \omega_y\frac{\delta\varphi}{\cos^2\varphi'}.\end{aligned}\right\} \tag{4.180}$$

Here $\Delta\omega_x$, $\Delta\omega_y$, $\Delta\omega_z$ are the gyro drift components of the stable platform about the corresponding axes; the angles α, β, γ characterize the perturbed position of the axes xyz fixed to the platform relative to the unperturbed axes $x_0y_0z_0$. The mutual orientation of the axes xyz and $x_0y_0z_0$ is described by the matrix of direction cosines (4.22). In accordance with this matrix, we obtain for the variations δn_{x_0}, δn_{y_0} in Eqs. (4.177)

$$\left.\begin{aligned}\delta n_{x_0} &= \Delta n_{x_0} + n_{y_0}\gamma - n_{z_0}\beta,\\ \delta n_{y_0} &= \Delta n_{y_0} + n_{z_0}\alpha - n_{x_0}\gamma.\end{aligned}\right\} \tag{4.181}$$

In (4.181), $\Delta n_{x_0}, \Delta n_{y_0}$ correspond to accelerometer errors; n_{x_0}, n_{y_0} are given by Eqs. (4.175), (4.176), and n_{z_0} is obtained from the equality

$$n_{z_0} = w_{z_0} - g_{z_0}. \tag{4.182}$$

Here

$$w_z = \ddot{h} + \omega_{x_0} v_{y_0} - \omega_{y_0} v_{x_0}, \tag{4.183}$$

and

$$g_{z_0} = g_{z_2} \cos(\varphi' - \varphi) + g_{y_2} \sin(\varphi' - \varphi), \tag{4.184}$$

where the functions $g_{y_2}(r, \varphi)$ and $g_{z_2}(r, \varphi)$ are given by (1.27).

Eqs. (4.177)—(4.180) together with Eqs. (4.181)—(4.183), (4.175), (4.176), (1.27) constitute a closed system of differential and algebraic equations for the errors $\delta\varphi'$, $\delta\lambda$ in the geographical coordinates. We will now show that these equations are equivalent to Eqs. (2.28), (1.96), (1.97).

Introducing, as in (4.37), new variables

$$\alpha_2 = -\delta\varphi', \quad \beta_2 = \delta\lambda \cos\varphi', \quad \gamma_2 = \delta\lambda \sin\varphi' \tag{4.184a}$$

and using the equalities

$$\gamma_2 = \beta_2 \operatorname{tg}\varphi', \quad \omega_{z_0} = \omega_{y_0} \operatorname{tg}\varphi', \tag{4.185}$$

we obtain from Eqs. (4.179)

$$\left.\begin{aligned} \dot{\alpha}_2 + \omega_{y_0}\gamma_2 - \omega_{z_0}\beta_2 &= \delta\omega_{x_0}, \\ \dot{\beta}_2 + \omega_{z_0}\alpha_2 - \omega_{x_0}\gamma_2 &= \delta\omega_{y_0}, \\ \dot{\gamma}_2 + \omega_{x_0}\beta_2 - \omega_{y_0}\alpha_2 &= \delta\omega_{z_0}, \end{aligned}\right\} \tag{4.186}$$

which coincide with Eqs. (4.38).

In (4.186) and (4.180) we make the substitution

$$\theta_{x_0} = \alpha_2 - \alpha, \quad \theta_{y_0} = \beta_2 - \beta, \quad \theta_{z_0} = \gamma_2 - \gamma. \tag{4.187}$$

and obtain equations which differ from Eqs. (1.96) only in that the right-hand sides contain the equivalent quantities $-\Delta\omega_{x_0}$, $-\Delta\omega_{y_0}$, $-\Delta\omega_{z_0}$ instead of the torques $\Delta m_{x_0}, \Delta m_{y_0}, \Delta m_{z_0}$.

Let us derive the first group of error equations. We will neglect, as before, the variation of the nonspherical gravitation component and products of variations with factors containing e^2. This is clearly equivalent to making the following simplifications in Eqs. (4.177), (4.178), (4.184):

$$\left.\begin{aligned} r_2 = r_3 = r, \quad \dot{h} = \dot{r}, \quad \ddot{h} = \ddot{r}, \\ \delta r_2 = \delta r_3 = \Delta r = \Delta h, \quad \delta\dot{r}_2 = \delta\dot{r}_3 = \Delta\dot{h}, \\ \delta g_{y_0} = 0, \quad g_{z_0} = -\mu/r^3. \end{aligned}\right\} \tag{4.188}$$

Eqs. (4.177) taken in conjunction with (4.178) are thus seen to coincide with Eqs. (4.25). They lead to Eqs. (4.28) and substitution of variables (4.32) gives Eqs. (4.35) which are equivalent to Eqs. (2.28).

4.3.2. Simplifications of the equations for geographical coordinates

The simplifications of the ideal operation equations (2.26) of an inertial system measuring the geographical coordinates are clearly analogous to the

simplifications in the case of orthodromic coordinates, treated in the previous section. We can again simplify the equations using the small eccentricity of the Earth, the closeness of the object to the Earth's surface, the low velocity, etc. In the orthodromic case we considered one further simplification associated with the proximity of the object trajectory to the orthodromy plane. The analog of this plane in geographical coordinates is the plane of the equator. In principle the results of the previous section are readily extended to the present case. However, the simplifications associated with the motion of the object near the equatorial plane are of no significance in practice, since clearly the object cannot always be constrained to move near the equator.

For small altitudes, when h/a is of the same order as e^2 (i.e., for $h \sim 50$—100 km), the expressions for r_2, r_3, g_{y_2} and the relations among the coordinate r, φ and h, φ' in Eqs. (2.26) can be simplified. We may take, for instance,

$$\left.\begin{aligned} r_2 &= a\left(1+\frac{e^2}{2}\sin^2\varphi'+h\right), \\ r_3 &= a\left(1-e^2+\frac{3e^2}{2}\sin^2\varphi'\right)+h. \end{aligned}\right\} \tag{4.189}$$

From the last three equations in (2.26) and from (1.27) we obtain an approximate expression for g_{y_2}:

$$g_{y_2} = \frac{geq}{2}\sin 2\varphi'. \tag{4.190}$$

The last three equations in (2.26) in this case drop out from the ideal operation equations.

Particularly significant are the simplifications associated with motion over the surface of the Clairaut ellipsoid, i.e., on the surface of the ocean. In this case $h = 0$. The altimeter becomes superfluous and we end up with an autonomous navigation system. The expressions for r_2, r_3, g_{y_2} and the integrands in the first two equations in (2.26) are markedly simplified in this case.

Note that for motion on the Earth's surface the last three equations in (2.26) can be entirely eliminated from the ideal operation equations. Indeed, consider the sum

$$-v_{x_2}\omega_{z_2}+g_{y_2}, \tag{4.191}$$

which enters the integrand in the second equation in (2.26). Since

$$\left.\begin{aligned} v_{x_2} &= r_2\omega_{y_2}, \quad \omega_{y_2} = (\dot\lambda+u)\cos\varphi', \\ \omega_{z_2} &= (\dot\lambda+u)\sin\varphi', \end{aligned}\right\} \tag{4.192}$$

we have

$$-v_{x_2}\omega_{z_2}+g_{y_2} = -r_2(\dot\lambda+u)^2\sin\varphi'\cos\varphi'. \tag{4.193}$$

Since the Clairaut ellipsoid is a level surface, we have

$$-r_2u^2\sin\varphi'\cos\varphi'+g_{y_2}=0, \tag{4.194}$$

so that

$$-v_{x_0}\omega_{z_0}+g_{y_0}=-r_2\dot{\lambda}(\dot{\lambda}+2u)\sin\varphi'\cos\varphi'. \tag{4.195}$$

The product $v_{y_0}\omega_{z_0}$ entering the first equation in (2.26) can be written in the form

$$v_{y_0}\omega_{z_0}=r_3\dot{\varphi}'(\dot{\lambda}+u\sin\varphi'). \tag{4.196}$$

For low speeds (e.g., a sea vessel), when $v_{y_0}\omega_{z_0}$, $-v_{x_0}\omega_{z_0}+g_{y_0}$ are small, r_2, r_3 can be replaced with a and $\dot{\varphi}'\dot{\lambda}$ and $\dot{\lambda}^2$ can be ignored in comparison to $\dot{\lambda}u$ and $\dot{\varphi}'u$. We thus get

$$\left.\begin{aligned}-v_{x_0}\omega_{z_0}+g_{y_0}&=-2au\dot{\lambda}\sin\varphi'\cos\varphi',\\ v_{y_0}\omega_{z_0}&=au\dot{\varphi}'\sin\varphi.\end{aligned}\right\} \tag{4.197}$$

The ideal operation equations in the case of slow motion on the Earth's surface thus take the form

$$\left.\begin{aligned}
v_{x_0}&=\int_0^t\left(n_{x_0}+a\dot{\varphi}'u\sin\varphi'\right)dt+v_x^0,\\
v_y&=\int_0^t\left(n_{y_0}-2au\dot{\lambda}\sin\varphi'\cos\varphi'\right)dt+v_y^0;\\
\omega_{y_0}&=\frac{v_{x_0}}{a}\left(1-\frac{e^2}{2}\sin^2\varphi'\right),\\
\omega_{x_0}&=-\frac{v_{y_0}}{a}\left(1+e^2-\frac{3e^2}{2}\sin^2\varphi'\right);\\
\varphi'&=\int_0^t\omega_{x_0}dt+\varphi'^0;\quad \lambda=\int_0^t\left(\frac{\omega_{y_0}}{\cos\varphi'}-u\right)dt+\lambda^0;\\
M_y^2&=H_2\omega_{x_0},\quad M_x^1=H_1\omega_{y_0},\quad M_x^3=-H_3\omega_{z_0};\\
&\omega_{z_0}=\omega_{y_0}\operatorname{tg}\varphi'.
\end{aligned}\right\} \tag{4.198}$$

For sufficiently low velocities, the terms $au\dot{\varphi}'\sin\varphi'$, $2au\dot{\lambda}\sin\varphi'\cos\varphi'$ in the integrands may prove to be quite negligible or, alternatively, they can be formed from approximate values of the coordinates. The terms containing the factor e^2 in the expressions for ω_{x_0}, ω_{y_0} can be treated similarly.

In view of the obvious similarity between the cases of orthodromic and geographical coordinates, we will not go into this problem in any greater detail. The corresponding equivalent errors can be found by comparing the simplified and the exact relations, as in the previous section in the discussion of orthodromic coordinates.

§4.4. DETERMINATION OF ORTHODROMIC COORDINATES WITH ACCELEROMETERS ORIENTED IN THE PLANE OF THE GEOGRAPHICAL HORIZON

An inertial system whose unperturbed platform axes x, y, z coincide with the corresponding axes of the co-moving geographical trihedron and which measures the geographical latitude and longitude has certain obvious advantages compared to other navigation systems. The main advantage naturally lies in the fact that the geographical coordinates are commonly

used in all fields. Further, the ideal operation equations of the inertial system are relatively simple. Last, it is also significant that the platform z axis of this system points along the normal to the level ellipsoid, and the y_0 axis lies in the plane of the geographical meridian. This, in particular, simplifies the initial setting and adjustment of the system, since the axis z_0 can be initially set by a plumb line using accelerometer signals and the axis y_0 can be brought to its proper meridional position with the aid of the system gyroscopes, using the gyrocompass effect. The z_0 axis caged to the normal of the level ellipsoid also simplifies the utilization of the altimeter readings h.

However, the geographical coordinate grid degenerates at the Earth's poles, and the inertial system therefore cannot function with satisfactory accuracy near the poles.

An inertial system measuring the orthodromic coordinates is free from this shortcoming. The polar axis ζ' of the orthodromic trihedron $\xi'\eta'\zeta'$ always can be chosen so that the object move near the plane $z = 0$, i.e., in the immediate neighborhood of the pole of the orthodromic grid. Now, the orthodromic coordinates are spherical coordinates. The axis z_0 of the platform should therefore point along the radius-vector r. This somewhat complicates the initial setting and adjustment of the platform.

In our discussion of the simplified ideal operation equations of an orthodromic system, we have shown that this shortcoming can be eliminated in the case of low speeds. It suffices to replace the first two equations in (4.72) with Eqs. (4.149) and add the terms ΔS and Δz from (4.152) to the right-hand sides of the fifth and the sixth equation in (4.72), while writing for ω_x, ω_y, ω_z in the last equation in (4.72) $\omega_{x_0} + \Delta\omega_{x_0}$, $\omega_{y_0} + \Delta\omega_{y_0}$, $\omega_{z_0} + \Delta\omega_{z_0}$, where $\Delta\omega_{x_0}$, $\Delta\omega_{y_0}$, $\Delta\omega_{z_0}$ are defined by (4.153). Then if the axis z_0 is initially oriented by the plumb line (along the normal to the level ellipsoid), it will preserve this attitude at any time, the orientation errors depending on the velocity of motion relative to the Earth. This approach, as we have seen in 4.2.5, is applicable to speeds comparable with the peripheral velocity on the equator. At velocities of the order of 1000 m/sec, however, it may lead to substantial errors (position errors ≈ 2.5 km, orientation errors $\approx 1'.4$).

There is another approach which permits bypassing this difficulty. Still working with the orthodromic coordinates z and S, we change the unperturbed orientation of the axes $x_0y_0z_0$ fixed to the platform so that the z_0 axis is directed along the normal to the level ellipsoid while the x_0 axis lies in a plane parallel to the orthodromy plane $z = 0$.

The unit vectors x_0, y_0, z_0 of the axes $x_0y_0z_0$ are given by

$$\left.\begin{aligned} z_0 &= z_3, \quad x_0 = \frac{\zeta' \times z_3}{|\zeta' \times z_3|}, \\ y_0 &= z_0 \times x_0 = \frac{\zeta' - z_0(\zeta' \cdot z_3)}{|\zeta \times z_3|}. \end{aligned}\right\} \tag{4.199}$$

Here the vector z_3 is the unit vector of the normal to the level ellipsoid, ζ' is the unit vector of the polar axis of the orthodromic frame $O_1\xi'\eta'\zeta'$.

Let $Ox_3y_3z_3$ be the co-moving orthodromic axes. The direction cosines of the axes x_0, y_0, z_0, defined by (4.199), relative to the axes x_3, y_3, z_3 are arranged in the matrix

$$\begin{array}{c|ccc} & x_0 & y_0 & z_0 \\ x_3 & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ y_3 & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ z_3 & \gamma_{31} & \gamma_{32} & \gamma_{33}. \end{array} \tag{4.200}$$

The orientation of the axis z_0 (normal to the Clairaut ellipsoid) relative to the axes ξ, η, ζ rigidly fixed to the Earth is specified by the third column in (4.161). The orientation of the axes x_3, y_3, z_3 relative to the axes ξ, η, ζ is determined by matrices (4.1) and (1.16). From these data we readily obtain the elements of the third column of the matrix (4.200). After obvious manipulations, using the identities

$$\left.\begin{aligned}\sin\varphi' &= \sin(\varphi'-\varphi)\cos\varphi + \cos(\varphi'-\varphi)\sin\varphi,\\ \cos\varphi' &= \cos(\varphi'-\varphi)\cos\varphi - \sin(\varphi'-\varphi)\sin\varphi,\end{aligned}\right\} \tag{4.201}$$

identities (4.3) and the orthogonality of matrices (4.1), (1.16), (4.161), we obtain the corresponding expressions for the elements γ_{i3} of the third column in (4.200):

$$\left.\begin{aligned}\gamma_{13} &= \frac{\sin(\varphi'-\varphi)}{\cos\varphi}(-\beta_{31}\sin S + \beta_{32}\cos S),\\ \gamma_{23} &= \frac{\sin(\varphi'-\varphi)}{\cos\varphi}(-\beta_{31}\sin z\cos S - \beta_{32}\sin z\sin S + \beta_{33}\cos z),\\ \gamma_{33} &= \cos(\varphi'-\varphi).\end{aligned}\right\} \tag{4.202}$$

Now from the second equality in (4.199)

$$\left.\begin{aligned}\gamma_{11} &= \frac{1}{\sin(\widehat{\zeta', z_0})}\left[\cos z\cos(\varphi'-\varphi) - \frac{\sin z\sin(\varphi'-\varphi)}{\cos\varphi}\times\right.\\ &\quad\left.\times(-\beta_{31}\sin z\cos S - \beta_{32}\sin z\cos S + \beta_{33}\cos z)\right],\\ \gamma_{21} &= \frac{\sin z\sin(\varphi'-\varphi)}{\sin(\widehat{\zeta', z_0})\cos\varphi}(-\beta_{31}\sin S + \beta_{32}\cos S),\\ \gamma_{31} &= -\frac{\cos z\sin(\varphi'-\varphi)}{\sin(\widehat{\zeta', z_0})\cos\varphi}(-\beta_{31}\sin S + \beta_{32}\cos S).\end{aligned}\right\} \tag{4.203}$$

Finally, from the third equality in (4.199)

$$\left.\begin{aligned}\gamma_{12} &= -\frac{1}{\sin(\widehat{\zeta', z_0})}\times\\ &\quad\times\frac{\sin(\varphi'-\varphi)\cos(\widehat{\zeta', z_0})(-\beta_{31}\sin S + \beta_{32}\cos S)}{\cos\varphi},\\ \gamma_{22} &= \frac{1}{\sin(\widehat{\zeta', z_0})}\left[\cos z - \frac{\sin(\varphi'-\varphi)\cos(\widehat{\zeta', z_0})}{\cos\varphi}\times\right.\\ &\quad\left.\times(-\beta_{31}\sin z\cos S - \beta_{32}\sin z\sin S + \beta_{33}\cos z)\right],\\ \gamma_{32} &= \frac{1}{\sin(\widehat{\zeta', z_0})}[\sin z - \cos(\varphi'-\varphi)\cos(\widehat{\zeta', z_0})].\end{aligned}\right\} \tag{4.204}$$

In (4.203), (4.204) the angle $(\widehat{\zeta', z_0})$ is given by the equality

$$\begin{aligned}\cos(\widehat{\zeta', z_0}) &= \zeta'\cdot z_0 = \sin z\cos(\varphi'-\varphi) +\\ &+\frac{\cos z\sin(\varphi'-\varphi)}{\cos\varphi}(-\beta_{31}\sin z\cos S - \beta_{32}\sin z\sin S + \beta_{33}\cos z).\end{aligned} \tag{4.205}$$

Let p_{x_0}, p_{y_0}, p_{z_0} be the projections on the axes x_0, y_0, z_0 of the angular velocity of $Ox_0y_0z_0$ relative to $Ox_3y_3z_3$. Using relations similar to (4.163), (4.165), we find

$$\left.\begin{aligned}p_{x_0} &= \dot\gamma_{12}\gamma_{13} + \dot\gamma_{22}\gamma_{23} + \dot\gamma_{32}\gamma_{33},\\ p_{y_0} &= \dot\gamma_{13}\gamma_{11} + \dot\gamma_{23}\gamma_{21} + \dot\gamma_{33}\gamma_{31},\\ p_{z_0} &= \dot\gamma_{11}\gamma_{12} + \dot\gamma_{21}\gamma_{22} + \dot\gamma_{31}\gamma_{32}.\end{aligned}\right\} \tag{4.206}$$

Since the expressions for the projections ω_{x_3}, ω_{y_3}, ω_{z_3} of the absolute angular velocity of the axes $Ox_3y_3z_3$ on the corresponding unit vectors are known (they are given by (4.15)), we can find the absolute angular velocity components of the axes $Ox_0y_0z_0$ in the corresponding directions. They are given by

$$\left.\begin{aligned}\omega_{x_3} &= p_{x_3} + \omega_{x_3}\gamma_{11} + \omega_{y_3}\gamma_{21} + \omega_{z_3}\gamma_{31},\\ \omega_{y_3} &= p_{y} + \omega_{x_3}\gamma_{12} + \omega_{y_3}\gamma_{22} + \omega_{z_3}\gamma_{32},\\ \omega_{z_3} &= p_{z_3} + \omega_{x_3}\gamma_{13} + \omega_{y_3}\gamma_{23} + \omega_{z_3}\gamma_{33}.\end{aligned}\right\} \tag{4.207}$$

Our further course of action is as follows: using (4.9) we should find a relation between the absolute velocity components v_{x_3}, v_{y_3} of the origin of the axes $Ox_0y_0z_0$ and the angular velocity components ω_{x_3}, ω_{y_3}, and also a relation between ω_{z_3} and ω_{x_3}, ω_{y_3}. Given these relations, we will be able to solve our problem, namely to construct a closed system of exact ideal operation equations for an inertial system with the axes z_0 of the platform pointing along the normal to the level ellipsoid which measures the orthodromic coordinates z and S.

To simplify the mathematics, we will only consider a highly important particular case: the orthodromic axes ξ', η' lie in the plane of the meridian, the axis η' pointing along the Earth's spin axis, i.e., the axis ζ.

The factors β_{3i} entering Eqs. (4.202)—(4.204) are then given by

$$\beta_{32} = 1,\ \beta_{31} = \beta_{33} = 0. \tag{4.208}$$

Eqs. (4.202)—(4.204) also include the trigonometric functions $\sin(\varphi' - \varphi)$, $\cos(\varphi' - \varphi)$. If they are expanded in powers of e, retaining only terms with e^2, we obtain

$$\sin(\varphi' - \varphi) = e^2 \sin\varphi\cos\varphi,\quad \cos(\varphi' - \varphi) = 1. \tag{4.209}$$

From the third equation in (4.3), using (4.208), we get

$$\sin\varphi = \cos z \sin S. \tag{4.210}$$

Inserting (4.208)—(4.210) in (4.202)—(4.204), we obtain the greatly simplified relations

$$\left.\begin{aligned}&\gamma_{11} = 1,\quad \gamma_{12} = -e^2 \sin z \sin S \cos S,\\ &\gamma_{13} = e^2 \cos z \sin S \cos S,\\ &\gamma_{21} = e^2 \sin z \sin S \cos S,\quad \gamma_{22} = 1,\\ &\gamma_{23} = -e^2 \sin z \cos z \sin^2 S,\\ &\gamma_{31} = -e^2 \cos z \sin S \cos S,\\ &\gamma_{32} = e^2 \sin z \cos z \sin^2 S,\\ &\gamma_{33} = 1.\end{aligned}\right\} \tag{4.211}$$

For p_{x_0}, p_{y_0}, p_{z_0} we now have (to terms of the order e^2)

$$\left.\begin{aligned}p_{x_0} &= \dot\gamma_{32} = e^2[\dot z(\cos^2 z - \sin^2 z) + 2\dot S \sin z \cos z \sin S \cos S],\\ p_{y_0} &= \dot\gamma_{13} = e^2[-\dot z \sin z \sin S \cos S + \dot S \cos z(\cos^2 S - \sin^2 S)],\\ p_{z_0} &= \gamma_{21} = e^2[\dot z \cos z \sin S \cos S + \dot S \sin z(\cos^2 S - \sin^2 S)].\end{aligned}\right\} \tag{4.212}$$

From (4.15), (4.208) we have

$$\left.\begin{aligned} \omega_{x_3} &= -\dot{z} + u\cos S, \quad \omega_{y_3} = \dot{S}\cos z - u\sin z\sin S, \\ \omega_{z_3} &= \dot{S}\sin z + u\cos z\sin S. \end{aligned}\right\} \tag{4.213}$$

Inserting (4.213), (4.212), (4.211) in (4.207), we get

$$\left.\begin{aligned} \omega_{x_0} &= -\dot{z} + u\cos S + \\ &\quad + e^2\sin S\,[\dot{z}\sin S(\cos^2 z - \sin^2 z) + \\ &\quad + 2\dot{S}\cos S\sin z\cos z - u\sin S\cos S], \\ \omega_{y_0} &= (\dot{S}\cos z - u\sin z\sin S)\times \\ &\quad \times[1 + e^2(\cos^2 S - \sin^2 S\cos^2 z)], \\ \omega_{z_0} &= \dot{S}\sin z + u\cos z\sin S + \\ &\quad + e^2[\dot{S}\sin z(\cos^2 S - \sin^2 S - \cos^2 z\sin^2 S) + \\ &\quad + u\cos z\sin S(\cos^2 S + \sin^2 z\sin^2 S)]. \end{aligned}\right\} \tag{4.214}$$

Let us establish a relation between v_{x_0}, v_{y_0} and ω_{x_0}, ω_{y_0}. We will use Eqs. (4.9). The point P is the Earth's center O_1, and its coordinates in the axes $Ox_3y_3z_3$ are

$$x_3^{O_1} = y_3^{O_1} = 0, \quad z_3^{O_1} = -r. \tag{4.215}$$

Since the mutual orientation of the axes $x_3y_3z_3$ and $x_0y_0z_0$ is described by the direction cosines (4.200), (4.211), and the two systems have a common origin, the coordinates $x_0^{O_1}$, $y_0^{O_1}$, $z_0^{O_1}$ of the Earth's center O_1 are given by

$$\left.\begin{aligned} x_0^{O_1} &= re^2\cos z\sin S\cos S, \\ y_0^{O_1} &= -re^2\sin z\cos z\sin^2 S, \quad z_0^{O_1} = -r. \end{aligned}\right\} \tag{4.216}$$

For r we can use the approximate expression

$$r = a\left(1 - \frac{e^2}{2}\sin^2\varphi\right) + h. \tag{4.217}$$

Taking h/a to be of the same order of magnitude as e^2, we obtain from (4.216)

$$\left.\begin{aligned} x_0^{O_1} &= ae^2\cos z\sin S\cos S, \\ y_0^{O_1} &= -ae^2\sin z\cos z\sin^2 S, \\ z_0^{O_1} &= -a\left(1 - \frac{e^2}{2}\sin^2\varphi\right) - h. \end{aligned}\right\} \tag{4.218}$$

Expressing $\sin^2\varphi$ from (4.210), we insert expressions (4.218) for $x_0^{O_1}$, $y_0^{O_1}$, $z_0^{O_1}$ in (4.9). Since O_1 is stationary in the axes $O_1\xi\eta\zeta$, we have

$$v_{x_0}^{O_1} = v_{y_0}^{O_1} = v_{z_0}^{O_1} = 0 \tag{4.219}$$

and Eqs. (4.9) yield

$$\left.\begin{aligned} &ae^2\frac{d}{dt}(\cos z\sin S\cos S) + \\ &\quad + v_x - a\omega_{y_0}\left(1 - \frac{e^2}{2}\cos^2 z\sin^2 S + \frac{h}{a}\right) + \\ &\quad + ae^2\omega_{z_0}\sin z\cos z\sin^2 S = 0, \end{aligned}\right\} \tag{4.220}$$

$$\left.\begin{aligned} &-ae^2\frac{d}{dt}\left(\sin z\cos z\sin^2 S\right)+\\ &\quad+v_{y_1}+ae^2\omega_z\cos z\sin S\cos S+\\ &\quad+a\omega_x\left(1-\frac{e^2}{2}\cos^2 z\sin^2 S+\frac{h}{a}\right)=0,\\ &-a\frac{d}{dt}\left(1-\frac{e^2}{2}\cos^2 z\sin^2 S+\frac{h}{a}\right)+\\ &\quad+v_{x_1}-ae^2\omega_x\sin z\cos z\sin^2 S-\\ &\quad-ae^2\omega_y\cos z\sin S\cos S=0. \end{aligned}\right\} \tag{4.220}$$

Carrying out the differentiation in the third equation in (4.220) and inserting ω_{x_1} and ω_y from (4.214), we find

$$v_z=\dot{h}. \tag{4.221}$$

The first two equations in (4.220) after differentiation and insertion for $\dot{S}$ and $\dot{z}$ of their expressions from the first and second equality in (4.214) yield

$$\left.\begin{aligned} v_x&=\omega_{y_1}a\left[1+\frac{h}{a}+e^2\left(\frac{1}{2}\cos^2 z\sin^2 S-\cos^2 S\right)\right]+\omega_x ae^2\sin z\sin S\cos S,\\ v_y&=-\omega_{x_1}a\left[1+\frac{h}{a}+e^2\sin^2 S\left(\frac{1}{2}\cos^2 z-\sin^2 z\right)\right]+\omega_y ae^2\sin z\sin S\cos S. \end{aligned}\right\} \tag{4.222}$$

Solving for ω_x and ω_{y_1}, we obtain

$$\left.\begin{aligned} \omega_{x_1}&=-\frac{v_y}{a}\left[1-\frac{h}{a}+e^2\sin^2 S\left(1-\frac{3}{2}\cos^2 z\right)\right]+\frac{v_x}{a}e^2\sin z\sin S\cos S,\\ \omega_y&=\frac{v_x}{a}\left[1-\frac{h}{a}+e^2\left(\cos^2 S-\frac{1}{2}\cos^2 z\sin^2 S\right)\right]-\frac{v_y}{a}e^2\sin z\sin S\cos S. \end{aligned}\right\} \tag{4.223}$$

From the second and third equations in (4.214), eliminating $\dot{S}$, we obtain a relation between ω_z and ω_{y_1}:

$$\omega_z=\omega_y\operatorname{tg} z\left(1-e^2\sin^2 S\right)+\frac{u\sin S}{\cos z}\left[1+e^2\left(\cos^2 S-\sin^2 z\sin^2 S\right)\right] \tag{4.224}$$

In what follows we will require the cosines of the angles between the axes x_0, y_0 and the co-moving geographical axis y_2. They are needed in calculations of the x_0 and y_0 projections of the horizontal gravitation component. Since this horizontal component is of the order e^2, we may write

$$\cos(\widehat{x_0, y_2})=\cos(\widehat{x_3, y_1}),\quad \cos(\widehat{y_0, y_2})=\cos(\widehat{y_3, y_1}), \tag{4.225}$$

where y_1 is the north axis of the co-moving geocentric system. Turning to Eqs. (1.20) and using (4.208), we obtain

$$\cos(\widehat{x_0, y_2})=\frac{\cos S}{\cos\varphi},\quad \cos(\widehat{y_0, y_2})=-\frac{\sin z\sin S}{\cos\varphi}. \tag{4.226}$$

The expression for g_{y_2} is given by the first equation in (1.30). Retaining only terms of the order e^2, we obtain

$$g_{y_2}=\frac{g_e q}{2}\sin 2\varphi. \tag{4.227}$$

Now, using (4.210), (4.225), we obtain for the x_0 and y_0 projections of gravitational acceleration

$$\left.\begin{aligned} g_{x_0} &= \frac{g_e q}{2}\cos z \sin 2S, \\ g_{y_0} &= -\frac{g_e q}{2}\sin^2 S \sin 2z. \end{aligned}\right\} \tag{4.228}$$

We have thus assembled all the relations needed for the derivation of the ideal operation equations.

The ideal operation equations will first include the equations for v_{x_0}, v_{y_0}, analogous to the first two equations in (4.26):

$$\left.\begin{aligned} v_{x_0} &= \int_0^t \left(n_{x_0} + v_{y_0}\omega_{z_0} - \dot{h}\omega_{y_0} - \frac{g_e q}{2}\cos z \sin 2S\right) dt + v_{x_0}^0, \\ v_{y_0} &= \int_0^t \left(n_{y_0} + \dot{h}\omega_{x_0} - v_{x_0}\omega_{z_0} + \frac{g_e q}{2}\sin^2 S \sin 2z\right) dt + v_{y_0}^0. \end{aligned}\right\} \tag{4.229}$$

To these equations we add (4.223), (4.224), which can be used to determine ω_{x_0}, ω_{y_0}, ω_{z_0} from known v_{x_0}, v_{y_0}:

$$\left.\begin{aligned} \omega_{x_0} &= -\frac{v_{y_0}}{a}\left[1 - \frac{h}{a} + e^2\sin^2 S\left(1 - \frac{3}{2}\cos^2 z\right)\right] + \frac{v_{x_0}}{a}e^2 \sin z \sin S \cos S, \\ \omega_{y_0} &= \frac{v_{x_0}}{a}\left[1 - \frac{h}{a} + e^2\left(\cos^2 S - \frac{1}{2}\cos^2 z \sin^2 S\right)\right] - \frac{v_{y_0}}{a}e^2 \sin z \sin S \cos S, \\ \omega_{z_0} &= \omega_{y_0}\operatorname{tg} z\,(1 - e^2\sin^2 S) + \frac{u \sin S}{\cos z}[1 + e^2(\cos^2 S - \sin^2 z \sin^2 S)]. \end{aligned}\right\} \tag{4.230}$$

Then come the equations for S and z using known ω_{x_0}, ω_{y_0}. They are obtained from the first two equations in (4.213) and have the form

$$\left.\begin{aligned} z &= \int_0^t [-\omega_{x_0} + u\cos S + e^2(-\omega_{x_0}\sin^2 S \cos 2z + \omega_{y_0}\sin z \sin 2S)]\, dt + z^0, \\ S &= \int_0^t \frac{1}{\cos z}\{\omega_{y_0}[1 - e^2(\cos^2 S - \sin^2 S \cos^2 z)] + u \sin z \sin S\}\, dt + S^0. \end{aligned}\right\} \tag{4.231}$$

Finally the ideal operation equations include, as is usual, suitable relations from which the control torques can be formed:

$$M_y^2 = -H_2\omega_{x_0}, \quad M_x^1 = H_1\omega_{y_0}, \quad M_x^3 = -H_3\omega_{z_0}. \tag{4.232}$$

The error equations of this system are Eqs. (2.28), (1.96), (1.97). They can be derived in the same way as for the systems discussed in the previous sections.

In conclusion, let us compare Eqs. (4.229)—(4.232) with Eqs. (2.26) using the simplifications (4.189), (4.190). If we take $z = 0$ in Eqs. (4.229)—(4.232), which corresponds to motion in the plane of the geographical meridian, and similarly assume meridional motion in Eqs. (2.26), we readily see that Eqs. (2.26) with simplifications (4.189), (4.190) are equivalent to Eqs. (4.229)—(4.232). To establish this equivalence, it suffices to note that the axes $x_0y_0z_0$

in the two systems make an angle of $\pi/2$ with each other. The variable z in Eqs. (4.229)—(4.232) therefore corresponds to the longitude λ in Eqs. (2.26), and the variable S corresponds to the geocentric latitude φ, which is related to the geographical latitude φ' by

$$\varphi' - \varphi = \frac{e'}{2} \sin 2\varphi' = \frac{e'}{2} \sin 2\varphi. \tag{4.233}$$

One further point should be remembered. Eqs. (4.229)—(4.232) were derived under assumption (4.208), i.e., for the case when the orthodromic polar axis lies in the plane of the terrestrial equator. This restriction obviously can be removed. The resulting equations, however, become more complex. But if the computing devices of the inertial system can function both according to Eqs. (2.26) and according to Eqs. (4.229)—(4.232), the two alternatives are sufficient to keep the object moving in any trajectory. At low latitudes, when $\varphi < \frac{\pi}{4}$, we should use Eqs. (2.26), and at high latitudes, when clearly $z < \frac{\pi}{4}$, we should use Eqs. (4.229)—(4.232).

Chapter 5

INERTIAL SYSTEMS WITH DOPPLER CORRECTION

§5.1. ASYMPTOTIC STABILITY OF INERTIAL SYSTEMS. VELOCITY ERRORS

In the previous three chapters we have considered inertial systems with altimeter correction. We have shown, in particular, that external information on the altitude h of the object above the Earth's surface ensures nonasymptotic stability of the inertial system for any motion at constant distance from the Earth's center and for Keplerian motion.

The information on the altitude h can be used in two ways. In one type of systems all the three accelerometers are retained and the external information on h is applied only to form the gravitational term μ/r^3 in the ideal operation equations. The homogeneous error equation of the first group in this case takes the form

$$\frac{d^2\delta r}{dt^2}+\frac{\mu\,\delta r}{r^3}=0. \tag{5.1}$$

The corresponding scalar equations in projection on the axes xyz, with the z axis directed along the radius vector r, are

$$\left.\begin{aligned}
&\delta\ddot{x}+\left(\tfrac{\mu}{r^3}-\omega_y^2-\omega_z^2\right)\delta x+(\omega_x\omega_y-\dot{\omega}_z)\,\delta y-\\
&\qquad-2\omega_z\,\delta\dot{y}+(\omega_x\omega_z+\dot{\omega}_y)\,\delta z+2\omega_y\,\delta\dot{z}=0,\\
&\delta\ddot{y}+\left(\tfrac{\mu}{r^3}-\omega_z^2-\omega_x^2\right)\delta y+(\omega_y\omega_z-\dot{\omega}_x)\,\delta z-\\
&\qquad-2\omega_x\,\delta\dot{z}+(\omega_y\omega_x+\dot{\omega}_z)\,\delta x+2\omega_z\,\delta\dot{x}=0,\\
&\delta\ddot{z}+\left(\tfrac{\mu}{r^3}-\omega_x^2-\omega_y^2\right)\delta z+(\omega_z\omega_x-\dot{\omega}_y)\,\delta x-\\
&\qquad-2\omega_y\,\delta\dot{x}+(\omega_z\omega_y+\dot{\omega}_x)\,\delta y+2\omega_x\,\delta\dot{y}=0.
\end{aligned}\right\} \tag{5.2}$$

In the other alternative, the z axis accelerometer is dispensed with and the first-group homogeneous error equations take the form

$$\left.\begin{aligned}
&\delta\ddot{x}+\left(\tfrac{\mu}{r^3}-\omega_y^2-\omega_z^2\right)\delta x+(\omega_x\omega_y-\dot{\omega}_z)\,\delta y-2\omega_z\,\delta\dot{y}=0,\\
&\delta\ddot{y}+\left(\tfrac{\mu}{r^3}-\omega_z^2-\omega_x^2\right)\delta y+(\omega_x\omega_y+\dot{\omega}_z)\,\delta x+2\omega_z\,\delta\dot{x}=0.
\end{aligned}\right\} \tag{5.3}$$

We have shown in Chapter 2 that the solutions of Eqs. (5.2), (5.3) in cases when their coefficients are constant constitute undamped oscillations. In the simplest case when $r=\text{const}$, $\omega_x=\omega_y=\omega_z=0$, these are harmonic oscillations

with Schuler's frequency. We are thus dealing with a case of nonasymptotic stability. The next step to consider is naturally the possibility of accomplishing asymptotically stable motion of the inertial system about the position specified by the ideal operation equations. This problem clearly can be solved by introducing terms proportional to $\delta\dot{x}$, $\delta\dot{y}$, $\delta\dot{z}$ in Eqs. (5.2), (5.3).

Eqs. (5.2), (5.3) are the variational forms of the corresponding ideal operation equations. Therefore, terms proportional to $\delta\dot{x}$, $\delta\dot{y}$, $\delta\dot{z}$ can be introduced in Eqs. (5.2), (5.3) if we appropriately alter the ideal operation equations; in other words, we should introduce in the ideal operation equations additional terms whose variations are proportional to $\delta\dot{x}$, $\delta\dot{y}$, $\delta\dot{z}$.

Turning to the first group of ideal operation equations (1.36) we change them as follows ($k_1 = \text{const}$):*

$$\boldsymbol{v}' = \int_0^t (\boldsymbol{n} - \boldsymbol{m}\times\boldsymbol{v}' + \boldsymbol{g} - k_1\boldsymbol{v}')\,dt + \boldsymbol{v}^0, \quad \boldsymbol{r}' = \int_0^t (\boldsymbol{v}' - \boldsymbol{m}\times\boldsymbol{r}')dt + \boldsymbol{r}^0. \tag{5.4}$$

Let the system be composed of ideal elements, without any instrumental errors. The function $r(t)$ used to form the magnitude of the vector $\boldsymbol{g}$ is assumed to be exactly known. Then from (5.4), (1.36), instead of Eq. (5.1), we obtain a new vector equation for the error $\delta\boldsymbol{r} = \boldsymbol{r}' - \boldsymbol{r}$:

$$\frac{d^2\delta\boldsymbol{r}}{dt^2} + k_1\frac{d\,\delta\boldsymbol{r}}{dt} + \frac{\mu\,\delta\boldsymbol{r}}{r^3} = -k_1\frac{d\boldsymbol{r}}{dt}. \tag{5.5}$$

Projecting this equation on the axes xyz with the z axis directed along $\boldsymbol{r}$, we obtain instead of (5.2)

$$\left.\begin{aligned}
&\delta\ddot{x} + k_1\delta\dot{x} + \left(\frac{\mu}{r^3} - \omega_y^2 - \omega_z^2\right)\delta x + \\
&\quad + (\omega_x\omega_y - \dot{\omega}_z - k_1\omega_z)\,\delta y - 2\omega_z\,\delta\dot{y} + \\
&\quad + (\omega_x\omega_z + \dot{\omega}_y + k_1\omega_y)\,\delta z + 2\omega_y\,\delta\dot{z} = -k_1 r\omega_y, \\
&\delta\ddot{y} + k_1\delta\dot{y} + \left(\frac{\mu}{r^3} - \omega_z^2 - \omega_x^2\right)\delta y + \\
&\quad + (\omega_y\omega_z - \dot{\omega}_x - k_1\omega_x)\,\delta z - 2\omega_x\,\delta\dot{z} + \\
&\quad + (\omega_y\omega_x + \dot{\omega}_z + k_1\omega_z)\,\delta x + 2\omega_z\,\delta\dot{x} = k_1 r\omega_x, \\
&\delta\ddot{z} + k_1\delta\dot{z} + \left(\frac{\mu}{r^3} - \omega_x^2 - \omega_y^2\right)\delta z + \\
&\quad + (\omega_z\omega_x - \dot{\omega}_y - k_1\omega_y)\,\delta x - 2\omega_y\,\delta\dot{x} + \\
&\quad + (\omega_z\omega_y + \dot{\omega}_x + k_1\omega_x)\,\delta y + 2\omega_x\,\delta\dot{y} = k_1\dot{r}.
\end{aligned}\right\} \tag{5.6}$$

For a two-accelerometer system we correspondingly obtain instead of Eqs. (5.3)

$$\left.\begin{aligned}
&\delta\ddot{x} + k_1\delta\dot{x} + \left(\frac{\mu}{r^3} - \omega_y^2 - \omega_z^2\right)\delta x + \\
&\quad + (\omega_x\omega_y - \dot{\omega}_z - k_1\omega_z)\,\delta y - 2\omega_z\,\delta\dot{y} = -k_1 r\omega_y, \\
&\delta\ddot{y} + k_1\delta\dot{y} + \left(\frac{\mu}{r^3} - \omega_z^2 - \omega_x^2\right)\delta y + \\
&\quad + (\omega_y\omega_x + \dot{\omega}_z + k_1\omega_z)\,\delta x + 2\omega_z\,\delta\dot{x} = k_1 r\omega_x.
\end{aligned}\right\} \tag{5.7}$$

Let us now consider the homogeneous equations (5.5)—(5.7). The solutions of the homogeneous equations (5.5), (5.6), as it follows from the form of

* Eqs. (5.4) contain $\boldsymbol{v}'$ and $\boldsymbol{r}'$ (as distinct from $\boldsymbol{v}$ and $\boldsymbol{r}$), since when the ideal operation equations are changed in accordance with (5.4), $\boldsymbol{r}' \neq \boldsymbol{r}$ and $\boldsymbol{v}' \neq \boldsymbol{v}$ even in the absence of instrumental errors and with exact initial conditions (see (5.5)).

Eq. (5.5), are damped oscillations. The vector δr, and hence δx, δy, δz in the homogeneous equations (5.6) for $k_1 > 0$, $r = \text{const}$, go to zero with time irrespective of the particular variation of ω_x, ω_y, ω_z. The situation is somewhat different as regard the solutions δx, δy of the homogeneous equations (5.7). These may be damped oscillations or, conversely, oscillations which grow with time depending on the particular relationship between k_1, ω_x, ω_y, ω_z; *in case of variable* ω_x, ω_y, ω_z, *the character of the solution further depends on* the particular form of the functions $\omega_x(t)$, $\omega_y(t)$, $\omega_z(t)$.

For $\omega_x = \omega_y = \omega_z = 0$, $r = \text{const}$, the solutions of Eqs. (5.7) evidently go to zero with time. For $\omega_x = \omega_z = 0$, $\omega_y = \text{const}$, $r = \text{const}$, i.e., for motion with constant velocity along the great circle of a sphere centered at O_1 which does not spin with the Earth, Eqs. (5.7) are separated, and for $k_1 > 0$ the solution is damped if

$$\omega_0^2 - \omega_y^2 > 0 \quad \left(\omega_0^2 = \frac{\mu}{r^3}\right). \tag{5.8}$$

For motion with constant velocity along a parallel (taking the axis y to lie in the plane of the meridian), we have

$$\omega_x = 0, \quad \omega_y = \text{const}, \quad \omega_z = \text{const}, \quad r = \text{const}. \tag{5.9}$$

The coefficients of the homogeneous equations (5.7) in this case are constant, so that Hurwitz's criterion is applicable to stability analysis.

The characteristic equation of (5.7) is

$$\begin{aligned} &p^4 + 2k_1 p^3 + (2\omega_0^2 + 2\omega_z^2 - \omega_y^2 + k_1^2)\, p^2 + \\ &+ k_1(2\omega_0^2 + 2\omega_z^2 - \omega_y^2)\, p + (\omega_0^2 - \omega_y^2 - \omega_z^2)(\omega_0^2 - \omega_z^2) + k_1^2\omega_z^2 = 0. \end{aligned} \tag{5.10}$$

The Hurwitz inequalities give the following conditions of asymptotic stability:

$$\left.\begin{aligned} k_1 > 0, \quad 2\omega_0^2 + 2\omega_z^2 + 2k_1^2 - \omega_y^2 > 0, \\ (8\omega_z^2 + 2k_1^2)(2\omega_0^2 - \omega_y^2) + \omega_y^4 > 0, \\ (\omega_0^2 - \omega_y^2 - \omega_z^2)(\omega_0^2 - \omega_z^2) + k_1^2\omega_z^2 > 0. \end{aligned}\right\} \tag{5.11}$$

In the region of the parameters k_1, ω_y^2, ω_z^2 where inequalities (5.11) are satisfied, the homogeneous equations (5.7) have a damped solution.

Let us compare regions (5.11) of asymptotic stability of the homogeneous equations (5.7) with the regions of nonasymptotic stability of the homogeneous equations (5.3) to which Eqs. (5.7) reduce for $k_1 = 0$. The conditions of nonasymptotic stability of Eqs. (5.3) are inequalities (2.45). The regions of nonasymptotic stability in the ω_z^2, ω_y^2 plane are shown in Figure 5.1. Clearly for $k_1 = 0$ conditions (5.11) reduce to conditions (2.45). Thus, in virtue of continuity for sufficiently small k_1 the regions of nonasymptotic stability in Figure 2.1 develop into regions of asymptotic stability of the homogeneous equations (5.7). It is also readily seen that as k_1 increases region 1 (Figure 2.1) does not contract. This is so because for

$$\omega_0^2 - \omega_y^2 - \omega_z^2 > 0 \tag{5.12}$$

conditions (5.11) are satisfied for any $k_1 > 0$. Conditions (5.11) are also satisfied for any $k_1 > 0$ in the half-strip $\omega_z^2 > \omega_0^2$, $\omega_y^2 < 2\omega_0^2$.

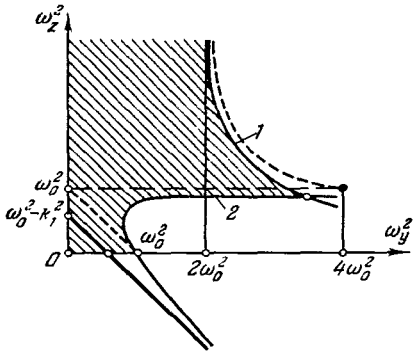

FIGURE 5.1.

In general the second inequality in (5.11) is satisfied when the third and the fourth inequalities are observed. The third inequality divides the ω_z^2, ω_y^2 plane into three regions (Figure 5.1). The boundary curve of these regions is the hyperbola 1 (only one branch is shown in the figure)

$$\omega_z^2 = -k_1^2/4 + \omega_y^4/[8(\omega_y^2 - 2\omega_0^2)]. \tag{5.13}$$

The asymptotes of this hyperbola are the lines

$$\omega_y^2 = 2\omega_0^2, \quad \omega_z^2 = (\omega_y^2 - k_1^2)/4 + \omega_y^2/8. \tag{5.14}$$

The inequality $(8\omega_z^2 + 2k_1^2)(2\omega_0^2 - \omega_y^2) + \omega_y^4 > 0$ corresponds to the region between the two branches of the hyperbola, and since one of the branches lies entirely in the lower halfplane* $\omega_z^2 < 0$, we are in fact left with the region between the branch 1 and the lines $\omega_z^2 = 0$, $\omega_y^2 = 0$.

The fourth inequality in (5.11) corresponds to the hyperbola 2

$$\omega_y^2 = \omega_0^2 - \omega_z^2 + k_1^2\omega_z^2/(\omega_0^2 - \omega_z^2), \tag{5.15}$$

with the asymptotes

$$-k_1^2 + \omega_0^2 - \omega_y^2 - \omega_z^2 = 0, \quad \omega_z^2 = \omega_0^2. \tag{5.16}$$

One of the branches of hyperbola (5.15) lies in the halfplane* $\omega_y^2 < 0$, the other in the halfplane $\omega_y^2 > 0$. The "physical" branch crosses the abscissa axis $\omega_z^2 = 0$ at the point $\omega_y^2 = \omega_0^2$ and meets the hyperbola (5.13) at a point with abscissa $\omega_y^2 \leqslant 4\omega_0^2$. The region of asymptotic stability of the homogeneous equations (5.7) for $k_1 > 0$ is diagonally hatched in Figure 5.1. For purposes of comparison the dashed lines mark the boundaries of the region of non-asymptotic stability of Eqs. (5.3). For $k_1 = 0$ the two regions naturally coincide. For $k_1 \to \infty$, the stability region is the strip $\omega_y^2 < 2\omega_0^2$, $\omega_z^2 > 0$.

* Clearly only the first quadrant of the ω_y^2, ω_z^2 plane is physically meaningful.

The parts of the hyperbolas (5.14), (5.15) which lie in the halfplanes $\omega_z^2 < 0$, $\omega_y^2 < 0$ are introduced only to elucidate the geometry of Figure 5.1. This was also the object of continuing the line $k_1^2 + \omega_0^2 - \omega_y^2 - \omega_z^2 = 0$ into the halfplane $\omega_z^2 < 0$.

Thus, having changed the ideal operation equations in accordance with (5.4), we managed to introduce damping in the first group of error equations. This was accomplished, however, at the expense of introducing additional velocity-proportional perturbations in the right-hand sides of the error equations (5.5)—(5.7). These perturbations constitute the so-called velocity errors of the system. The situation here is entirely analogous to the damping of the Schuler pendulum considered in Chapter 3.

Let us estimate the magnitude of these velocity errors. Consider equatorial motion. Let the x axis of the system xyz point along the equator. Then $\omega_z = \omega_x = 0$, $\omega_y = \text{const}$. Eqs. (5.7) are separated. The first equation takes the form

$$\delta\ddot{x} + k_1\delta\dot{x} + \omega_0^2\delta x = -k_1 r\omega_y. \tag{5.17}$$

If $\omega_y = \text{const}$, the steady-state velocity error is

$$\delta\tilde{x} = -\frac{k_1 r\omega_y}{\omega_0^2}. \tag{5.18}$$

To ensure effective ("noticeable") damping, k_1 should be of the order of ω_0; if we take $k_1 = \omega_0$, we get

$$\delta\tilde{x} = -\frac{r\omega_y}{\omega_0}. \tag{5.19}$$

For an object stationary relative to the Earth, when $\omega_y/\omega_0 \approx 1/16$, $|\delta\tilde{x}|$ is thus of the order of 400 km, which is clearly inadmissible.

To maintain the desired accuracy of the damped inertial system, these velocity errors should be eliminated. This clearly requires additional information on the magnitude and the direction of the absolute velocity of the object. One of the possible sources of this aiding information is a Doppler velocity meter, which measures the velocity of the object relative to the Earth. Since the angular velocity of the Earth's rotation is known to high accuracy, the Doppler meter readings plus the transportation velocity due to the Earth's spin give the absolute velocity needed for guidance purposes.

For sea vessels, a normal log can be used as a source of external guidance information on the relative velocity (this presupposes knowledge of the sea current velocities).

Before proceeding with a discussion of correction using Doppler velocity meters (or logs), it is worthwhile considering one further possibility which arises in this context.

Let us change the ideal operation equations (1.36) as follows:

$$\left.\begin{aligned} \frac{dr}{dt} &= \int_0^t \left[n - m\times\frac{dr}{dt} + g - k_1\left(\frac{dr}{dt} - m\times r\right)\right]dt + \frac{dr^0}{dt}, \\ r &= \int_0^t \left(\frac{dr}{dt} - m\times r\right)dt + r^0. \end{aligned}\right\} \tag{5.20}$$

We ignore the accelerometer errors and the errors of the gyroscopic absolute angular velocity meters. Let further the axes xyz fixed to the

inertial platform be maintained in an invariable attitude with the axis z caged to r. This can be accomplished, say, by using suitable gravimetric arrangements with accelerometers.* In this case the coordinates determined by the inertial system from the solutions of Eqs. (5.20) are not used for orienting the platform z axis along the radius-vector r. The error equations (5.6) are now replaced by

$$\left.\begin{aligned}
&\delta\ddot{x} + k_1\delta\dot{x} + \left(\frac{\mu}{r^3} - \omega_y^2 - \omega_z^2\right)\delta x + \\
&\quad + (\omega_x\omega_y - \dot{\omega}_z)\delta y - 2\omega_z\delta\dot{y} + \\
&\quad + (\omega_x\omega_z + \dot{\omega}_y)\delta z + 2\omega_y\delta\dot{z} = 0, \\
&\delta\ddot{y} + k_1\delta\dot{y} + \left(\frac{\mu}{r^3} - \omega_z^2 - \omega_x^2\right)\delta y + \\
&\quad + (\omega_y\omega_z - \dot{\omega}_x)\delta z - 2\omega_x\delta\dot{z} + \\
&\quad + (\omega_y\omega_x + \dot{\omega}_z)\delta x + 2\omega_z\delta\dot{x} = 0, \\
&\delta\ddot{z} + k_1\delta\dot{z} + \left(\frac{\mu}{r^3} - \omega_x^2 - \omega_y^2\right)\delta z + \\
&\quad + (\omega_z\omega_x - \dot{\omega}_y)\delta x - 2\omega_y\delta\dot{x} + \\
&\quad + (\omega_z\omega_y + \dot{\omega}_x)\delta y + 2\omega_x\delta\dot{y} = -k_1\dot{r}.
\end{aligned}\right\} \qquad (5.21)$$

If r is exactly known, Eqs. (5.7) can be replaced by

$$\left.\begin{aligned}
&\delta\ddot{x} + k_1\delta\dot{x} + \left(\frac{\mu}{r^3} - \omega_y^2 - \omega_z^2\right)\delta x + (\omega_x\omega_y - \dot{\omega}_z)\delta y - 2\omega_z\delta\dot{y} = 0, \\
&\delta\ddot{y} + k_1\delta\dot{y} + \left(\frac{\mu}{r^3} - \omega_z^2 - \omega_x^2\right)\delta y + (\omega_y\omega_x + \dot{\omega}_z)\delta x + 2\omega_z\delta\dot{x} = 0.
\end{aligned}\right\} \qquad (5.22)$$

The structure of the left-hand sides of Eqs. (5.21), (5.22) is somewhat different from the left-hand sides of Eqs. (5.6), (5.7). However, Eqs. (5.21), (5.22), like Eqs. (5.6), (5.7), contain the damping terms $k_1\delta\dot{x}$, $k_1\delta\dot{y}$, $k_1\delta\dot{z}$. At the same time, the right-hand sides of Eqs. (5.22) are zero, so that there are no velocity errors. Velocity errors in Eqs. (5.21) are entirely attributable to the vertical velocity component $\dot{r}$ of the object and therefore do not arise for $r = \text{const}$.

At a first glance it would seem that the velocity errors have been eliminated without using additional information on the velocity of the object, anyhow without any information on its horizontal component. In fact, however, this is not so. Remember that we assumed gravimetric attitude stabilization caging the z axis of the platform to r and postulated ideal gyroscopic angular velocity meters. Now this implies from the start knowledge of ω_x, ω_y, and hence (seeing that r is known) knowledge of the horizontal components of the absolute velocity $v_x = r\omega_y$, $v_y = -r\omega_x$. We thus in fact did use additional information on the velocity, although this information was provided only by inertial elements (not to speak about r).

§5.2. BASIC INFORMATION ON DOPPLER VELOCITY METERS

The principle of the Doppler velocity meter is naturally the Doppler effect. We will briefly recapitulate the elementary theory of this effect here.

* See my "Theory of Inertial Navigation: Autonomous Systems." — Nauka. 1966.

A source of electromagnetic waves is situated at point A (Figure 5.2). A detector which receives waves reflected from the plane NN is situated at the same point. Suppose the point A moves with velocity V relative to the plane NN, the direction of V being normal to the reflecting plane. The emitted signal can be written in the form

$$E_1 = E_1^0 \sin \omega t, \tag{5.23}$$

where ω is the radian frequency of the radiated signal.

The detected signal is then written in the form

$$E_2 = E_2^0 \sin\left[\omega t - \omega \frac{2\left(l^0 - \int_0^t V\,dt\right)}{c}\right], \tag{5.24}$$

where V is the speed of the point A, l^0 the initial distance from A to the plane NN, c the velocity of light

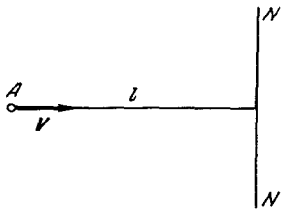

FIGURE 5.2.

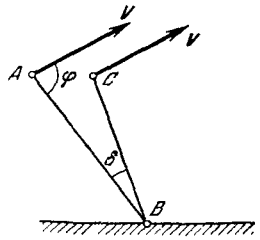

FIGURE 5.3.

If ω is the radian frequency of the radiated signal, the radian frequency of the detected signal is

$$\omega' = \frac{d}{dt}\left[\omega t - \omega \frac{2\left(l^0 - \int_0^t V\,dt\right)}{c}\right] = \omega\left(1 + \frac{2V}{c}\right). \tag{5.25}$$

The Doppler frequency shift is given by

$$\Delta\omega = \omega' - \omega = \frac{2\omega}{c} V, \tag{5.26}$$

i.e., the Doppler frequency shift is proportional to the speed V of the point relative to the reflecting plane.

A similar relation is applicable to a tight beam of radiation directed at the Earth's surface from a moving object.* Indeed, suppose a packet of electromagnetic waves is beamed from point A (Figure 5.3) in the direction AB. After reflection at point B of the Earth's surface, the wave packet is

* See, e.g., Vinnitskii, A.S. Ocherk osnov radiolokatsii pri nepreryvnom izluchenii radiovolon (Continuous Radar Techniques).—Sovetskoe Radio. 1961; Astaf'ev, G.P., V.S. Shebshaevich, and Yu. A. Yurkov. Radionavigatsionnye ustroistva i sistemy (Radio Navigation Systems). — Sovetskoe Radio. 1958.

detected by a receiver at point C; this is the point to which the object will have moved during the time it takes the signal to travel to the Earth and back. Suppose the velocity V of the object makes an angle ψ with the beaming direction AB; both V and ψ can be regarded as constant during the time it takes the signal to travel to the Earth and back, and the angle δ is small. The Doppler shift is thus

$$\Delta\omega = \frac{2\omega}{c} V \cos\psi. \tag{5.27}$$

i.e., the frequency shift is proportional to the projection of the vector V on the beaming direction.

Thus, if there are three transceivers on board the moving object beaming in the directions q_1, q_2, q_3, the system can measure the three projections V_1, V_2, V_3 of the relative velocity vector on these directions. Let $q^1 q^2 q^3$ be the contravariant base vectors corresponding to $q_1 q_2 q_3$, so that the vector V can be written in the form

$$V = q^1 V_1 + q^2 V_2 + q^3 V_3. \tag{5.28}$$

The orientation of the Doppler velocity meter antennas relative to the body of the object is known, i.e., the orientation of the unit vectors q_1, q_2, q_3 is given. The orientation of the inertial platform relative to the body of the object is also known. From V_1, V_2, V_3 we can thus find the projections V_x, V_y, V_z of the relative velocity vector V along the accelerometer sensitive axes.

Note that the inertial system measures the absolute velocity of the point where (or near which) the accelerometer sensitive masses are located. The Doppler velocity meter, on the other hand, measures the projections of the relative velocity of those points where (or near which) the beaming antennas are mounted. The different antennas (or pairs of receiving and transmitting antennas) are placed at different points, all at certain distances from the point of the sensitive elements. As a result, the relative velocity vector measured by the inertial system may differ from that measured by the Doppler even if both systems are ideal. However, the integral over the difference in these velocities is clearly small, and the effect of the error is therefore negligible. In what follows we will therefore assume that the Doppler in fact measures the relative velocity of that point whose coordinates are determined by the inertial system.

Further note that the antennas radiate a beam of finite width, so that the radiation is reflected by a certain area of the Earth's surface, and not by a single point. The reflected signal therefore contains a whole spectrum of Doppler shifts. A similar effect results when V and ψ are changed between the sending and the reception of the wave packet (or the pulse train). On the other hand, the Doppler has a fairly high random noise level, and the effective passband of the processing channel in the receiver is therefore made fairly narrow. As a result, the Doppler is incapable of measuring the instantaneous relative velocity; it only produces a certain average figure, which is equivalent to a certain delay. In other words, the Doppler readings V_{D1}, V_{D2}, V_{D3} are not equal to the projections V_1, V_2, V_3 of the relative velocity on the directions q_1, q_2, q_3. The two sets of quantities are related to first

approximation by differential expressions of the form

$$T\frac{dV_{D1}}{dt}+V_{D1}=V_1. \tag{5.29}$$

Here $T=1/\Delta\Omega$ is the time constant, $\Delta\Omega$ being the effective passband of the processing channel. The time constant T is at most a few seconds. When the Doppler velocity meter is considered in conjunction with the inertial system, this time constant is ignorable, since the period of natural oscillations of the inertial system is of the order of the Schuler period, i.e., tens of minutes.

In what follows we will assume that the Doppler measures the velocity V_D of the object relative to the Earth's surface below the moving object. Since this is the velocity relative to the neighborhood of the point where the current radius-vector r of the object crosses the surface of the geoid, we have

$$V_D=v-\omega\times r\frac{\rho}{r}, \tag{5.30}$$

where v is the absolute velocity of the object, ω is the angular velocity of the Earth's rotation, and ρ is the distance from the Earth's center to the Doppler reference point. For motion near the Earth's surface $\rho/r\approx 1$.

The Doppler error will be designated ΔV_D. V'_D is the actual Doppler reading, i.e., $V'_D=V_D+\Delta V_D$. The variation δV_D of the Doppler readings is

$$\delta V_D=V'_D-V_D=\Delta V_D. \tag{5.31}$$

$|\Delta V_D|$ is of the order* of $0.001\,|V|$.

§5.3. DOPPLER DAMPING OF AN INERTIAL SYSTEM

5.3.1. Unperturbed (ideal) operation equations. Error equations

We have from the preceding

$$V_D=v-\omega\times r\frac{\rho}{r}. \tag{5.32}$$

The first integration in the inertial system, according to the ideal operation equations (1.36), gives the velocity vector v, and the second integration provides the radius-vector r. We can thus find the relative velocity

$$V=v-\omega\times r\frac{\rho}{r}. \tag{5.33}$$

We now form the difference

$$\Delta V=V-V_D. \tag{5.34}$$

* See, e.g., McClure, C.L. Theory of Inertial Guidance. — Prentice Hall. 1960.

If $\Delta V_D = 0$ (i.e., an ideal Doppler velocity meter), and the inertial system is error-free, we have $\Delta V \equiv 0$.

For perturbed operation and $\Delta V_D \neq 0$, the variation $\Delta V' = \delta \Delta V$ is given by the equality

$$\begin{aligned}\delta \Delta V = &-\left(v - u \times r \frac{\rho}{r} + \Delta V_D\right) + \\ &+\left[v + \delta v - u \times r \frac{\rho}{r} - \delta\left(u \times r \frac{\rho}{r}\right)\right] = \\ &= \delta v - \delta u \times r \frac{\rho}{r} - u \times \delta\left(r \frac{\rho}{r}\right) - \Delta V_D. \end{aligned} \tag{5.35}$$

The vector δu is obtained from the following considerations. The angular velocity vector of the Earth is given in the fundamental Cartesian system of coordinates, whose orientation on board the moving object is known to within a small rotation angle θ, which is obtained from the second group of error equations. Therefore,

$$\delta u = -\theta \times u. \tag{5.36}$$

Now

$$\delta\left(r \frac{\rho}{r}\right) = \delta r \frac{\rho}{r} + r \frac{\delta \rho}{r} - r \frac{\rho \delta r}{r^2}.$$

Retaining terms which contain the product of the variations and the square of the eccentricity of the Clairaut ellipsoid, we find

$$\left.\begin{aligned} \delta\left(r \frac{\rho}{r}\right) &= \delta r \frac{a}{r} + r \frac{\delta \rho}{r} - r \frac{a \Delta r}{r^2}, \\ \Delta r &= \Delta h, \\ \delta u \frac{\rho}{r} &= -\theta \times u \frac{a}{r}. \end{aligned}\right\} \tag{5.37}$$

Inserting (5.37) in (5.35) and dropping the small product $u \delta \rho$, we finally have

$$\left.\begin{aligned} \delta \Delta V = \delta v + \left(\theta \times u \frac{a}{r}\right) \times r - u \times \left(\delta r \frac{a}{r} - r \frac{a \Delta r}{r^2}\right) - \Delta V_D \\ (\Delta r = \Delta h). \end{aligned}\right\} \tag{5.38}$$

The difference ΔV between the relative velocity measured by the inertial system and the Doppler velocity can be put to work for damping the inertial system. It suffices to alter the ideal operation equations (1.36) as we did in (5.4), namely we should take

$$\left.\begin{aligned} v &= \int_0^t (n - m \times v + g - k_1 \Delta V)\, dt + v^0, \\ r &= \int_0^t (v - m \times r)\, dt + r^0, \\ \Delta V &= V - V_D. \end{aligned}\right\} \tag{5.39}$$

Here, as in (5.4), the dimensions of k_1 are $\sec^{-1}$.

Our aim can be achieved also by writing (1.36) in the form

$$\left.\begin{aligned} v = \int_0^t (n - m \times v + g)\, dt + v^0, \end{aligned}\right\} \tag{5.40}$$

$$\left.\begin{aligned} r &= \int_0^t [v - m \times r + k_1'(\Delta\dot{V} + m \times \Delta V)]\, dt + r^0, \\ \Delta V &= V - V_D. \end{aligned}\right\} \tag{5.40}$$

Here the dimensions of k_1' are obviously sec. Dot denotes differentiation in the axes fixed to the inertial platform, i.e., in the same axes in which we integrate in Eqs. (5.39), (5.40).

If an ideal Doppler is assumed, the inertial system elements are error-free, and the initial conditions are exact, we have, as before, $\Delta V = 0$, and Eqs. (5.39), (5.40) therefore coincide with the corresponding equations (1.36). Otherwise, the motion of the system is perturbed. This perturbed motion, in general, is different for Eqs. (5.39) and (5.40). In either case, however, it differs from the perturbed motion for $k_1 = 0$, $k_1' = 0$, discussed in the previous chapters.

Let us consider the perturbed motion with Eqs. (5.39).

Varying these equations, we get

$$\left.\begin{aligned} \delta\dot{v} &= \Delta n - \Delta m \times v - \omega \times \delta v + \delta g - k_1 \delta\, \Delta V, \\ \delta\dot{r} &= \delta v - \Delta m \times r - \omega \times \delta r. \end{aligned}\right\} \tag{5.41}$$

We will assume a three-accelerometer system in which the gravitational acceleration is formed using external guidance information on the altitude h of the object. Then, by (2.12),

$$\delta g = -\delta\left(\frac{\mu r}{r^3}\right) = -\frac{\mu}{r^3}\delta r - r\mu\delta\left(\frac{1}{r^3}\right) = -\frac{\mu\,\delta r}{r^3} + \frac{3\mu\,\Delta r r}{r^4}. \tag{5.42}$$

Inserting δv from the second equation in (5.41) in the first, we at the same time substitute for $\delta\,\Delta V$ and δg their expressions from (5.38) and (5.42). After obvious manipulations, we obtain the vector equation

$$\left.\begin{aligned} &\frac{d^2\delta r}{dt^2} + k_1\frac{d\,\delta r}{dt} - k_1 u \times \delta r\frac{a}{r} + \mu\frac{\delta r}{r^3} = \\ &= \Delta n - \Delta m \times \frac{dr}{dt} - \frac{d}{dt}(\Delta m \times r) + \\ &+ 3\mu\frac{r\,\Delta r}{r^4} + k_1\left[\Delta V_D - u \times r\frac{a\,\Delta r}{r^2} - \right. \\ &\left. - (\theta \times u) \times r\frac{a}{r} - \Delta m \times r\right] \\ &(\Delta r = \Delta h). \end{aligned}\right\} \tag{5.43}$$

Eq. (5.43) gives the first group of scalar error equations for an inertial system with simultaneous altimeter and Doppler correction. It clearly replaced Eq. (2.15a) to which it is reduced when $k_1 = 0$. The second group of error equations does not change: it retains the same form $\left(\frac{d\theta}{dt} = \Delta m\right)$ as for autonomous inertial systems.

Eq. (5.43) is the error equation of a three-accelerometer system. The first group of error equations for a two-accelerometer system (with the accelerometers lying in the plane of the horizon) can also be derived from Eq. (5.43). To this end, the vector equation (5.43) should be projected onto the axis x, y of a coordinate system in which the z axis points along the

radius-vector r. In the resulting scalar equations, δz should be replaced with $\Delta r (= \Delta h)$. Finally, we obtain

$$\left.\begin{aligned}
&\delta\ddot{x} + k_1\delta\dot{x} + \left(\frac{\mu}{r^3} - \omega_y^2 - \omega_z^2\right)\delta x + \\
&\quad + \left[\omega_x\omega_y - \dot{\omega}_z - k_1\left(\omega_z - u_z\frac{a}{r}\right)\right]\delta y - 2\omega_z\delta\dot{y} = \\
&\quad = \Delta n_x - 2\Delta m_y\dot{r} - \Delta\dot{m}_y r - \omega_x r\,\Delta m_z - \omega_z r\,\Delta m_x - \\
&\qquad - (\omega_x\omega_z + \dot{\omega}_y + k_1\omega_y)\Delta r - 2\omega_y\Delta\dot{r} + \\
&\qquad + k_1(\Delta V_{Dx} + au_z\theta_x - au_x\theta_z - r\,\Delta m_y), \\
&\delta\ddot{y} + k_1\delta\dot{y} + \left(\frac{\mu}{r^3} - \omega_z^2 - \omega_x^2\right)\delta y + \\
&\quad + \left[\omega_y\omega_x + \dot{\omega}_z + k_1\left(\omega_z - u_z\frac{a}{r}\right)\right]\delta x + 2\omega_z\delta\dot{x} = \\
&\quad = \Delta n_y + 2\Delta m_x r + \Delta\dot{m}_x r - \omega_y r\,\Delta m_z - \\
&\qquad - \omega_z r\,\Delta m_y - (\omega_y\omega_z - \dot{\omega}_x - k_1\omega_x)\Delta r + 2\omega_x\Delta\dot{r} + \\
&\qquad + k_1(\Delta V_{Dy} + au_z\theta_y - au_y\theta_z + r\,\Delta m_x) \\
&\qquad (\Delta r = \Delta h).
\end{aligned}\right\} \tag{5.44}$$

For $k_1 = 0$, these equations are reduced to Eqs. (2.28).

The error equations (5.43), (5.44) were derived for the case of Doppler correction in accordance with (5.39). The error equations for a system functioning in accordance with (5.40) can be similarly derived.

By varying the first two equations in (5.40), we get

$$\left.\begin{aligned}
&\delta\dot{v} = \Delta n - \Delta m \times v - \omega \times \delta v + \delta g, \\
&\delta\dot{r} = \delta v - \Delta m \times r - \omega \times \delta r + k_1'(\delta\Delta\dot{V} + \omega \times \delta\Delta V).
\end{aligned}\right\} \tag{5.45}$$

Inserting for $\delta\Delta V$ in the second equation its expression from (5.38) and solving the resulting equation for $\delta\dot{v}$, we get

$$\left.\begin{aligned}
\delta\dot{v} = &\frac{1}{k_1'}(\delta\dot{r} - \delta v + \Delta m \times r + \omega \times \delta r) - \omega \times \delta v + \\
&+ \frac{d}{dt}\left[u \times \delta r\,\frac{a}{r} - (\theta \times u) \times r\,\frac{a}{r} - u \times r\,\frac{a\,\Delta r}{r^2} + \Delta V_D\right].
\end{aligned}\right\} \tag{5.46}$$

From (5.46) and the first equality in (5.45) we now find

$$\left.\begin{aligned}
\delta v = &- k_1'(\Delta n - \Delta m \times v + \delta g) + \delta\dot{r} + \omega \times \delta r - \\
&- k_1'\frac{d}{dt}\left[u \times \delta r\,\frac{a}{r} - (\theta \times u) \times r\,\frac{a}{r} - u \times r\,\frac{a\,\Delta r}{r^2} + \Delta V_D\right].
\end{aligned}\right\} \tag{5.47}$$

This expression for δv is inserted together with δg from (5.42) in Eq. (5.46) or into the first equation in (5.45). After obvious simplifications, we obtain the vector equation

$$\left.\begin{aligned}
&\frac{d^2\delta r}{dt^2} + k_1'\frac{\mu}{r^3}\frac{d\,\delta r}{dt} + k_1'\frac{d^2}{dt^2}\left(u \times \delta r\,\frac{a}{r}\right) + \\
&\quad + \frac{\mu}{r^3}\left(1 - 3k_1'\frac{\dot{r}}{r}\right)\delta r = \Delta n - \Delta m \times \frac{dr}{dt} - \\
&\qquad - \frac{d}{dt}(\Delta m \times r) + 3\mu r\,\frac{\Delta r}{r^5} + \\
&\quad + k_1'\left\{\frac{d}{dt}\left(\Delta n - \Delta m \times \frac{dr}{dt} + 3\mu r\,\frac{\Delta r}{r^5}\right) + \right. \\
&\quad \left. + \frac{d^2}{dt^2}\left[(\theta \times u) \times r\,\frac{a}{r} + u \times r\,\frac{a\,\Delta r}{r^2} - \Delta V_D\right]\right\} \\
&\qquad (\Delta r = \Delta h).
\end{aligned}\right\} \tag{5.48}$$

For $k_1' = 0$ this equation reduces to Eq. (2.15a).

Eq. (5.48) is the error equation for a three-accelerometer system functioning according to (5.40). To obtain the corresponding equation for a two-accelerometer system, we should project the vector equation (5.48) on the axes x, y of a coordinate system whose z axis points along the radius-vector r. In the resulting equations δz is replaced with Δr and the projections of the vector $\frac{d}{dt}\left(\frac{3\mu r\,\Delta r}{r^4}\right)$ on the right are omitted.

5.3.2. Ideal operation equations and error equations for an orthodromic system

Before proceeding with the analysis of the error equations (5.43), (5.44), (5.48), it will be useful to repeat the derivation of these equations for some particular inertial navigation system. As an example, we choose a two-accelerometer system which measures the orthodromic coordinates. The ideal operation equations of this system are Eqs. (2.23).

We introduce damping with the aid of Eqs. (5.39). According to (5.39), (5.33), (5.34), (5.31), the first two equations in (2.23) take the form

$$\left.\begin{aligned} v_x &= \int_0^t (n_x + v_y\omega_z - \dot{r}\omega_y + g_x - k_1\,\Delta V_x)\,dt + v_x^0, \\ v_y &= \int_0^t (n_y + \dot{r}\omega_x - v_x\omega_z + g_y - k_1\,\Delta V_y)\,dt + v_y^0. \end{aligned}\right\} \tag{5.49}$$

In (5.49)

$$\left.\begin{aligned} &\Delta V_x = V_x - V_{Dx}, \quad \Delta V_y = V_y - V_{Dy}, \\ &V_x = v_x - \rho u\,(-\beta_{31}\sin z\cos S - \beta_{32}\sin z\sin S + \\ &\qquad + \beta_{33}\cos z), \\ &V_y = v_y + \rho u\,(-\beta_{31}\sin S + \beta_{32}\cos S). \end{aligned}\right\} \tag{5.50}$$

We have already mentioned in the preceding that it is undesirable to perform any computational operations on the accelerometer readings before their integration. This remark is equally applicable to Eqs. (5.49) and to the first two equations in (2.23). The reason for this is that n_x, n_y contain rapidly varying components. In the actual design stages it may prove more convenient to partition each of the integrals on the right in Eqs. (5.49) at least into two.

The first will be integrals over n_x, n_y. In the computations we will therefore operate with fairly slowly varying quantities $\int n_x\,dt$, $\int n_y\,dt$, so that slow computers can be used. Similarly, ΔV_x, ΔV_y can be integrated separately; a further simplifying step is to expand ΔV_x, ΔV_y in accordance with the last three equalities in (5.50) and to proceed with separate integrations of V_{Dx}, V_{Dy}. The different alternatives are entirely equivalent from the point of view of the final result and the error equations, and in what follows we will therefore consider the ideal operation equations in the form (5.49).

The error equations of our damped system follow from the error equations of the general system considered in the previous section. These are Eqs. (5.44). However, to better understand the meaning of these equations,

we will derive them independently for an orthodromic inertial system from the corresponding operation equations. We will confine the treatment to the first group of error equations, since the second group obviously does not change.

To obtain the first group of error equations, we should vary Eqs. (5.49), (5.50) together with the remaining relations in (2.23), which complete Eqs. (5.49), (5.50) to a closed system. The solution of the problem is simplified by the previous treatment in §4.2, where we derived the error equations of an orthodromic system without Doppler correction. There we started with ideal operation equations (4.20). Putting $\varepsilon = 0$ in Eqs. (4.20), we see that they differ from Eqs. (4.49), (4.50), (2.23) only in that the integrands in the first two equations in (4.20) do not contain the terms $k_1 \Delta V_x$, $k_1 \Delta V_y$. The error equations correspondingly differ only in the variations $k_1 \delta \Delta V_x$, $k_1 \delta \Delta V_y$ of these extra terms. The problem thus reduces to a computation of these variations, after which the results of §4.2 can be applied.

As before, $x_0 y_0 z_0$ stand for the unperturbed axes fixed to the inertial platform. The orientation of the perturbed axes xyz relative to $x_0 y_0 z_0$ is defined by the small angles α, β, γ in the matrix (4.22). From this matrix and the equality $V_D' = V_D + \Delta V_D$ we have

$$\left.\begin{aligned} V'_{Dx} &= V_{x_0} + V_{y_0}\gamma - V_{z_0}\beta + \Delta V_{Dx_0}, \\ V'_{Dy} &= V_{y_0} + V_{z_0}\alpha - V_{x_0}\gamma + \Delta V_{Dy_0}, \end{aligned}\right\} \tag{5.51}$$

where ΔV_{Dx_0}, ΔV_{Dy_0} are the Doppler errors. They also include the errors associated with the conversion of the relative velocity components along the Doppler antennas to the velocity components along the platform axes.

From the last two equalities in (5.50) we see that the relative velocity projections V'_x, V'_y determined by the perturbed inertial system are

$$\left.\begin{aligned} V'_x &= v_{x_0} + \delta v_{x_0} - (\rho + \delta\rho)\, u\,(-\beta_{31}\sin z\cos S - \\ &\quad - \beta_{32}\sin z\sin S + \beta_{33}\cos z) - \rho u\delta\,(-\beta_{31}\sin z\cos S - \\ &\quad - \beta_{32}\sin z\sin S + \beta_{33}\cos z), \\ V'_y &= v_{y_0} + \delta v_{y_0} + (\rho + \delta\rho)\, u\,(-\beta_{31}\sin S + \beta_{32}\cos S) + \\ &\quad + \rho u\delta\,(-\beta_{31}\sin S + \beta_{32}\cos S). \end{aligned}\right\} \tag{5.52}$$

But

$$\left.\begin{aligned} u\,(-\beta_{31}\sin z\cos S - \beta_{32}\sin z\sin S + \beta_{33}\cos z) &= u_{y_0}, \\ u\,(\beta_{31}\cos z\cos S + \beta_{32}\cos z\sin S + \beta_{33}\sin z) &= u_{z_0}, \\ u\,(\beta_{31}\sin S - \beta_{32}\cos S) &= -u_{x_0}, \\ u\,(\beta_{31}\cos S + \beta_{32}\sin S) &= u_{z_0}\cos z - u_{y_0}\sin z. \end{aligned}\right\} \tag{5.53}$$

Further

$$\left.\begin{aligned} u\delta\,(-\beta_{31}\sin z\cos S - \beta_{32}\sin z\sin S + \beta_{33}\cos z) &= \\ &= -(u_{z_0}\delta z + u_{x_0}\sin z\delta S), \\ u\delta\,(-\beta_{31}\sin S + \beta_{32}\cos S) &= \\ &= -(u_{z_0}\cos z - u_{y_0}\sin z)\,\delta S. \end{aligned}\right\} \tag{5.54}$$

Inserting (5.53), (5.54) in Eqs. (5.52), we introduce new variables α_2, β_2, γ_2 defined by (4.37) with $\varepsilon = 0$. This gives

$$\left.\begin{aligned} V'_x &= v_{x_0} - \rho u_{y_0} + \delta v_{x_0} - u_{y_0}\delta\rho + \rho(-u_{z_0}\alpha_2 + u_{x_0}\gamma_2), \\ V'_y &= v_{y_0} + \rho u_{x_0} + \delta v_{y_0} + u_{x_0}\delta\rho - \rho(u_{z_0}\beta_2 - u_{y_0}\gamma_2). \end{aligned}\right\} \tag{5.55}$$

Finally inserting for the angles α_2, β_2, γ_2 their expressions in terms of α, β, γ, θ_{x_0}, θ_{y_0}, θ_{z_0} from (4.39) and dropping the factors $u_{x_0}\delta\rho$, $u_{y_0}\delta\rho$, we obtain from (5.55)

$$\left.\begin{aligned} V'_x &= v_{x_0} - \rho u_{y_0} + \delta v_{x_0} + \rho(-u_{z_0}\theta_{x_0} + u_{x_0}\theta_{z_0}) + \rho(-u_{z_0}\alpha + u_{x_0}\gamma), \\ V'_y &= v_{y_0} + \rho u_{x_0} + \delta v_{y_0} - \rho(u_{z_0}\theta_{y_0} - u_{y_0}\theta_{z_0}) - \rho(u_{z_0}\beta - u_{y_0}\gamma). \end{aligned}\right\} \tag{5.56}$$

From (5.56), (5.51), seeing that

$$V_{x_0} = v_{x_0} - \rho u_{y_0},\quad V_{y_0} = v_{y_0} + \rho u_{x_0},\quad V_{z_0} = \dot{r}, \tag{5.57}$$

we obtain the variations $\delta\Delta V_{x_0} = V'_x - V'_{Dx}$, $\delta\Delta V_{y_0} = V'_y - V'_{Dy}$. They are given by

$$\left.\begin{aligned} \delta\Delta V_{x_0} &= \delta v_{x_0} + \rho(-u_{z_0}\theta_{x_0} + u_{x_0}\theta_{z_0}) - \rho u_{z_0}\alpha + r\omega_{x_0}\gamma + \dot{r}\beta - \Delta V_{Dx_0}, \\ \delta\Delta V_{y_0} &= \delta v_{y_0} - \rho(u_{z_0}\theta_{y_0} - u_{y_0}\theta_{z_0}) - \rho u_{z_0}\beta + r\omega_{y_0}\gamma - \dot{r}\alpha - \Delta V_{Dy_0}. \end{aligned}\right\} \tag{5.58}$$

To further simplify Eqs. (5.58), we use the first two equalities in (4.25). Seeing that in these equalities we may take $\delta r = \Delta r$ and inserting $\delta\omega_{x_0}$, $\delta\omega_{y_0}$ from (4.24), we obtain expressions for δv_{x_0}, δv_{y_0} in terms of α, β, γ, Δr, $\Delta\omega_{x_0}$, $\Delta\omega_{y_0}$, r, ω_{x_0}, ω_{y_0}, ω_{z_0}. Inserting these expressions in (5.58) and substituting the variables δx and δy for α and β in accordance with (4.32), we get

$$\left.\begin{aligned} \delta\Delta V_{x_0} &= \Delta r\omega_{y_0} - r\Delta\omega_{y_0} + \delta\dot{x} - \left(\omega_{z_0} - \frac{\rho}{r}u_{z_0}\right)\delta y + \rho(u_{x_0}\theta_{z_0} - u_{z_0}\theta_{x_0}) - \Delta V_{Dx_0}, \\ \delta\Delta V_{y_0} &= -\Delta r\,\omega_{x_0} + r\Delta\omega_{x_0} + \delta\dot{y} + \left(\omega_{z_0} - \frac{\rho}{r}u_{z_0}\right)\delta x + \rho(u_{y_0}\theta_{z_0} - u_{z_0}\theta_{y_0}) - \Delta V_{Dy_0}. \end{aligned}\right\} \tag{5.59}$$

It is readily seen that in an unperturbed system, when the instrumental errors vanish and $\delta x = \delta y = \theta_{x_0} = \theta_{y_0} = \theta_{z_0} = 0$, we have zero on the right in Eqs. (5.59). Therefore, $\delta\Delta V_{x_0} = \delta\Delta V_{y_0} = 0$, as is proper.

Let us compare Eqs. (5.59) with the expressions obtained from Eqs. (5.38), (5.41) for the case of a two-accelerometer system. From (5.38) and the second equality in (5.41) we get

$$\begin{aligned} \delta\Delta V = {} & \delta\dot{r} + \omega\times\delta r + \Delta m\times r + \\ & + (\theta\times u)\times r\frac{a}{r} - u\times\left(\delta r\frac{a}{r} - r\frac{\Delta r a}{r^2}\right) - \Delta V_D. \end{aligned} \tag{5.60}$$

In our case

$$\delta r = \delta x\, x_0 + \delta y\, y_0 + \delta z z_0, \tag{5.61}$$

where x_0, y_0, z_0 are the unit vectors of the corresponding axes. Therefore, taking the projections of Eq. (5.60) onto the axes x_0, y_0, putting $\rho = a$, and seeing that $\Delta m_{x_0} = -\Delta\omega_{x_0}$, $\Delta m_{y_0} = -\Delta\omega_{y_0}$, $\Delta m_{z_0} = -\Delta\omega_{z_0}$, we obtain Eqs. (5.59).

Varying Eqs. (5.49), we get

$$\left.\begin{aligned} \delta\dot{v}_x &= \delta n_x + \omega_z\delta v_y + v_y\delta\omega_z - \dot{r}\delta\omega_y - \omega_y\delta\dot{r} + \delta g_x - k_1\delta\Delta V_x, \\ \delta\dot{v}_y &= \delta n_y + \omega_x\delta\dot{r} + \dot{r}\,\delta\omega_x - v_x\delta\omega_z - \omega_z\delta v_x + \delta g_y - k_1\delta\Delta V_y. \end{aligned}\right\} \tag{5.62}$$

These equations differ from the first two equations in (4.25) only in that they contain additional terms $-k_1\delta\Delta V_x$, $-k_1\delta\Delta V_y$ on the right. From Eqs. (4.25) we derived Eqs. (4.28) and then Eqs. (4.35). Hence, Eqs. (5.44) should be

obtained from Eqs. (4.35) by adding these terms to the right-hand sides. Using (5.59), we readily see that introduction of the terms $-k_1\delta\Delta V_{x_3}$, $-k_1\delta\Delta V_{y_3}$ on the right in (4.35) indeed leads to Eqs. (5.44), since $\Delta\omega_{x_3} = -\Delta m_{x_3}$, $\Delta\omega_{y_3} = -\Delta m_{y_3}$, $\Delta\omega_{z_3} = -\Delta m_{z_3}$.

We have given a direct derivation of the error equations for an orthodromic inertial system with ideal operation equations in the form (5.39). If the damping is introduced into the system in accordance with Eqs. (5.40), and not (5.39), the first two equations in (2.23) do not change, but the next two equations in (2.23) are definitely altered. They take the form

$$\left.\begin{aligned} \omega_{x_3} &= -\frac{1}{r}\left[v_y + k_1'(\Delta\dot{V}_y + \omega_z\Delta V_x - \omega_x\Delta V_z)\right], \\ \omega_{y_3} &= \frac{1}{r}\left[v_x + k_1'(\Delta\dot{V}_x + \omega_y\Delta V_z - \omega_z\Delta V_y)\right], \end{aligned}\right\} \tag{5.63}$$

where ΔV_x and ΔV_y are expressed by (5.50).

The first group of error equations is obtained from the relations

$$\left.\begin{aligned} \delta\dot{v}_{x_3} &= \delta n_{x_0} + \omega_{z_3}\delta v_{y_3} + v_{y_3}\delta\omega_{z_3} - \Delta\dot{r}\,\omega_{y_0} - \dot{r}\delta\omega_{y_3}, \\ \delta\dot{v}_{y_3} &= \delta n_{y_3} + \Delta\dot{r}\omega_{x_3} + \dot{r}\,\delta\omega_{x_0} - \omega_{x_3}\delta v_{x_3} - v_{x_0}\delta\omega_{z_3}, \end{aligned}\right\} \tag{5.63a}$$

which are the variational form of the first two equations in (2.23), and from the variations of Eqs. (5.63)

$$\left.\begin{aligned} -\Delta r\omega_{x_3} - r\,\delta\omega_{x_3} &= \delta v_{y_3} + k_1'(\delta\Delta\dot{V}_{y_3} + \omega_{z_3}\delta\Delta V_{x_3} - \omega_{x_3}\delta\Delta V_{z_3}), \\ \Delta r\omega_{y_3} + r\,\delta\omega_{y_3} &= \delta v_{x_3} + k_1'(\delta\Delta\dot{V}_{x_3} + \omega_{y_3}\delta\Delta V_{z_3} - \omega_{z_3}\delta\Delta V_{y_3}). \end{aligned}\right\} \tag{5.64}$$

To Eqs. (5.63a), (5.64) we should naturally add Eqs. (5.58) for $\delta\Delta V_{x_3}$, $\delta\Delta V_{y_3}$, Eqs. (4.24), (4.26) for $\delta\omega_{x_3}$, $\delta\omega_{y_3}$, $\delta\omega_{z_3}$, δn_{x_3}, δn_{y_3}, and also Eqs. (4.32).

5.3.3. Stability analysis

Consider the homogeneous equation

$$\frac{d^2\delta r}{dt^2} + k_1\frac{d\,\delta r}{dt} - k_1\boldsymbol{u}\times\delta r\,\frac{a}{r} + \frac{\mu}{r^3}\delta r = 0, \tag{5.65}$$

corresponding to the vector equation (5.43), i.e., the error equation of an inertial system with altimeter and Doppler correction, when the Doppler correction follows Eqs. (5.39).

Our object is to investigate the stability of the motion described by Eq. (5.65). First note that the structure of this equation is somewhat different from the homogeneous equation (5.5).

An additional term $k_1\boldsymbol{u}\times\delta r\,\frac{a}{r}$ appeared in Eq. (5.65). Its origin is self-evident. After all, the damping in Eq. (5.5) uses information on the absolute velocity, whereas in Eq. (5.65) it is based on Doppler measurements of the relative velocity. If the Earth did not spin ($\boldsymbol{u} = 0$), the two equations would coincide.

For $r = \text{const}$, i.e., motion at constant distance from the Earth's surface, the projections of the vector equation (5.65) onto the axes ξ_3, η_3, ζ_3 or ξ, η, ζ are equations with constant coefficients. Thus, projecting onto constant-

orientation axes ξ_*, η_*, ζ_*, which do not spin with the Earth, and seeing that $u_{\xi_*} = u_{\eta_*} = 0$, $u_{\zeta_*} = u = \text{const}$, $\delta r = \delta\xi_*\xi_* + \delta\eta_*\eta_* + \delta\zeta_*\zeta_*$, we find

$$\left.\begin{aligned} \delta\ddot{\xi}_* + k_1\delta\dot{\xi}_* + \omega_0^2\delta\xi_* + k_1\frac{a}{r}u\,\delta\eta_* = 0,\\ \delta\ddot{\eta}_* + k_1\delta\dot{\eta}_* + \omega_0^2\delta\eta_* - k_1\frac{a}{r}u\delta\xi_* = 0,\\ \delta\ddot{\zeta}_* + k_1\delta\dot{\zeta}_* + \omega_0^2\delta\zeta_* = 0. \end{aligned}\right\} \tag{5.66}$$

The third equation in (5.66) is separated. The roots of the corresponding characteristic equation ($k_1 - 4\omega_0^2 < 0$) are

$$p_{1,2} = \frac{1}{2}(-k_1 \pm j\nu),\ \nu = \sqrt{4\omega_0^2 - k_1^2}. \tag{5.67}$$

For $k_1 > 0$, the error $\delta\zeta_*$ decays with time.

To determine the stability of the first two equations in (5.66), we write the corresponding characteristic equation:

$$p^4 + 2k_1p^3 + (k_1^2 + 2\omega_0^2)\,p^2 + 2k_1\omega_0^2p + \omega_0^4 + k_1^2u^2\frac{a^2}{r^2} = 0. \tag{5.68}$$

Hurwitz's conditions for this equation lead to the inequalities

$$\left.\begin{aligned} k_1 > 0,\ 2k_1(k_1^2 + \omega_0^2) > 0,\\ 4k_1^4\left(\omega_0^2 - u^2\frac{a^2}{r^2}\right) > 0,\ \omega_0^4 + k_1^2u^2\frac{a^2}{r^2} > 0. \end{aligned}\right\} \tag{5.69}$$

Since $\omega_0^2 > u^2\frac{a^2}{r^2}$, these inequalities are always satisfied for $k_1 > 0$. Thus, for $k_1 > 0$, Eq. (5.65) leads to asymptotic stability of the three-accelerometer system with altimeter and Doppler correction for any motion at constant distance from the Earth's center.

We can also reach this conclusion by examining the projections of Eq. (5.65) onto the axes fixed to the Earth. Eqs. (5.66) in this case are replaced by

$$\left.\begin{aligned} \delta\ddot{\xi} + k_1\delta\dot{\xi} + (\omega_0^2 - u^2)\delta\xi - 2u\delta\dot{\eta} - k_1u\left(1 - \frac{a}{r}\right)\delta\eta = 0,\\ \delta\ddot{\eta} + k_1\delta\dot{\eta} + (\omega_0^2 - u^2)\delta\eta + 2u\delta\dot{\xi} + k_1u\left(1 - \frac{a}{r}\right)\delta\xi = 0,\\ \delta\ddot{\zeta} + k_1\delta\dot{\zeta} + \omega_0^2\delta\zeta = 0. \end{aligned}\right\} \tag{5.70}$$

The third equation in (5.70) has the same form as the third equation in (5.66). This is quite obvious, since the axis ζ coincides with ζ_*, and therefore $\delta\zeta = \delta\zeta_*$.

The characteristic equation of the system comprising the two first equations in (5.70) is

$$\begin{aligned} p^4 + 2k_1p^3 + \left[k_1^2 + 2(\omega_0^2 + u^2)\right]p^2 +\\ + 2k_1\left[\omega_0^2 + u^2\left(1 - 2\frac{a}{r}\right)\right]p + (\omega_0^2 - u^2)^2 + k_1^2u^2\left(1 - \frac{a}{r}\right)^2 = 0. \end{aligned} \tag{5.71}$$

Hurwitz's inequalities give the following conditions for negative real parts of the roots of the characteristic equation (5.71):

$$\left.\begin{aligned} 2k_1 > 0,\ 2k_1\left[k_1^2+\omega_0^2+u^2\left(1+2\frac{a}{r}\right)\right] > 0,\\ 4k_1^2\left(\omega_0^2-\frac{u^2a^2}{r^2}\right)\left(k_1^2+4u^2\right) > 0,\\ \left(\omega_0^2-u^2\right)^2+k_1^2u^2\left(1-\frac{a}{r}\right)^2 > 0. \end{aligned}\right\} \tag{5.72}$$

These conditions, like (5.69), are always satisfied for $k_1 > 0$.

We have so far considered motion at constant distance from the Earth's center. We will now show that an appropriate choice of k_1 will ensure asymptotic stability also for a variable r. Indeed, let

$$r = r^0 + \rho(t),\ r^0 = \text{const},\quad \frac{|\rho(t)|}{r^0} \ll 1, \tag{5.73}$$

i.e., the object moves near some constant distance r^0 from the Earth's center. Eqs. (5.66) then can be written in the form

$$\left.\begin{aligned} \delta\ddot{\xi}_* + k_1\delta\dot{\xi}_* + \omega_0^2\delta\xi_* = \omega_0^2\frac{3\rho}{r^0}\delta\xi_* - k_1\frac{a}{r}u\,\delta\eta_*,\\ \delta\ddot{\eta}_* + k_1\delta\dot{\eta}_* + \omega_0^2\delta\eta_* = \omega_0^2\frac{3\rho}{r^0}\delta\eta_* + k_1\frac{a}{r}u\,\delta\dot{\xi}_*,\\ \delta\ddot{\zeta}_* + k_1\delta\dot{\zeta}_* + \omega_0^2\delta\zeta_* = \omega_0^2\frac{3\rho}{r^0}\delta\zeta_*, \end{aligned}\right\} \tag{5.74}$$

where

$$\omega_0^2 = \frac{\mu}{r^{0^3}} = \text{const}. \tag{5.75}$$

First let us concentrate on the third equation in (5.74). We write it in Cauchy form, as a set of two first-order differential equations:

$$\left.\begin{aligned} \dot{x}_1 = -k_1x_1 - \omega_0^2x_2 + \varphi_x,\\ \dot{x}_2 = x_1, \end{aligned}\right\} \tag{5.76}$$

where

$$x_2 = \delta\zeta_*,\ \varphi_x = \omega_0^2\frac{3\rho}{r^0}x_2. \tag{5.77}$$

Changing over to the normal coordinates* b_3, u_3, we make the substitution

$$\left.\begin{aligned} x_1 = b_3(\varepsilon\cos u_3 - \nu\sin u_3),\\ x_2 = b_3\cos u_3,\\ \varepsilon = -\frac{k_1}{2},\quad \nu = \sqrt{\omega_0^2-\frac{k_1^2}{4}}. \end{aligned}\right\} \tag{5.78}$$

* See Bulgakov, B.V. Kolebaniya (Oscillations). — Gostekhizdat. 1954.

In terms of the normal variables b_3, u_3, we have

$$\left.\begin{aligned}\frac{db_3}{dt} &= \varepsilon b_3 - \frac{1}{\nu}\tilde{\varphi}_x \sin u_3,\\ \frac{du_3}{dt} &= \nu - \frac{1}{b_3\nu}\tilde{\varphi}_x \cos u_3,\end{aligned}\right\} \tag{5.79}$$

where now

$$\tilde{\varphi}_x = \omega_0^2 \frac{3\rho}{r^0} b_3 \cos u_3. \tag{5.80}$$

Inserting $\tilde{\varphi}_x$ in the first equation in (5.79), we get

$$\frac{db_3}{dt} = b_3\left(\varepsilon - \frac{1}{\nu}\omega_0^2 \frac{3\rho}{r^0} \sin u_3 \cos u_3\right). \tag{5.81}$$

To ensure stability, it is clearly sufficient for the expression in parentheses to be always negative. Hence, and from (5.78), we get

$$q = \frac{k_1}{2} + \frac{1}{\sqrt{\omega_0^2 - \frac{k_1^2}{4}}}\omega_0^2 \frac{3\rho}{2r^0} \sin 2u_3 > 0. \tag{5.82}$$

This condition is satisfied if

$$\frac{k_1}{2} - \frac{\omega_0^2 c}{\sqrt{\omega_0^2 - \frac{k_1^2}{4}}} > 0, \tag{5.83}$$

where

$$c = \frac{3 \max|\rho|}{2r^0} \ll 1. \tag{5.84}$$

We see from inequality (5.83) that to ensure asymptotic stability, k_1 should be chosen between the limits

$$2\omega_0 c < k_1 < \omega_0(2 - c^2). \tag{5.85}$$

Here q can be used as a measure of stability.

Turning to the first two equations in (5.74), we write them in Cauchy form:*

$$\left.\begin{aligned}\dot{y}_1 &= -k_1 y_1 - \omega_0^2 y_2 + \varphi_y,\\ \dot{y}_2 &= y_1,\\ \dot{z}_1 &= -k_1 z_1 - \omega_0^2 z_2 + \varphi_z,\\ \dot{z}_2 &= z_1,\end{aligned}\right\} \tag{5.86}$$

where

$$\left.\begin{aligned}&y_2 = \delta\xi_*,\quad z_2 = \delta\eta_*,\\ &\varphi_y = \omega_0^2 \frac{3\rho}{r^0} y_2 - k_1 \frac{a}{r} u z_2,\\ &\varphi_z = \omega_0^2 \frac{3\rho}{r^0} z_2 + k_1 \frac{a}{r} u y_2.\end{aligned}\right\} \tag{5.87}$$

* The method of stability analysis that follows was developed jointly with N.A. Parusnikov.

We change over to normal coordinates:

$$\left.\begin{aligned} y_1 &= b_1(\varepsilon\cos u_1 - \nu\sin u_1),\\ y_2 &= b_1\cos u_1,\\ z_1 &= b_2(\varepsilon\cos u_2 - \nu\sin u_2),\\ z_2 &= b_2\cos u_2. \end{aligned}\right\} \tag{5.88}$$

In these variables, Eqs. (5.86) take the form

$$\left.\begin{aligned} \frac{db_1}{dt} &= \varepsilon b_1 - \frac{1}{\nu}\tilde{\varphi}_y\sin u_1,\\ \frac{du_1}{dt} &= \nu - \frac{1}{\nu b_1}\tilde{\varphi}_y\cos u_1,\\ \frac{db_2}{dt} &= \varepsilon b_2 - \frac{1}{\nu}\tilde{\varphi}_z\sin u_2,\\ \frac{du_2}{dt} &= \nu - \frac{1}{b_2\nu}\tilde{\varphi}_z\cos u_2. \end{aligned}\right\} \tag{5.89}$$

In accordance with (5.87), (5.88), the functions $\tilde{\varphi}_y$, $\tilde{\varphi}_z$ are given by

$$\left.\begin{aligned} \tilde{\varphi}_y &= \omega_0^2\frac{3\rho}{r^5}b_1\cos u_1 - k_1\frac{a}{r}ub_2\cos u_2,\\ \tilde{\varphi}_z &= \omega_0^2\frac{3\rho}{r^5}b_2\cos u_2 + k_1\frac{a}{r}ub_1\cos u_1. \end{aligned}\right\} \tag{5.90}$$

We introduce a new function

$$U = b_1^2 + b_2^2. \tag{5.91}$$

For asymptotic stability it is clearly sufficient that this function go to zero for t going to infinity. Then b_1, b_2, and also y_1, y_2, z_1, z_2, and hence $\delta\xi_*$, $\delta\dot{\xi}_*$, $\delta\eta_*$, $\delta\dot{\eta}_*$ will asymptotically approach zero with time.

Let us now establish sufficient conditions for

$$\lim_{t\to\infty} U = 0. \tag{5.92}$$

To this end, we take the total derivative dU/dt. Using the definition (5.91) of U, we obtain from the first and third equations in (5.89) and Eqs. (5.90)

$$\frac{dU}{dt} = 2\varepsilon U - \frac{2}{\nu}\left[\omega_0^2\frac{3\rho}{r^5}(b_1^2\sin u_1\cos u_1 + b_2^2\sin u_2\cos u_2) - \right.$$
$$\left. - k_1\frac{a}{r}ub_1b_2\sin(u_1 - u_2)\right]. \tag{5.93}$$

Let

$$b_1 = b\sin\chi, \quad b_2 = b\cos\chi, \quad b^2 = b_1^2 + b_2^2. \tag{5.93a}$$

From (5.93) we then have

$$\frac{dU}{dt} = 2U\left\{\varepsilon - \frac{1}{\nu}\left[\omega_0^2\frac{3\rho}{r^5}(\sin u_1\cos u_1\sin^2\chi + \right.\right.$$
$$\left.\left. + \sin u_2\cos u_2\cos^2\chi) - k_1\frac{a}{r}u\sin(u_1 - u_2)\sin\chi\cos\chi\right]\right\}. \tag{5.94}$$

For the function in braces we write

$$q_1 = \varepsilon - \frac{1}{\nu}\left[\omega_0^2\frac{3\rho}{r^5}(\sin u_1\cos u_1\sin^2\chi + \sin u_2\cos u_2\cos^2\chi) - \right.$$
$$\left. - k_1\frac{a}{r}u\sin(u_1 - u_2)\sin\chi\cos\chi\right]. \tag{5.94a}$$

U will go to zero for $t \to \infty$ if the sign of the derivative dU/dt is always minus that of U. A sufficient condition for this is clearly

$$q_1 < 0. \tag{5.95}$$

We have the obvious estimates

$$\left.\begin{aligned} &|\sin u_1 \cos u_1 \sin^2\chi + \sin u_2 \cos u_2 \cos^2\chi| \leqslant \tfrac{1}{2}, \\ &|\sin(u_1 - u_2)\sin\chi\cos\chi| \leqslant \tfrac{1}{2}, \quad \tfrac{a}{r} \leqslant 1. \end{aligned}\right\} \tag{5.96}$$

Therefore inequality (5.95) is definitely satisfied for

$$\frac{k_1}{2} - \frac{1}{\sqrt{\omega_0^2 - \frac{k_1^2}{4}}}\left(c\omega_0^2 + u\frac{k_1}{2}\right) > 0, \tag{5.97}$$

where c is expressed by (5.84)

Hence, seeing that c and u/ω_0 are small compared to unity, we obtain the limits for k_1:

$$2c(\omega_0 + u) < k_1 < 2(\omega_0 + uc). \tag{5.98}$$

Since $c \ll 1$ and $u/\omega_0 \ll 1$, the limits (5.98) for k_1 can be taken to coincide with the limits (5.85).

q_2 provides a measure of stability. The damping in the system is no worse than that corresponding to the factor $e^{-\min|q_2|}$. This follows from the definition of U and Eq. (5.94). Indeed, by (5.94),

$$\frac{dU}{dt} \leqslant -2U\min|q_2|. \tag{5.99}$$

Hence

$$U \leqslant U^0 e^{-2\min|q_2|}, \tag{5.100}$$

But then

$$b = \sqrt{b_1^2 + b_2^2} \leqslant b^0 e^{-\min|q_2|}, \tag{5.101}$$

where b^0 is the initial value of b.

At this point we will end our analysis of the stability of a three-accelerometer system operating in accordance with Eqs. (5.39), and proceed to an inertial system using Eqs. (5.40). In this case the vector error equation is Eq. (5.48). The corresponding homogeneous equation is

$$\frac{d^2\delta r}{dt^2} + k_1'\frac{\mu}{r^3}\frac{d\delta r}{dt^2} + k_1'\frac{d^2}{dt^2}\left(u \times \delta r\,\frac{a}{r}\right) + \frac{\mu}{r^3}\left(1 - 3k_1'\frac{r}{r}\right)\delta r = 0. \tag{5.102}$$

For $r = \text{const}$ the projections of this equation on the axes ξ_*, η_*, ζ_* give three scalar differential equations with constant coefficients:

$$\left.\begin{aligned} \delta\ddot{\xi}_* + k_1'\omega_0^2\delta\dot{\xi}_* + \omega_0^2\delta\xi_* - k_1'u\frac{a}{r}\delta\ddot{\eta}_* = 0, \\ \delta\ddot{\eta}_* + k_1'\omega_0^2\delta\dot{\eta}_* + \omega_0^2\delta\eta_* + k_1'u\frac{a}{r}\delta\ddot{\xi}_* = 0, \\ \delta\ddot{\zeta}_* + k_1'\omega_0^2\delta\dot{\zeta}_* + \omega_0^2\delta\zeta_* = 0. \end{aligned}\right\} \tag{5.103}$$

For $k_1'\omega_0^2 = k_1$, the third equation in (5.103) reduces to the third equation in (5.74). The characteristic equation for the first two equations in (5.103) is

$$\left(1 + k_1' \frac{a^3 u_z}{r^2}\right) p^4 + 2k_1'\omega_0^2 p^3 + \omega_0^2 (2 + k_1'\omega_0^2)\, p^2 - 2k_1'\omega_0^4 p + \omega_0^4 = 0. \qquad (5.104)$$

Applying Hurwitz's criterion we find that in virtue of the inequality $\omega_0^2 - \frac{u_z a^2}{r^3} > 0$ the real parts of the roots of Eq. (5.104) are always negative for $k_1' > 0$. The solution of the homogeneous equation (5.102) for $r = \text{const}$ is thus damped. The system is asymptotically stable for $k_1' > 0$.

If r is a function of time (and not constant), asymptotic stability can still be ensured by an appropriate choice of k_1'. This choice can be made as above, for Eq. (5.65), i.e., by changing over to normal coordinates. Note, however, that unlike Eq. (5.65), the coefficients of Eq. (5.102) not only contain the distance r from the Earth's center but also its time derivatives. Therefore, in choosing k_1' for $r = r(t)$ we have to consider not only the deviation ρ of r from some constant value r^0 but also the rate of change $\dot{\rho}$ and the corresponding acceleration $\ddot{\rho}$.

Eqs. (5.65) and (5.102) describe the perturbed motion of an error-free three-accelerometer system. The corresponding equations for two-accelerometer systems (with the accelerometers in the plane of the geographical or geocentric horizon) are obtained, as we have noted before, from Eqs. (5.65), (5.102) by taking their projections on the axes x, y of the co-moving system whose z axis points along the radius-vector r and the y axis lies in the plane of the meridian, say. In this way we pass from Eqs. (5.65) to Eqs. (5.44) (if only the terms on the left are retained). The corresponding homogeneous equations are

$$\left.\begin{aligned} &\delta\ddot{x} + k_1\delta\dot{x} + \left(\frac{\mu}{r^3} - \omega_y^2 - \omega_z^2\right)\delta x + \\ &+ \left[\omega_x\omega_y - \dot{\omega}_z - k_1\left(\omega_z - u_z\frac{a}{r}\right)\right]\delta y - 2\omega_z\delta\dot{y} = 0, \\ &\delta\ddot{y} + k_1\delta\dot{y} + \left(\frac{\mu}{r^3} - \omega_z^2 - \omega_x^2\right)\delta y + \\ &+ \left[\omega_y\omega_x + \dot{\omega}_z + k_1\left(\omega_z - u_z\frac{a}{r}\right)\right]\delta x + 2\omega_z\delta\dot{x} = 0. \end{aligned}\right\} \qquad (5.105)$$

Let us investigate the stability of these equations. First we consider motion with constant velocity along a parallel, when the coefficients of Eqs. (5.105) are constant. Then $\omega_x = u_x = 0$. The characteristic equation of (5.105) takes the form

$$\begin{aligned} &p^4 + 2k_1 p^3 + (k_1^2 + 2\omega_0^2 + 2\omega_z^2 - \omega_y^2)\, p^2 + \\ &\quad + k_1\left(2\omega_0^2 + 2\omega_z^2 - \omega_y^2 - 4\omega_z u_z\frac{a}{r}\right) p + \\ &+ (\omega_0^2 - \omega_z^2)(\omega_0^2 - \omega_z^2 - \omega_y^2) + k_1^2\left(\omega_z - u_z\frac{a}{r}\right)^2 = 0. \end{aligned} \qquad (5.106)$$

Eq. (5.106) differs from (5.10) in the coefficient before p and in the free term. For $u = 0$ this equation reduces to (5.10), and for $k_1 = 0$ it reduces to (2.43).

Application of Hurwitz's criterion to Eq. (5.106) gives the following conditions of asymptotic stability:

$$\left.\begin{aligned} k_1 &> 0, \\ 2k_1^2 + 2\omega_0^2 + 2\omega_z^2 - \omega_y^2 + 4\omega_z u_z \frac{a}{r} &> 0, \\ (8\omega_z^2 + 2k_1^2)\left(2\omega_0^2 - \omega_y^2 - 2u_z^2 \frac{a^2}{r^2}\right) + \omega_y^4 &> 0, \\ (\omega_0^2 - \omega_z^2)(\omega_0^2 - \omega_z^2 - \omega_y^2) + k_1^2\left(\omega_z - u_z \frac{a}{r}\right)^2 &> 0. \end{aligned}\right\} \tag{5.107}$$

We have previously established that in the region

$$\omega_0^2 - \omega_y^2 - \omega_z^2 > 0 \tag{5.108}$$

a two-accelerometer system is asymptotically stable without Doppler correction. It is readily seen that for $k_1 > 0$ all the conditions (5.107) of asymptotic stability are observed in the region (5.108). For the first, third, and fourth conditions, this is obvious. From the second condition we have

$$2k_1^2 + 2\omega_0^2 + \omega_z^2 - \omega_y^2 - 2\frac{u_z^2 a^2}{r^2} + \left(\omega_z + 2\frac{u_z a}{r}\right)^2 > 0. \tag{5.109}$$

Since

$$\omega_0^2 > 2\frac{u_z^2 a^2}{r^2}, \tag{5.110}$$

inequality (5.109) is also satisfied in the region (5.108).

For equatorial motion, when $\omega_z = 0$, $u_z = 0$, inequalities (5.107) reduce to two inequalities

$$k_1 > 0, \quad \omega_0^2 - \omega_y^2 > 0. \tag{5.111}$$

In this case, (5.105) falls into two second-order equations. Their characteristic equations are

$$p^2 + k_1 p + (\omega_0^2 - \omega_y^2) = 0, \qquad p^2 + k_1 p + \omega_0^2 = 0, \tag{5.112}$$

whence again follow inequalities (5.111).

Note that conditions (5.107) of asymptotic stability are analogous to (5.11): the two sets of inequalities coincide for $u = 0$. As for (5.11), we can construct the stability region corresponding to (5.107) in the ω_y, ω_z plane. Since the Earth's spin u is small compared to ω_0, these regions will be close to the stability regions (5.11) shown in Figure 5.1.

Conditions (5.107) provide a solution to the problem of stability of inertial system for motion with constant velocity along a parallel. Exact stability analysis of Eqs. (5.105) in case of more general motion is not so easy, since we should investigate a system of fourth-order differential equations of quite a general form with variable coefficients, which are essentially arbitrary functions of time. No regular methods for the solution of this problem are available. We can, however, find sufficient conditions of asymptotic stability for the case of general motion provided the velocity of the object is much less than the circular velocity. This covers a fairly wide class of objects, such as sea vessels, aircraft, winged rockets.

To find the sufficient conditions of asymptotic stability for these cases, consider the projections of Eq. (5.105) onto azimuthally free axes with $\omega_z = 0$. Eqs. (5.105) thus take the form

$$\left.\begin{aligned} \delta\ddot{x} + k_1\delta\dot{x} + \left(\frac{\mu}{r^3} - \omega_y^2\right)\delta x + \left(\omega_x\omega_y + k_1 u_z\frac{a}{r}\right)\delta y = 0, \\ \delta\ddot{y} + k_1\delta\dot{y} + \left(\frac{\mu}{r^3} - \omega_x^2\right)\delta y + \left(\omega_x\omega_y - k_1 u_z\frac{a}{r}\right)\delta x = 0. \end{aligned}\right\} \tag{5.113}$$

Here r, u_z, ω_x, ω_y are some functions of time, determined by the law of motion of the object. In what follows we will take

$$\left.\begin{aligned} \Omega^2 = \omega_x^2 + \omega_y^2 \ll \omega_0^2, \quad r = r^0 + \rho(t), \\ r^0 = \text{const}, \quad \frac{|\rho|}{r^0} \ll 1. \end{aligned}\right\} \tag{5.114}$$

The first inequality in (5.114) indicates that the velocity of the object is much less than the circular velocity. The other conditions in (5.114) restrict the variation of r. They imply that the object moves near a sphere of constant radius r^0 concentric with the Earth.

If $\frac{|\rho|}{r^0} \ll 1$, Eqs. (5.113) can be written in a somewhat different form:

$$\left.\begin{aligned} \delta\ddot{x} + k_1\delta\dot{x} + \omega_0^2\delta x = \left(\omega_0^2\frac{3\rho}{r^0} + \omega_y^2\right)\delta x - \left(\omega_x\omega_y + k_1 u_z\frac{a}{r}\right)\delta y, \\ \delta\ddot{y} + k_1\delta\dot{y} + \omega_0^2\delta y = \left(\omega_0^2\frac{3\rho}{r^0} + \omega_x^2\right)\delta y - \left(\omega_x\omega_y - k_1 u_z\frac{a}{r}\right)\delta x. \end{aligned}\right\} \tag{5.114a}$$

where

$$\omega_0^2 = \frac{\mu}{r^{0\,3}}.$$

Substituting

$$\delta\dot{x} = x_1, \quad \delta x = x_2, \quad \delta\dot{y} = y_1, \quad \delta y = y_2, \tag{5.115}$$

we write Eqs. (5.114a) in the form

$$\left.\begin{aligned} \dot{x}_1 = -k_1 x_1 - \omega_0^2 x_2 + \varphi_1, \\ \dot{x}_2 = x_1, \\ \dot{y}_1 = -k_1 y_1 - \omega_0^2 y_2 + \varphi_2, \\ \dot{y}_2 = y_1, \end{aligned}\right\} \tag{5.116}$$

where, by (5.114a) and (5.115),

$$\left.\begin{aligned} \varphi_1 = \left(\omega_0^2\frac{3\rho}{r^0} + \omega_y^2\right)x_2 - \left(\omega_x\omega_y + k_1 u_z\frac{a}{r}\right)y_2, \\ \varphi_2 = \left(\omega_0^2\frac{3\rho}{r^0} + \omega_x^2\right)y_2 - \left(\omega_x\omega_y - k_1 u_z\frac{a}{r}\right)x_2. \end{aligned}\right\} \tag{5.117}$$

We now change to the normal coordinates b_1, u_1, b_2, u_2:

$$\left.\begin{aligned} x_1 = b_1(e\cos u_1 - \nu\sin u_1), \\ x_2 = b_1\cos u_1, \\ y_1 = b_2(e\cos u_2 - \nu\sin u_2), \\ y_2 = b_2\cos u_2, \end{aligned}\right\} \tag{5.117a}$$

where

$$\varepsilon = -\frac{k_1}{2}, \quad \nu = \sqrt{\omega_0^2 - \frac{k_1^2}{4}}.$$

Eqs. (5.116) in normal coordinates take the form

$$\left.\begin{aligned} \frac{db_1}{dt} &= \varepsilon b_1 - \frac{1}{\nu}\tilde{\varphi}_1 \sin u_1, \\ \frac{du_1}{dt} &= \nu - \frac{1}{b_1\nu}\tilde{\varphi}_1 \cos u_1, \\ \frac{db_2}{dt} &= \varepsilon b_2 - \frac{1}{\nu}\tilde{\varphi}_2 \sin u_2, \\ \frac{du_2}{dt} &= \nu - \frac{1}{b_2\nu}\tilde{\varphi}_2 \cos u_2. \end{aligned}\right\} \tag{5.118}$$

Here

$$\left.\begin{aligned} \tilde{\varphi}_1 &= \left(\omega_0^2 \frac{3\rho}{r^5} + \omega_y^2\right) b_1 \cos u_1 - \left(\omega_x\omega_y + k_1 u_z \frac{a}{r}\right) b_2 \cos u_2, \\ \tilde{\varphi}_2 &= \left(\omega_0^2 \frac{3\rho}{r^5} + \omega_x^2\right) b_2 \cos u_2 - \left(\omega_x\omega_y - k_1 u_z \frac{a}{r}\right) b_1 \cos u_1. \end{aligned}\right\} \tag{5.119}$$

We introduce the function

$$U = b_1^2 + b_2^2. \tag{5.120}$$

In virtue of Eqs. (5.118)

$$\frac{dU}{dt} = 2\left[\varepsilon U - \frac{1}{\nu}(\tilde{\varphi}_1 b_1 \sin u_1 + \tilde{\varphi}_2 b_2 \sin u_2)\right]. \tag{5.121}$$

Inserting $\tilde{\varphi}_1$, $\tilde{\varphi}_2$ from (5.119), we get

$$\begin{aligned} \frac{dU}{dt} = 2\Big\{ \varepsilon U - \frac{1}{\nu}\Big[& b_1^2\omega_y^2 \sin u_1 \cos u_1 + b_2^2\omega_x^2 \cos u_2 \sin u_2 - \\ & - b_1 b_2 \omega_x\omega_y \sin(u_1 + u_2) + k_1 u_z \frac{a}{r} b_1 b_2 \sin(u_2 - u_1)\Big] + \\ & + \omega_0^2 \frac{3\rho}{r^5}\left(b_1^2 \sin u_1 \cos u_1 + b_2^2 \sin u_2 \cos u_2\right)\Big\}. \end{aligned} \tag{5.122}$$

The projections ω_x, ω_y can be written in the form

$$\omega_x = \Omega \sin\psi_1, \quad \omega_y = \Omega \cos\psi_1, \tag{5.123}$$

where the function $\Omega(t)$ is defined by the first relation in (5.114), and ψ_1 is some angle, which is a function of time. Clearly

$$\sin\psi_1 = \frac{\omega_x}{\sqrt{\omega_x^2 + \omega_y^2}}, \quad \cos\psi_1 = \frac{\omega_y}{\sqrt{\omega_x^2 + \omega_y^2}}.$$

As in (5.123), we may write

$$b_1 = b \sin\psi_2, \quad b_2 = b \cos\psi_2, \quad b = \sqrt{b_1^2 + b_2^2}. \tag{5.124}$$

Introducing (5.123), (5.124) on the right in (5.122), we get

$$\left.\begin{aligned}\frac{dU}{dt} = 2U\Big\{\varepsilon - \frac{1}{\nu}\Big[\Omega^2(&\sin u_1\cos u_1\sin^2\psi_2\cos^2\psi_1 + \\ &+\sin u_2\cos u_2\cos^2\psi_2\sin^2\psi_1 - \\ &-\sin(u_1+u_2)\sin\psi_1\cos\psi_1\sin\psi_2\cos\psi_2) + \\ &+k_1 u_z\frac{a}{r}\sin(u_2-u_1)\sin\psi_2\cos\psi_2 + \\ &+\omega_0^2\frac{3\varrho}{r^3}(\sin u_1\cos u_1\sin^2\psi_1+\sin u_2\cos u_2\cos^2\psi_1)\Big]\Big\}.\end{aligned}\right\} \tag{5.125}$$

A sufficient condition of asymptotic stability is that the function in braces on the right in Eq. (5.125) remain always negative.

We have the obvious estimates

$$\left.\begin{aligned}&|\sin u_1\cos u_1\sin^2\psi_1+\sin u_2\cos u_2\cos^2\psi_1|\leqslant\frac{1}{2},\\ &|\sin u_1\cos u_1\sin^2\psi_2\cos^2\psi_1 + \\ &\quad+\sin u_2\cos u_2\cos^2\psi_2\sin^2\psi_1 - \\ &\quad-\sin(u_1+u_2)\sin\psi_1\cos\psi_1\sin\psi_2\cos\psi_2|\leqslant\frac{3}{4},\\ &|\sin(u_2-u_1)\sin\psi_2\cos\psi_2|\leqslant\frac{1}{2},\\ &\frac{a}{r}\leqslant 1,\quad |u_z|\leqslant u.\end{aligned}\right\} \tag{5.126}$$

We also write

$$\left.\begin{aligned}\max\Omega^2 &= \Omega_*^2,\\ \frac{3\max|\varrho|}{2r^3} &= c.\end{aligned}\right\} \tag{5.127}$$

Using (5.126), (5.127), we reach the following conclusion: the expression in braces in Eq. (5.125) is positive if

$$q_2 = -\frac{k_1}{2} + \frac{1}{\sqrt{\omega_0^2-\frac{k_1^2}{4}}}\left(\frac{3\Omega_*^2}{4}+\omega_0^2 c + u\frac{k_1}{2}\right) < 0. \tag{5.128}$$

Comparing inequality (5.128) with inequality (5.97) we conclude, by analogy with (5.98), that inequality (5.128) is satisfied whenever k_1 lies between the limits

$$2\left(\frac{3\Omega_*^2}{4\omega_0^2}+c\right)(\omega_0+u) < k_1 < 2\left[\omega_0+u\left(\frac{3\Omega_*^2}{4\omega_0^2}+c\right)\right]. \tag{5.129}$$

Here $\min|q_2|$ can be used as a measure of stability, since as in (5.101) we have the relation

$$b = \sqrt{b_1^2+b_2^2} < b^0 e^{-\min|q_2|t}. \tag{5.130}$$

We have so far analyzed the stability of a two-accelerometer system with the error equations obtained by taking the projections of the vector equation (5.65) on the axes x, y. The stability analysis of a two-accelerometer system for the vector equation (5.102) can be performed along the same lines.

5.3.4. Solution of the error equations. Effect of instrumental errors

We will now concentrate on the error equations (5.43), (5.48) of three-accelerometer systems with Doppler correction. We will find the solutions of these equations for the motion at constant distance from the Earth, when the corresponding homogeneous equations are reduced to equations with constant coefficients.

Projecting the vector equation (5.43) onto the axes ξ_*, η_*, ζ_* we obtain the set of scalar equations

$$\left.\begin{aligned} \delta\ddot{\xi}_* + k_1\,\delta\dot{\xi}_* + \omega_0^2\,\delta\xi_* + k_1\frac{a}{r}u\,\delta\eta_* &= f_{\xi_*},\\ \delta\ddot{\eta}_* + k_1\,\delta\dot{\eta}_* + \omega_0^2\,\delta\eta_* - k_1\frac{a}{r}u\,\delta\xi_* &= f_{\eta_*},\\ \delta\ddot{\zeta}_* + k_1\,\delta\dot{\zeta}_* + \omega_0^2\,\delta\zeta_* &= f_{\zeta_*}. \end{aligned}\right\} \tag{5.131}$$

where

$$\left.\begin{aligned} f_{\xi_*} &= \Delta n_{\xi_*} - 2(\Delta m_{\eta_*}\dot{\zeta}_* - \Delta m_{\zeta_*}\dot{\eta}_*) - \Delta\dot{m}_{\eta_*}\zeta_* + \\ &+ \Delta\dot{m}_{\zeta_*}\eta_* + 3\frac{\mu\Delta r}{r^4}\xi_* + k_1\Big[\Delta V_{D\xi_*} - \Delta m_{\eta_*}\zeta_* + \\ &+ \Delta m_{\zeta_*}\eta_* + \frac{au}{r}\Big(\frac{\Delta r\eta_*}{r} + \theta_{\xi_*}\zeta_*\Big)\Big],\\ f_{\eta_*} &= \Delta n_{\eta_*} - 2(\Delta m_{\zeta_*}\dot{\xi}_* - \Delta m_{\xi_*}\dot{\zeta}_*) - \Delta\dot{m}_{\zeta_*}\xi_* + \\ &+ \Delta\dot{m}_{\xi_*}\zeta_* + 3\frac{\mu\Delta r}{r^4}\eta_* + k_1\Big[\Delta V_{D\eta_*} - \Delta m_{\zeta_*}\xi_* + \\ &+ \Delta m_{\xi_*}\zeta_* + \frac{au}{r}\Big(-\frac{\Delta r\xi_*}{r} + \theta_{\eta_*}\zeta_*\Big)\Big],\\ f_{\zeta_*} &= \Delta n_{\zeta_*} - 2(\Delta m_{\xi_*}\dot{\eta}_* - \Delta m_{\eta_*}\dot{\xi}_*) - \Delta\dot{m}_{\xi_*}\eta_* + \\ &+ \Delta\dot{m}_{\eta_*}\xi_* + 3\frac{\mu\Delta r}{r^4}\zeta_* + k_1\Big[\Delta V_{D\zeta_*} - \Delta m_{\xi_*}\eta_* + \\ &+ \Delta m_{\eta_*}\xi_* - \frac{au}{r}(\theta_{\xi_*}\xi_* + \theta_{\eta_*}\eta_*)\Big]. \end{aligned}\right\} \tag{5.132}$$

The solution of the third equation in (5.131) is obvious:

$$\left.\begin{aligned} \delta\zeta_* &= e^{-\frac{k_1 t}{2}}\Big[\delta\zeta_*^0\cos\nu t + \frac{1}{\nu}\Big(\delta\dot{\zeta}_*^0 + \frac{k_1}{2}\delta\zeta_*^0\Big)\sin\nu t\Big] + \\ &+ \frac{1}{\nu}\int_0^t f_{\zeta_*}(\tau)\,e^{-\frac{k_1}{2}(t-\tau)}\sin\nu(t-\tau)\,d\tau,\\ \nu &= \sqrt{\omega_0^2 - \frac{k_1^2}{4}}. \end{aligned}\right\} \tag{5.133}$$

The first two equations in (5.131) constitute a coupled system. To solve this system, we require the roots of the characteristic equation (5.68). Since u^2 is small compared to ω_0^2, sufficiently simple and exact expressions can be obtained for these roots.

As we know, the roots of the fourth-degree algebraic equation

$$p^4 + a_1p^3 + a_2p^2 + a_3p + a_4 = 0 \tag{5.134}$$

coincide with the roots of two quadratic equations

$$x^2+(a_1+A_{1,2})\frac{x}{2}+\left(y+\frac{a_1y-a_3}{A_{1,2}}\right)=0, \tag{5.135}$$

where

$$A_{1,2}=\pm\sqrt{8y+a_1^2-4a_2}, \tag{5.136}$$

and y is some real root of the cubic equation

$$8y^3-4a_2y^2+(2a_1a_3-8a_4)\,y+a_4(4a_2-a_1^2)-a_3^2=0. \tag{5.137}$$

In our case this auxiliary cubic equation has the form

$$f(y)=2y^3-(k_1^2+2\omega_0^2)\,y^2+\left[2\omega_0^2(k_1^2-\omega_0^2)-2k_1^2\frac{a^2u^2}{r^2}\right]y+$$
$$+\omega_0^4(2\omega_0^2-k_1^2)+2\omega_0^2k_1^2\frac{a^2u^2}{r^2}=0. \tag{5.138}$$

For $u=0$ the roots of this equation are

$$y_1=\frac{k_1^2}{2}-\omega_0^2,\quad y_{2,3}=\omega_0^2. \tag{5.139}$$

Taking the root y_1 as the first approximation, we find by Newton's technique the improved root of Eq. (5.138)

$$y_1'=\frac{k_1^2}{2}-\omega_0^2-\frac{2k_1^2\frac{a^2u^2}{r^2}}{4\omega_0^2-k_1^2}. \tag{5.140}$$

Now to terms of the order u^2 we have

$$A_{1,2}=\pm 2j\sqrt{4\omega_0^2-k_1^2}\left[1+\frac{2k_1^2\frac{a^2u^2}{r^2}}{(4\omega_0^2-k_1^2)^2}\right]. \tag{5.141}$$

For the coefficients of the quadratic equations (5.135) we obtain

$$\left.\begin{aligned}&\frac{a_1+A_{1,2}}{2}=k_1\pm j\sqrt{4\omega_0^2-k_1^2}\left[1+\frac{2k_1^2\frac{a^2u^2}{r^2}}{(4\omega_0^2-k_1^2)^2}\right],\\&y_1'+\frac{a_1y_1'-a_3}{A_{1,2}}=\frac{k_1^2}{2}-\omega_0^2-\\&-\frac{2k_1^2\frac{a^2u^2}{r^2}}{4\omega_0^2-k_1^2}\pm j\frac{k_1}{2}\sqrt{4\omega_0^2-k_1^2}\left[1+\frac{2k_1^2\frac{a^2u^2}{r^2}}{(4\omega_0^2-k_1^2)^2}\right].\end{aligned}\right\} \tag{5.142}$$

Solving the quadratic equations (5.135), we obtain to terms of the order u^2 inclusive, for the roots of the characteristic equation (5.68):

$$\left.\begin{aligned}p_{1,2}&=-\frac{k_1}{2}\left(1\pm\frac{2\frac{ua}{r}}{\sqrt{4\omega_0^2-k_1^2}}\right)-\frac{j}{2}\sqrt{4\omega_0^2-k_1^2}\left[1+\frac{2k_1^2\frac{a^2u^2}{r^2}}{(4\omega_0^2-k_1^2)^2}\right],\\p_{3,4}&=-\frac{k_1}{2}\left(1\pm\frac{2\frac{ua}{r}}{\sqrt{4\omega_0^2-k_1^2}}\right)+\frac{j}{2}\sqrt{4\omega_0^2-k_1^2}\left[1+\frac{2k_1^2\frac{a^2u^2}{r^2}}{(4\omega_0^2-k_1^2)^2}\right],\end{aligned}\right\} \tag{5.143}$$

or to first approximation

$$p_{1,2,3,4} = -\frac{k_1}{2}\left(1 \pm \frac{2\frac{au}{r}}{\sqrt{4\omega_0^2 - k_1^2}}\right) \pm \frac{j}{2}\sqrt{4\omega_0^2 - k_1^2}. \tag{5.144}$$

Note that this method can also be applied to find the roots of the characteristic equation (5.71). To first approximation we have

$$p_{1,2,3,4} = -\frac{k_1}{2} \pm \frac{j}{2}\left(\sqrt{4\omega_0^2 - k_1^2} \pm 2u\right). \tag{5.145}$$

Having found the roots of the characteristic equation (5.68), we proceed in the usual way with the construction of the general solutions of the first two equations in (5.131).

The form of the roots (5.144), (5.145) of the characteristic equations (5.68), (5.71) suggests that the fourth-order system of differential equations comprising the first two equations in (5.131) can be replaced by two second-order equations

$$\left.\begin{aligned} \delta\ddot{\xi}_* + k_1\delta\dot{\xi}_* + \omega_0^2\delta\xi_* &= f_{\xi_*}, \\ \delta\ddot{\eta}_* + k_1\delta\dot{\eta}_* + \omega_0^2\delta\eta_* &= f_{\eta_*}. \end{aligned}\right\} \tag{5.146}$$

This is so because u being small the coupling between the first and the second differential equations in (5.131) is weak and in a damped system the effects associated with this coupling do not develop fast enough to become noticeable during the transient phase.

The solution of Eqs. (5.146) is analogous to the solution (5.133) of the last equation in (5.131). These three solutions can be combined into a single vector solution

$$\left.\begin{aligned} \delta r = e^{-\frac{k_1 t}{2}}\Big[\delta r^0\cos\nu t + \frac{1}{\nu}\Big(\frac{d\,\delta r^0}{dt} - \frac{k_1}{2}\delta r^0\Big)\sin\nu t + \\ + \frac{1}{\nu}\int_0^t f_1(\tau)\, e^{-\frac{k_1}{2}(t-\tau)}\sin\nu(t-\tau)\,d\tau \\ \left(\nu = \sqrt{\omega_0^2 - \frac{k_1^2}{4}}\right). \end{aligned}\right\} \tag{5.147}$$

which corresponds to the vector equation (5.43) with $u = 0$ on the left, i.e., to the equation

$$\frac{d^2\,\delta r}{dt^2} + k_1\frac{d\,\delta r}{dt} + \omega_0^2\delta r = f_1, \tag{5.148}$$

where the vector function f_1 is the right-hand side of Eq. (5.43). The projections of this function on the axes ξ_*, η_*, ζ_* are given by Eqs. (5.132). The integration in the right-hand side of Eq. (5.147) is in the axes ξ_*, η_*, ζ_*.

Let us consider in more detail the right-hand side of Eq. (5.43). It contains an additional term which did not appear in undamped inertial systems, namely

$$k_1\Big[\Delta V_D - u \times r\,\frac{a\,\Delta r}{r^2} - (\theta \times u) \times r\,\frac{a}{r} - \Delta m \times r\Big]. \tag{5.149}$$

Besides the Doppler errors, this expression contains a vector θ, which is the solution of the second group of error equations. However, since a is small compared to ω_0 and k_1, this does not lead to any additional errors in position determination. To first approximation, we need retain only the first and the last terms in (5.149). In this case, the function f_1 in (5.147) is taken in the form

$$f_1 = \Delta n - \Delta m \times \frac{dr}{dt} + \frac{d}{dt}(r \times \Delta m) + 3\frac{\omega r \Delta r}{r^3} + k_1(\Delta V_D - \Delta m \times r). \tag{5.150}$$

Note that the term $-k_1 \Delta m \times r$ appears on the right in error equations of damped systems even if $\Delta V_D = 0$ (an ideal Doppler velocity meter).

To obtain a quantitative estimate of the effect of instrumental errors on position errors in a damped inertial system, let us consider the case when the function f_1 is a constant vector in the axes ξ_1, η_1, ζ_1. Let $f_1 = a = \overline{\text{const}}$. Solution (5.147) then takes the form

$$\delta r = e^{-\frac{k_1 t}{2}}\left[\delta r^0 \cos \nu t + \frac{1}{\nu}\left(\frac{d\delta r^0}{dt} - \frac{k_1}{2}\delta r^0\right)\sin \nu t\right] + \\ + \frac{a}{\omega_0^2}\left[1 - \frac{1}{\nu}e^{-\frac{k_1 t}{2}}\left(-\frac{k_1}{2}\sin \nu t + \nu \cos \nu t\right)\right]. \tag{5.151}$$

The steady-state error is

$$\delta r = \frac{a}{\omega_0^2}. \tag{5.152}$$

If the system is error-free, except for the Doppler error ΔV_D, we have $a = k_1 \Delta V_D$ and

$$\delta r = \frac{k_1}{\omega_0^2}\Delta V_D. \tag{5.153}$$

If the ratio k_1/ω_0 is of the order of unity, we have $\delta r \approx \Delta V_D/\omega_0$. Let, for example, $\Delta V_D = 0.001V$. Then $\delta r \approx V/1.25$. For velocities of the order of 1000 m/sec we then have $|\delta r| \approx 0.8$ km. The term $-k_1 \Delta m \times r$ on the right in (5.150) has a similar effect.

As regards the other instrumental errors, i.e., the first four terms on the right in (5.150), note that they enter the error equations of a damped system in the same form as they appear in the error equations of a system without Doppler correction. However, because of their contribution, the steady-state error δr in a damped system is half the maximum value of this error in an undamped system. Indeed, let the only instrumental error be the constant accelerometer error $\Delta n = \overline{\text{const}}$. Then from (5.151) we get the steady-state value

$$|\delta r| = \frac{|\Delta n|}{\omega_0^2}. \tag{5.154}$$

In an undamped system, on the other hand

$$\delta r = \frac{\Delta n}{\omega_0^2}(1 - \cos \omega_0 t) \tag{5.155}$$

and therefore

$$\max|\delta r| = 2\frac{|\Delta n|}{\omega_0^2}. \tag{5.156}$$

We have so far derived the solution of Eq. (5.43). The solution of Eq. (5.48) for a three-accelerometer system with Doppler correction in accordance with Eqs. (5.40) can be similarly obtained. It can be further shown that for $r = \text{const}$ Eq. (5.48) can be replaced to first approximation by

$$\frac{d^2\delta r}{dt^2} + k_1'\omega_0^2 \frac{d\delta r}{dt} + \omega_0^2 \delta r = f_2, \tag{5.157}$$

where f_2 is the right-hand side of Eqs. (5.48) for $u = 0$:

$$f_2 = \Delta n - \Delta m \times \frac{dr}{dt} - \frac{d}{dt}(\Delta m \times r) + \\ + 3\frac{\mu r \Delta r}{r^4} + k_1'\left[\frac{d}{dt}\left(\Delta n - \Delta m \times \frac{dr}{dt} + 3\frac{\mu r \Delta r}{r^4}\right) - \frac{d^2}{dt^2}\Delta V_D\right]. \tag{5.158}$$

For $k_1'\omega_0^2 = k_1$ the left-hand side of Eq. (5.157) coincides with the left-hand side of Eq. (5.148). The solution of Eq. (5.157) is given by (5.147) with f_2 substituted for f_1 and $k_1'\omega_0^2$ for k_1.

Let us now turn to error equations of two-accelerometer damped systems. First we will consider Eqs. (5.44). The analysis will be confined to the case of small ω_x, ω_y, i.e., the case of slow motion with velocities less than the circular velocity at a constant distance from the Earth's center (allowing small variation of this distance).

We have shown in connection with the stability of the homogeneous equations (5.44) that their projections on the azimuthally free axes have the form (5.114a), and the right-hand sides of these equations do not have a pronounced effect on the solution. The coupling between Eqs. (5.114a) (associated with terms on the right) is weak for small ω_x, ω_y, ρ. The effects produced by this coupling cannot develop to a considerable extent during the relatively short transient time in effectively damped systems. Therefore the terms on the right in Eqs. (5.114a) can be dropped in projections onto azimuthally free axes. This is analogous to neglecting the last terms on the left in the first two equations in (5.131) (as we did in constructing the solution (5.147) of Eq. (5.43)).

It follows from the preceding that in projection onto azimuthally free axes Eqs. (5.44) can be replaced to first approximation by the equations

$$\left.\begin{aligned} \delta\ddot{x} + k_1\delta\dot{x} + \omega_0^2\delta x = f_x, \\ \delta\ddot{y} + k_1\delta\dot{y} + \omega_0^2\delta y = f_y. \end{aligned}\right\} \tag{5.159}$$

The functions f_x and f_y on the right are the right-hand sides of Eqs. (5.44) with $\omega_z = 0$ (the case of azimuthally free axes). To first approximation we can drop all the small terms containing ω_y, ω_x, u_x, u_y, u_z in the right-hand sides of Eqs. (5.44), retaining terms with $\dot{\omega}_x$, $\dot{\omega}_y$. We then have

$$\left.\begin{aligned} f_x = \Delta n_x - \Delta\dot{m}_y r - \dot{\omega}_y \Delta r + k_1(\Delta V_{Dx} - r\,\Delta m_y), \\ f_y = \Delta n_y + \Delta\dot{m}_x r + \dot{\omega}_x \Delta r + k_1(\Delta V_{Dy} + r\,\Delta m_x). \end{aligned}\right\} \tag{5.160}$$

The solution of Eqs. (5.159) can be obtained from (5.147) putting δx, δy for the vector δr and f_x, f_y for f_1.

Eqs. (5.44) are the error equations of two-accelerometer systems operating in accordance with (5.39). For two-accelerometer systems operating in accordance with (5.40), the error equations are obtained from the vector

equation (5.48) by forming its projections onto the axes x, y of the co-moving system with the axis z pointing along the vector r. In the process, as we have seen at the end of 5.3.1, δz is replaced by Δr and the projections of the vector $\frac{d}{dt}\left(3\frac{\mu r\,\Delta r}{r^5}\right)$ are omitted.

For slow motion at constant distance from the Earth's center we obtain to first approximation the following equations in projection on the azimuthally free axes (making the same assumptions as in the derivation of Eqs. (5.159)):

$$\left.\begin{aligned}\delta\ddot{x}+k_1'\omega_0^2\delta x-\omega_0^2\delta x&=f_x',\\ \delta\ddot{y}+k_1'\omega_0^2\delta y+\omega_0^2\delta y&=f_y',\end{aligned}\right\} \tag{5.161}$$

where

$$\left.\begin{aligned}f_x'&=\Delta n_x-\Delta\dot{m}_y r-\dot{\omega}_y\Delta r+k_1'(\Delta\dot{n}_x-r\dot{\omega}_x\Delta m_z-\Delta\ddot{V}_{Dx}),\\ f_y'&=\Delta n_y+\Delta\dot{m}_x r+\dot{\omega}_x\Delta r+k_1'(\Delta\dot{n}_y-r\dot{\omega}_y\Delta m_z-\Delta\ddot{V}_{Dy}).\end{aligned}\right\} \tag{5.162}$$

For $k_1'\omega_0^2=k_1$, Eqs. (5.161) and (5.159) differ from one another only in the terms containing the factors k_1 and k_1' on the right. The most significant difference is that only the second derivatives $\Delta\ddot{V}_{Dx}$, $\Delta\ddot{V}_{Dy}$ of the Doppler errors enter the functions f_x', f_y', whereas the functions f_x, f_y depend directly on ΔV_{Dx}, ΔV_{Dy}.

§5.4. CHANGING THE NATURAL FREQUENCIES OF A DOPPLER-AIDED SYSTEM

5.4.1. Ideal operation equations. Error equations

We have established in §5.3 that the first group of error equations of a damped inertial system is reduced in the simplest case to second-order differential equations with constant coefficients. The characteristic equation is

$$p^2+k_1p+\omega_0^2=0. \tag{5.163}$$

Its roots are

$$p_{1,2}=-\frac{k_1}{2}\pm\frac{1}{2}\sqrt{k_1^2-4\omega_0^2}. \tag{5.164}$$

If $4\omega_0^2>k_1^2$, the roots of the characteristic equation are

$$p_{1,2}=-\mu\pm j\nu,\quad \mu=\frac{k_1}{2},\quad \nu=\frac{1}{2}\sqrt{4\omega_0^2-k_1^2}. \tag{5.165}$$

The maximum value of μ is evidently attained for $k_1=2\omega_0$, being

$$\mu_0=\omega_0. \tag{5.166}$$

As k_1 is further increased, the roots (5.164) become real and negative (note that $k_1 > 0$), namely $p_1 = -\mu_1$, $p_2 = -\mu_2$, where $\mu_1 < \mu_0$, $\mu_2 > \mu_0$. Therefore for $\mu = \mu_0 = \omega_0$ we achieve the maximum attainable damping in the system.

Since $\mu_0 = \omega_0 \approx 1.25 \cdot 10^{-3}$ is small, this damping may prove to be insufficient, especially during short runs and for large setting errors. Indeed, $\omega_0 \approx 1.25 \cdot 10^{-3}$, and the damping function $e^{-\omega_0 t}$ therefore decreases slowly. It drops to $1/e \approx 1/3$ for $t \approx 800$ sec. In real systems we generally have $k_1 < 2\omega_0$, and the damping of the natural oscillations is even less pronounced. This difficulty can be avoided by artificially increasing the natural frequency ω_0 of the inertial system.

In an autonomous inertial system, any increase in the natural frequency will lead to ballistic errors. This is understandable if we recall the analogy between inertial systems using accelerometers and gyroscopes to measure the object position and the Schuler pendulum. After all, it was the particular choice of the pendulum frequency $\omega_0 = \sqrt{g/r}$ that ensured unperturbable relative equilibrium for arbitrary acceleration of the pendulum pivot.

However, external information from a Doppler velocity meter can be applied to increase the natural frequency of the inertial system without the risk of ballistic errors. The situation here is analogous to the elimination of velocity errors in damped systems with Doppler guidance.

The natural frequency of an inertial system can be altered in two ways. One is to replace the first group of ideal operation equations (1.36) by the equations

$$\left.\begin{aligned} v &= \int_0^t (n - m \times v + g)\, dt + v^0, \\ r &= \int_0^t (v - m \times r + k_2 \Delta V)\, dt + r^0 \\ &\qquad (\Delta V = V - V_D). \end{aligned}\right\} \tag{5.167}$$

An alternative solution is to replace Eqs. (1.36) by the following set of equations:

$$\left.\begin{aligned} v &= \int_0^t [n - m \times v + g + k_2'(\Delta \dot{V} + m \times \Delta V)]\, dt + v^0, \\ r &= \int_0^t (v - m \times r)\, dt + r^0 \\ &\qquad (\Delta V = V - V_D). \end{aligned}\right\} \tag{5.168}$$

In Eqs. (5.167), (5.168), ΔV is the difference in the relative velocities as measured by the inertial system and the Doppler, respectively, and the coefficients k_2, k_2 are nondimensional constants. For $k_2 = 0$, $k_2 = 0$, Eqs. (5.167), (5.168) reduce to Eqs. (1.36).

To verify that the alteration of the ideal operation equations in accordance with (5.167), (5.168) does indeed cause a change in the natural frequency of the inertial system, we should write the equations of perturbed motion.

We start with the case (5.167). From these equations and (5.38) we get

$$\left.\begin{aligned} \delta\dot{v} &= \Delta n - \Delta m \times v - \omega \times \delta v + \delta g, \\ \delta\dot{r} &= \delta v - \Delta m \times r - \omega \times \delta r + k_2 \delta \Delta V, \\ \delta\Delta V &= \delta v + \left(\theta \times u \frac{a}{r}\right) \times r - u \times \left(\delta r \frac{a}{r} - r \frac{\Delta r a}{r^2}\right) - \Delta V_D. \end{aligned}\right\} \tag{5.169}$$

From the second and the third equation in (5.169) we get

$$\delta v = \frac{1}{1+k_2}\left\{\delta\dot{r} + \omega \times \delta r + k_2\left(u \times \delta r \frac{a}{r}\right) + \right.$$
$$\left. + \Delta m \times r + k_2\left[-\left(\theta \times u \frac{a}{r}\right)\times r - u \times r \frac{a\,\Delta r}{r^2} + \Delta V_D\right]\right\}. \tag{5.170}$$

Inserting this result in the first equation in (5.169) together with expression (5.42) for δg, we obtain the vector equation

$$\frac{d^2\delta r}{dt^2} + k_2 \frac{d}{dt}\left(u \times \delta r \frac{a}{r}\right) + (1+k_2)\frac{\mu}{r^3}\delta r =$$
$$= (1+k_2)\left(\Delta n - \Delta m \times \frac{dr}{dt} + 3\frac{\mu r\,\Delta r}{r^4}\right) - \frac{d}{dt}(\Delta m \times r) +$$
$$+ k_2 \frac{d}{dt}\left[(\theta \times u)\times r \frac{a}{r} + u \times r \frac{a\,\Delta r}{r^2} - \Delta V_D\right]. \tag{5.171}$$

This is the sought error equation of a system operating according to Eqs. (5.167). For $k_2 = 0$, Eq. (5.171) reduces to Eq. (2.15a), i.e., the error equation of a system operating according to (1.36). Note that alteration (5.167) refers only to the first group of ideal operation equations, used to determine the coordinates x, y, z in the moving axes O_1xyz. The second group of ideal operation equations which is concerned with the direction cosines β_{ij} between the axes ξ, η, ζ and x, y, z and with the coordinates ξ, η, ζ remains unchanged. The second group of error equations therefore does not change either.

We have so far derived the error equations of a system operating according to Eqs. (5.167). Let us now consider the second case, that of Eqs. (5.168). Varying these equations and using (5.38), we get

$$\left.\begin{aligned} \delta\dot{v} &= \Delta n - \Delta m \times v - \omega \times \delta v + \delta g + k_2'(\delta\,\Delta\dot{V} + \omega \times \delta\,\Delta V), \\ \delta\dot{r} &= \delta v - \Delta m \times r - \omega \times \delta r, \\ \delta\,\Delta V &= \delta v + (\theta \times u)\times r \frac{a}{r} - u \times \left(\delta r \frac{a}{r} - r\frac{\Delta r a}{r^2}\right) - \Delta V_D. \end{aligned}\right\} \tag{5.172}$$

From the second and the third equation in (5.172) we get

$$(1-k_2')\frac{d\,\delta v}{dt} - \delta g + k_2' \frac{d}{dt}\left(u \times \delta r \frac{a}{r}\right) = \Delta n - \Delta m \times \frac{dr}{dt} -$$
$$- k_2' \frac{d}{dt}\left[-(\theta \times u)\times r \frac{a}{r} - u \times r \frac{a\,\Delta r}{r^2} + \Delta V_D\right]. \tag{5.173}$$

Inserting δv from the second equation in (5.172), δg from (5.42), and dividing through by $1-k_2'$, we obtain the final form of the error equation for alternative (5.168):

$$\frac{d^2\delta r}{dt^2} + \frac{k_2'}{1-k_2'}\frac{d}{dt}\left(u \times \delta r \frac{a}{r}\right) + \frac{\mu}{r^3(1-k_2')}\delta r =$$
$$= \frac{1}{1-k_2'}\left(\Delta n - \Delta m \times \frac{dr}{dt} + 3\frac{\mu r\,\Delta r}{r^4}\right) - \frac{d}{dt}(\Delta m \times r) +$$
$$+ \frac{k_2'}{1-k_2'}\frac{d}{dt}\left[(\theta \times u)\times r \frac{a}{r} + u \times r \frac{a\,\Delta r}{r^2} - \Delta V_D\right]. \tag{5.174}$$

5.4.2. Analysis of error equations

Let us compare Eqs. (5.171) and (5.174). They have identical structure and differ only in coefficients containing k_2 and k_2'.

If these coefficients are so chosen that

$$\frac{1}{1-k_2'} = 1 + k_2, \tag{5.175}$$

we also have

$$\frac{k_2'}{1-k_2'} = k_2. \tag{5.176}$$

Note that substitution (5.175) reduces Eq. (5.174) to Eq. (5.171). Thus, changed forms (5.167) and (5.168) of the ideal operation equations are dynamically equivalent. They lead to the same error equation. In what follows we therefore need consider only one of the two equations (5.171), (5.174), Eq. (5.171) say.

First let us analyze the homogeneous equation (5.171):

$$\frac{d^2\delta r}{dt^2} + k_2 \frac{d}{dt}\left(u \times \delta r \frac{a}{r}\right) + (1+k_2)\frac{\mu}{r^3}\delta r = 0. \tag{5.177}$$

If the Earth did not spin, this equation would take the form ($u = 0$)

$$\frac{d^2\delta r}{dt^2} + (1+k_2)\frac{\mu}{r^3}\delta r = 0. \tag{5.178}$$

For $r = \text{const}$, putting as is usual $\mu/r^3 = \omega_0^2$, we obtain the solution of Eq. (5.178) in the form

$$\delta r = \delta r^0 \cos\omega_0\sqrt{1+k_2}\,t + \\ + \frac{1}{\omega_0\sqrt{1+k_2}}\frac{d\delta r^0}{dt}\sin\omega_0\sqrt{1+k_2}\,t. \tag{5.179}$$

The real part of the frequency thus changed: instead of ω_0 we now have $\omega_0\sqrt{1+k_2}$.

The assumption $u = 0$ signifies that the relative velocity measured by the Doppler is equal to the absolute velocity, i.e., the velocity in the axes $O_1\xi_0\eta_0\zeta_0$ which do not rotate with the Earth. In fact $u \neq 0$ and the Doppler only measures the relative velocity. This explains the origin of the term $k_2 \frac{d}{dt}\left(u \times \delta r \frac{a}{r}\right)$ in Eq. (5.177).

This relative velocity term is responsible for the appearance of gyroscopic forces in the error equation, which do not affect the (nonasymptotic) stability. Indeed, dot-multiplying Eq. (5.177) by $d\delta r/dt$, remembering that the vector u is invariable and $r = \text{const}$, so that

$$\frac{d\delta r}{dt} \cdot \frac{d}{dt}\left(u \times \delta r \frac{a}{r}\right) = \frac{a}{r}\frac{d\delta r}{dt} \cdot \left(u \times \frac{d\delta r}{dt}\right), \tag{5.180}$$

we get

$$\frac{d^2\delta r}{dt^2}\ \frac{d\delta r}{dt} + \omega_0^2(1+k_2)\delta r \cdot \frac{d\delta r}{dt} = 0, \tag{5.181}$$

which can be integrated to give

$$\left(\frac{d\,\delta r}{dt}\right)^2 + \omega_0^2(1+k_2)\,(\delta r)^2 = \text{const.} \tag{5.182}$$

The nonasymptotic stability of the system immediately follows from this equation.

In projection on the axes ξ_*, η_*, ζ_* the vector equation (5.177) gives three scalar equations

$$\left.\begin{aligned} \delta\ddot{\xi}_* + (1+k_2)\,\omega_0^2\,\delta\xi_* - k_2 u\,\frac{a}{r}\,\delta\dot{\eta}_* = 0, \\ \delta\ddot{\eta}_* + (1+k_2)\,\omega_0^2\,\delta\eta_* + k_2 u\,\frac{a}{r}\,\delta\dot{\xi}_* = 0, \\ \delta\ddot{\zeta}_* + (1+k_2)\,\omega_0^2\,\delta\zeta_* = 0. \end{aligned}\right\} \tag{5.183}$$

The solution of the third equation in (5.183) is a harmonic oscillation of frequency $\nu = \omega_0\sqrt{1+k_2}$. The characteristic equation of the system comprising the first two equations in (5.183) is

$$p^4 + p^2\left[2\omega_0^2(1+k_2) + k_2^2\,\frac{a^2u^2}{r^2}\right] + \omega_0^4(1+k_2)^2 = 0. \tag{5.184}$$

The roots of this biquadratic equation are

$$p_{1,2,3,4} = \pm\sqrt{-\omega_0^2(1+k_2) - k_2^2\frac{a^2u^2}{2r^2} \pm k_2\frac{au}{r}\sqrt{\omega_0^2(1+k_2) + k_2^2\frac{a^2u^2}{4r^2}}}. \tag{5.185}$$

Since $\omega_0^2 \gg \frac{a^2u^2}{r^2}$, we retain only the terms linear in au/r and obtain the approximate roots

$$p_{1,2,3,4} = \pm j\left(\omega_0\sqrt{1+k_2} \mp \frac{k_2}{2}\,\frac{au}{r}\right).$$

Thus, the solution of the first two equations in (5.183) is a superposition of two harmonic modes with frequencies $\omega_0\sqrt{1+k_2}$ and $\frac{k_2}{2}\frac{au}{r}$.

Having found the roots of the characteristic equation (5.184), we can readily construct the general solution of the homogeneous and the inhomogeneous system (5.183).

If k_2 is relatively small (e.g., $k_2 \approx 10$), $\frac{k_2}{2}\frac{au}{r}$ will be small compared to $\omega_0\sqrt{1+k_2}$ (since u is small compared to ω_0). The roots of the characteristic equation (5.184) are not markedly different from $\pm j\omega_0\sqrt{1+k_2}$. The coupling between the first and the second equation in (5.183) is weak, and over small periods of time we can consider the separated equations

$$\delta\ddot{\xi}_* + (1+k_2)\,\omega_0^2\,\delta\xi_* = 0, \quad \delta\ddot{\eta}_* + (1+k_2)\,\omega_0^2\,\delta\eta_* = 0,$$

which are obtained from (5.183) taking $u = 0$.

The vector equation (5.171) in this case is replaced by

$$\frac{d^2\,\delta r}{dt^2} + (1+k_2)\,\omega_0^2\,\delta r = f, \tag{5.185a}$$

where f stands for the right-hand side of Eq. (5.171). Since u is small, we may take

$$f=(1+k_2)\left(\Delta n-\Delta m\times\frac{dr}{dt}\right)-\frac{d}{dt}(\Delta m\times r)-k_2\frac{d\,\Delta V_D}{dt}+3\frac{\mu r\,\Delta r}{r^4}. \tag{5.186}$$

The general solution of (5.185a) is obtained in the form

$$\delta r=\delta r^0\cos\nu t+\frac{1}{\nu}\frac{d\,\delta r^0}{dt}\sin\nu t+\frac{1}{\nu}\int_0^t f(\tau)\sin\nu(t-\tau)\,d\tau, \tag{5.187}$$

where

$$\nu=\omega_0\sqrt{1+k_2}. \tag{5.188}$$

Eq. (5.171) is the error equation of a three-accelerometer system operating according to Eqs. (5.167). The error equation of the corresponding two-accelerometer system, without the z accelerometer, which in the unperturbed state is directed along the radius-vector r, is obtained from the vector equation (5.171) by taking its projections onto the co-moving axes x and y and putting

$$\left.\begin{aligned}&\delta z=\Delta r\,(=\Delta h),\\&\Delta V_{Dz}=\Delta\dot r-(V_D\times\theta)\cdot z=\\&\quad=\Delta\dot r-\left(\omega_y-\frac{\rho}{r}u_y\right)\delta x+\left(\omega_x-\frac{\rho}{r}u_x\right)\delta y,\end{aligned}\right\} \tag{5.188a}$$

where we may write $\rho/r=a/r$. Since the vector u is constant in the axes $O_1\xi_*\eta_*\zeta_*$, in which the derivatives in (5.171) are taken, we have for $r=\text{const}$,

$$\frac{d}{dt}\left(u\times\delta r\frac{a}{r}\right)=\frac{a}{r}u\times\frac{d\,\delta r}{dt}. \tag{5.189}$$

The error equations for a two accelerometer system are thus obtained in the form

$$\left.\begin{aligned}&\delta\ddot x+\left[(1+k_2)(\omega_0^2-\omega_y^2)-\omega_z^2-k_2\frac{a}{r}u_z\omega_z\right]\delta x+\\&+\left[(1+k_2)\,\omega_x\omega_y-\dot\omega_z+k_2\frac{a}{r}(u_y\omega_x-u_x\omega_y)\right]\delta y-\\&\qquad-\left(2\omega_z+k_2\frac{a}{r}u_z\right)\delta\dot y=f_x,\\&\delta\ddot y+\left[(1+k_2)(\omega_0^2-\omega_x^2)-\omega_z^2-k_2\frac{a}{r}u_z\omega_z\right]\delta y+\\&+\left[(1+k_2)\,\omega_y\omega_x+\dot\omega_z+k_2\frac{a}{r}(u_x\omega_y-u_y\omega_x)\right]\delta x+\\&\qquad+\left(2\omega_z+k_2\frac{a}{r}u_z\right)\delta\dot x=f_y,\end{aligned}\right\} \tag{5.190}$$

where f_x and f_y are expressed by

$$\left.\begin{aligned}f_x=&-(\omega_x\omega_z+\dot\omega_y)\,\Delta r-(2+k_2)\,\omega_y\Delta\dot r+\\&+(1+k_2)\,\Delta n_x-\Delta m_y r-(1+k_2)\,r\omega_x\,\Delta m_z-\\&-r\omega_z\,\Delta m_x-k_2\left[\frac{d}{dt}(\Delta V_{Dx}+au_z\theta_x-au_x\theta_z)-\right.\\&\left.-\omega_z(\Delta V_{Dy}+au_z\theta_y-au_y\theta_z)\right].\end{aligned}\right\} \tag{5.191}$$

$$\left.\begin{aligned} f_y = &-(\omega_y\omega_z - \dot{\omega}_x)\Delta r + (2+k_2)\omega_x\Delta\dot{r} + \\ &+(1+k_2)\Delta n_y + \Delta\dot{m}_x r - (1+k_2) r\omega_y \Delta m_z - \\ &- r\omega_z \Delta m_y - k_2\left[\frac{d}{dt}(\Delta V_{Dy} + au_z\theta_y - au_y\theta_z) + \right. \\ &\left. + \omega_z(\Delta V_{Dx} + au_z\theta_x - au_x\theta_z)\right]. \end{aligned}\right\} \qquad (5.191)$$

For an object moving with constant velocity along a parallel, the homogeneous equations (5.190) take the form

$$\left.\begin{aligned} \delta\ddot{x} + &\left[(1+k_2)(\omega_0^2 - \omega_y^2) - \omega_z^2 - k_2\frac{a}{r}u_z\omega_z\right]\delta x - \\ &-\left(2\omega_z + k_2\frac{a}{r}u_z\right)\delta\dot{y} = 0, \\ \delta\ddot{y} + &\left[(1+k_2)\omega_0^2 - \omega_z^2 - k_2\frac{a}{r}u_z\omega_z\right]\delta y + \\ &+\left(2\omega_z + k_2\frac{a}{r}u_z\right)\delta\dot{x} = 0. \end{aligned}\right\} \qquad (5.192)$$

The characteristic equation of this system reduces to a biquadratic equation

$$\begin{aligned} p^4 + &\left[(2\omega_0^2 - \omega_y^2)(1+k_2) + 2\omega_z^2 + 2k_2\frac{a}{r}u_z\omega_z + \right. \\ &\left. + k_2^2\frac{a^2u_z^2}{r^2}\right]p^2 + \left[(\omega_0^2 - \omega_y^2)(1+k_2) - \omega_z^2 - \right. \\ &\left. - k_2\frac{a}{r}u_z\omega_z\right]\left[\omega_0^2(1+k_2) - \omega_z^2 - k_2\frac{a}{r}u_z\omega_z\right] = 0. \end{aligned} \qquad (5.193)$$

It can be shown that at least in the region

$$\omega_0^2 - \omega_y^2 - \omega_z^2 > 0$$

Eq. (5.193) has two pairs of pure imaginary conjugate roots, i.e., the system is nonasymptotically stable in this region.

In case of slow motion at constant distance from the Earth's center, the projections on the azimuthally free axes are replaced to first approximation by the equations

$$\left.\begin{aligned} \delta\ddot{x} + \omega_0^2(1+k_2)\delta x = f'_x, \\ \delta\ddot{y} + \omega_0^2(1+k_2)\delta y = f'_y, \end{aligned}\right\} \qquad (5.194)$$

where

$$\left.\begin{aligned} f'_x = &-\dot{\omega}_y\Delta r + (1+k_2)\Delta n_x - \Delta\dot{m}_y r - k_2\Delta\dot{V}_{Dx}, \\ f'_y = &\dot{\omega}_x\Delta r + (1+k_2)\Delta n_y + \Delta\dot{m}_x r - k_2\Delta\dot{V}_{Dy}. \end{aligned}\right\} \qquad (5.195)$$

§5.5. DAMPING WITH SIMULTANEOUS FREQUENCY CHANGE

5.5.1. Ideal operation equations. Error equations

In §5.2 and §5.3 we have considered the damping of an inertial system using external information from a Doppler velocity meter. In §5.4 we have discussed the possibilities of changing the natural frequencies of inertial

systems using Doppler guidance information. We can now proceed with a discussion of damping with a simultaneous change in frequency.

The two damping alternatives (Eqs. (5.39), (5.40)) and the two frequency-changing alternatives (Eqs. (5.167), (5.168)) give four damped inertial systems with changed frequency. We will start by writing out the corresponding ideal operation equations.

Combining Eqs. (5.39), (5.167), we obtain the ideal operation equations for the first of the four alternatives:

$$\left.\begin{aligned} v &= \int_0^t (n - m \times v + g - k_1 \Delta V)\, dt + v^0, \\ r &= \int_0^t (v - m \times r + k_2 \Delta V)\, dt + r^0 \\ &\qquad (\Delta V = V - V_D). \end{aligned}\right\} \tag{5.196}$$

The second alternative is produced by combining Eqs. (5.39), (5.168). This gives

$$\left.\begin{aligned} v &= \int_0^t [n - m \times v + g - k_1 \Delta V + k_2'(\Delta \dot{V} + m \times \Delta V)]\, dt + v^0, \\ r &= \int_0^t (v - m \times r)\, dt + r^0 \\ &\qquad (\Delta V = V - V_D). \end{aligned}\right\} \tag{5.197}$$

The third alternative is obtained from Eqs. (5.40), (5.167):

$$\left.\begin{aligned} v &= \int_0^t (n - m \times v + g)\, dt + v^0, \\ r &= \int_0^t [v - m \times r + k_1'(\Delta \dot{V} + m \times \Delta V) + k_2 \Delta V]\, dt + r^0 \\ &\qquad (\Delta V = V - V_D). \end{aligned}\right\} \tag{5.198}$$

Finally, the ideal operation equations of the fourth system are obtained by combining Eqs. (5.40) and (5.168):

$$\left.\begin{aligned} v &= \int_0^t [n - m \times v + g + k_2'(\Delta \dot{V} + m \times \Delta V)]\, dt + v^0, \\ r &= \int_0^t [v - m \times r + k_1'(\Delta \dot{V} + m \times \Delta V)]\, dt + r^0 \\ &\qquad (\Delta V = V - V_D). \end{aligned}\right\} \tag{5.199}$$

We will now derive the error equations for these alternatives. Varying Eqs. (5.196), we obtain the variational equations

$$\left.\begin{aligned} \delta \dot{v} &= \Delta n - \omega \times \delta v - \Delta m \times v + \delta g - k_1 \delta \Delta V, \\ \delta \dot{r} &= \delta v - \Delta m \times r - \omega \times \delta r + k_2 \delta \Delta V, \\ \delta \Delta V &= \delta v + (\theta \times u) \times r \frac{a}{r} - u \times \left| \left(\delta r \frac{a}{r} - r \frac{a \Delta r}{r^2} \right) - \Delta V_D. \right. \end{aligned}\right\} \tag{5.200}$$

Eliminating from Eqs. (5.200) first $\delta\Delta V$ and then δv and inserting for δg its expression from (5.42), we get the vector equation

$$\begin{aligned}
&\frac{d^2\delta r}{dt^2}+k_1\frac{d\delta r}{dt}+\frac{\mu}{r^3}(1+k_2)\delta r+\\
&\quad+k_2\frac{d}{dt}\left(u\times\delta r\frac{a}{r}\right)-k_1u\times\delta r\frac{a}{r}=\\
&\quad=(1+k_2)\left(\Delta n-\Delta m\times\frac{dr}{dt}+3\frac{\mu r\,\Delta r}{r^4}\right)-\\
&\quad-\frac{d}{dt}(\Delta m\times r)-k_1(\Delta m\times r)+\\
&\quad+k_2\frac{d}{dt}\left[(\theta\times u)\times r\frac{a}{r}+u\times r\frac{a\,\Delta r}{r^2}-\Delta V_D\right]-\\
&\quad-k_1\left[(\theta\times u)\times r\frac{a}{r}+u\times r\frac{a\,\Delta r}{r^2}-\Delta V_D\right].
\end{aligned}\tag{5.201}$$

For $k_2=0$ this equation is clearly reduced to Eq. (5.43), for $k_1=0$ it is reduced to Eq. (5.171), and if both $k_1=0$ and $k_2=0$, Eq. (5.201) is reduced to Eq. (2.15a).

Similarly from Eq. (5.197) we obtain the variational equations

$$\left.\begin{aligned}
\delta\dot v&=\Delta n-\Delta m\times v-\omega\times\delta v+\delta g-\\
&\quad-k_1\delta\Delta V+k_2'(\delta\Delta\dot V+\omega\times\delta\Delta V),\\
\delta\dot r&=\delta v-\Delta m\times r-\omega\times\delta r,\\
\delta\Delta V&=\delta v+(\theta\times u)\times r\frac{a}{r}-u\times\left(\delta r\frac{a}{r}-r\frac{a\,\Delta r}{r^2}\right)-\Delta V_D.
\end{aligned}\right\}\tag{5.202}$$

Inserting $\delta\Delta V$ from the third equation into the first equation in (5.202), and taking δv from the second equation, we obtain the error equation for the case (5.197):

$$\begin{aligned}
&\frac{d^2\delta r}{dt^2}+\frac{k_1}{1-k_2'}\frac{d\delta r}{dt}+\frac{\mu}{r^3(1-k_2')}\delta r+\\
&+\frac{k_2'}{1-k_2'}\frac{d}{dt}\left(u\times\delta r\frac{a}{r}\right)-\frac{k_1}{1-k_2'}\left(u\times\delta r\frac{a}{r}\right)=\\
&=\frac{1}{1-k_2'}\left(\Delta n-\Delta m\times\frac{dr}{dt}+3\frac{\mu r\,\Delta r}{r^4}\right)-\\
&-\frac{d}{dt}(\Delta m\times r)-\frac{k_1}{1-k_2'}\Delta m\times r-\frac{k_1}{1-k_2'}\left[(\theta\times u)\times r\frac{a}{r}+\right.\\
&\left.+u\times r\frac{a\,\Delta r}{r^2}-\Delta V_D\right]+\frac{k_2'}{1-k_2'}\frac{d}{dt}\left[(\theta\times u)\times r\frac{a}{r}+u\times r\frac{a\,\Delta r}{r^2}-\Delta V_D\right].
\end{aligned}\tag{5.203}$$

For $k_2'=0$ this equation reduces to Eq. (5.43), for $k_1=0$ to Eq. (5.174), and for $k_2'=k_1=0$, to Eq. (2.15a).

Let us now derive the error equation for the case (5.198). Varying Eqs. (5.198), we get

$$\left.\begin{aligned}
\delta\dot v&=\Delta n-\Delta m\times v-\omega\times\delta v+\delta g,\\
\delta\dot r&=\delta v-\Delta m\times r-\omega\times\delta r+\\
&\quad+k_1'(\delta\Delta\dot V+\omega\times\delta\Delta V)+k_2\delta\Delta V,\\
\delta\Delta V&=\delta v+(\theta\times u)\times r\frac{a}{r}-u\times\left(\delta r\frac{a}{r}-r\frac{a\,\Delta r}{r^2}\right)-\Delta V_D.
\end{aligned}\right\}\tag{5.204}$$

To reduce these equations to the usual form, we proceed as follows. $\delta\Delta V$ is inserted from the third equation in the second and the resulting equation

is solved for $d\,\delta v/dt$:

$$\frac{d\,\delta v}{dt}=\frac{1}{k_1'}\left\{\frac{d\,\delta r}{dt}+\Delta m\times r-(1+k_2)\,\delta v+\right.$$
$$\left.+k_2\left[-(\theta\times u)\times r\,\frac{a}{r}+u\times\left(\delta r\,\frac{a}{r}-r\,\frac{a\,\Delta r}{r^2}\right)+\Delta V_D\right]\right\}+$$
$$+\frac{d}{dt}\left[-(\theta\times u)\times r\,\frac{a}{r}+u\times\left(\delta r\,\frac{a}{r}-r\,\frac{a\,\Delta r}{r^2}\right)+\Delta V_D\right]. \tag{5.205}$$

On the other hand, from the first equation in (5.204) we have

$$\frac{d\,\delta v}{dt}=\Delta n-\Delta m\times v+\delta g. \tag{5.206}$$

Equating the right-hand sides of Eqs. (5.205), (5.206), we get

$$\delta v=-\frac{k_1'}{1+k_2}\left(\Delta n-\Delta m\times\frac{dr}{dt}+\delta g\right)+$$
$$+\frac{1}{1+k_2}\left(\frac{d\,\delta r}{dt}+\Delta m\times r\right)-\frac{k_2}{1+k_2}\left[(\theta\times u)\times r\,\frac{a}{r}-\right.$$
$$\left.-u\times\left(\delta r\,\frac{a}{r}-r\,\frac{a\,\Delta r}{r^2}\right)-\Delta V_D\right]-\frac{k_1'}{1+k_2}\,\frac{d}{dt}\left[(\theta\times u)\times\right.$$
$$\left.\times r\,\frac{a}{r}-u\times\left(\delta r\,\frac{a}{r}-r\,\frac{a\,\Delta r}{r^2}\right)-\Delta V_D\right]. \tag{5.207}$$

Inserting this δv in (5.206) or (5.205) and using expression (5.42) for δg, we obtain after some manipulations

$$\frac{d^2\,\delta r}{dt^2}+k_1'\,\frac{\mu}{r^3}\,\frac{d\,\delta r}{dt}+(1+k_2)\,\frac{\mu}{r^3}\left(1-3k_1'\,\frac{\dot r}{r}\right)\delta r+$$
$$+k_2\,\frac{d}{dt}\left(u\times\delta r\,\frac{a}{r}\right)+k_1'\,\frac{d^2}{dt^2}\left(u\times\delta r\,\frac{a}{r}\right)=$$
$$=(1+k_2)\left(\Delta n-\Delta m\times\frac{dr}{dt}+3\,\frac{\mu r\,\Delta r}{r^4}\right)-\frac{d}{dt}\,(\Delta m\times r)+$$
$$+k_2\,\frac{d}{dt}\left[(\theta\times u)\times r\,\frac{a}{r}+u\times r\,\frac{a\,\Delta r}{r^2}-\Delta V_D\right]+$$
$$+k_1'\left\{\frac{d^2}{dt^2}\left[(\theta\times u)\times r\,\frac{a}{r}+u\times r\,\frac{a\,\Delta r}{r^2}-\Delta V_D\right]+\right.$$
$$\left.+\frac{d}{dt}\left(\Delta n-\Delta m\times\frac{dr}{dt}+\frac{3\mu r\,\Delta r}{r^4}\right)\right\}. \tag{5.208}$$

For $k_2=0$, Eq. (5.208) reduces, as expected, to Eq. (5.48). For $k_1'=0$ it reduces to Eq. (5.171). For $k_1'=k_2=0$ it reduces to Eq. (2.15a).

The error equation for the fourth alternative is obtained by taking the variations of Eq. (5.199). This gives

$$\left.\begin{aligned}
\delta\dot v&=\Delta n-\Delta m\times v-\omega\times\delta v+\delta g+k_2'(\delta\,\Delta\dot V+\omega\times\delta\,\Delta V),\\
\delta\dot r&=\delta v-\Delta m\times r-\omega\times\delta r+k_1'(\delta\,\Delta\dot V+\omega\times\delta\,\Delta V),\\
\delta\,\Delta V&=\delta v+(\theta\times u)\times r\,\frac{a}{r}-u\times\left(\delta r\,\frac{a}{r}-r\,\frac{a\,\Delta r}{r^2}\right)-\Delta V_D.
\end{aligned}\right\} \tag{5.209}$$

From the second and the third equation in (5.209)

$$\frac{d\,\delta v}{dt}=\frac{1}{k_1'}\left(\frac{d\,\delta r}{dt}+\Delta m\times r-\delta v\right)+$$
$$+\frac{d}{dt}\left[-(\theta\times u)\times r\,\frac{a}{r}+u\times\left(\delta r\,\frac{a}{r}-r\,\frac{a\,\Delta r}{r^2}\right)+\Delta V_D\right]. \tag{5.210}$$

From the first and the third equation in (5.209)

$$\frac{d\delta v}{dt} = \frac{1}{1-k_2'}\left(\Delta n - \Delta m \times \frac{dr}{dt} + \delta g\right) - \\ - \frac{k_2'}{1-k_2'}\frac{d}{dt}\left[-(\theta \times u)\times r\frac{a}{r} + u \times \left(\delta r \frac{a}{r} - r\frac{a\,\Delta r}{r^2}\right) + \Delta V_D\right]. \qquad (5.211)$$

Comparison of (5.210) and (5.211) gives an expression for δv:

$$\delta v = \frac{d\,\delta r}{dt} + \Delta m \times r - \frac{k_1'}{1-k_2'}\left\{\Delta n - \Delta m \times \frac{dr}{dt} + \delta g + \right. \\ \left. + \frac{d}{dt}\left[(\theta \times u)\times r\frac{a}{r} - u \times \left(\delta r \frac{a}{r} - r\frac{a\,\Delta r}{r^2}\right) - \Delta V_D\right]\right\}. \qquad (5.212)$$

To complete the derivation of the error equations for case (5.199), it remains to insert this expression in Eq. (5.210) or Eq. (5.211), using Eq. (5.42). In either case we end up with the equation

$$\frac{d^2\delta r}{dt^2} + \frac{k_1'}{1-k_2'}\frac{\mu}{r^3}\frac{d\,\delta r}{dt} + \frac{1}{1-k_2'}\frac{\mu}{r^3}\left(1 - 3k_1'\frac{\dot r}{r}\right)\delta r + \\ + \frac{k_1'}{1-k_2'}\frac{d^2}{dt^2}\left(u \times \delta r \frac{a}{r}\right) + \frac{k_2'}{1-k_2'}\frac{d}{dt}\left(u \times \delta r \frac{a}{r}\right) = \\ = \frac{1}{1-k_2'}\left(\Delta n - \Delta m \times \frac{dr}{dt} + 3\frac{\mu r\,\Delta r}{r^4}\right) - \frac{d}{dt}(\Delta m \times r) + \\ + \frac{k_1'}{1-k_2'}\frac{d}{dt}\left(\Delta n - \Delta m \times \frac{dr}{dt} + 3\frac{\mu r\,\Delta r}{r^4}\right) + \\ + \frac{k_1'}{1-k_2}\frac{d^2}{dt^2}\left[(\theta \times u)\times r\frac{a}{r} + u \times r\frac{a\,\Delta r}{r^2} - \Delta V_D\right] + \\ + \frac{k_2'}{1-k_2}\frac{d}{dt}\left[(\theta \times u)\times r\frac{a}{r} + u \times r\frac{a\,\Delta r}{r^2} - \Delta V_D\right]. \qquad (5.213)$$

For $k_1' = 0$ this equation is readily seen to reduce to Eq. (5.174) and for $k_2' = 0$ it is reduced to Eq. (5.48). If now both $k_1' = 0$ and $k_2' = 0$, Eq. (5.213) is reduced to Eq. (2.15a).

We have thus found the error equations for the four alternative sets of ideal operation equations (5.196)—(5.199). These are respectively Eqs. (5.201), (5.203), (5.208), (5.213). Comparing these four equations with one another, we readily see that (5.201) and (5.203) are essentially the same. Indeed, their structure is identical. They differ only in the coefficients of the corresponding terms. These coefficients, in their turn, depend on k_1, k_2, k_2'. Writing k_1'' for k_1 in Eq. (5.203), and putting

$$\frac{1}{1-k_2'} = 1 + k_2, \quad \frac{k_1''}{1-k_2'} = k_1, \qquad (5.214)$$

we see that Eqs. (5.201) and (5.203) coincide. The perturbed motions of the inertial systems based on Eqs. (5.196) and (5.197) are thus indistinguishable from one another. Comparing Eqs. (5.208) and (5.213) we see that they also have identical structure and under condition (5.214) they fully coincide. The perturbed motion of systems operating according to Eqs. (5.198) and (5.199) is thus also identical.

It follows from the preceding that we need only consider one of the two equations (5.201), (5.203) and one of the pair (5.208), (5.213). In principle,

of course it does not matter exactly which equations we choose. In practice, however, the ideal operation equations (5.196), (5.198) have a simpler structure than Eqs. (5.197), (5.199). Moreover, as we see from the corresponding error equations, by changing the coefficients k_1, k_2 and k_1', k_2' in Eqs. (5.196), (5.198) we can change the damping and the frequency independently of each other. For these reasons the ideal operation equations (5.196), (5.198) are preferred for practical implementation over the other two alternatives, (5.197), (5.199). In what follows we will therefore concentrate on error equations (5.201), (5.208), which correspond to the two simpler alternatives.

Eqs. (5.201), (5.208) are error equations of three-accelerometer inertial systems. The corresponding error equations of two-accelerometer systems can be found from the vector equations (5.201), (5.202) by taking their projections on the co-moving axes x, y with the z axis pointing along the radius-vector r. We should moreover take $\delta z = \Delta r\,(= \Delta h)$, and insert for ΔV_{Dz} its expression from the second equality in (5.188a). From the vector equation (5.201) we thus obtain the following set of scalar equations:

$$\left.\begin{aligned}
&\delta\ddot{x} + k_1\,\delta\dot{x} + \\
&\quad + \left[(1+k_2)\left(\frac{\mu}{r^3} - \omega_y^2\right) - \omega_z^2 - k_2\frac{a}{r}u_z\omega_z\right]\delta x + \\
&\quad + \left[(1+k_2)\,\omega_x\omega_y - \dot{\omega}_z - k_1\left(\omega_z - u_z\frac{a}{r}\right) - \right. \\
&\quad \left. - k_2 u_z\frac{a\dot{r}}{r^2} + k_2\frac{a}{r}(\omega_x u_y - \omega_y u_x)\right]\delta y - \\
&\qquad\qquad - \left(2\omega_z + k_2\frac{a}{r}u_z\right)\delta\dot{y} = f_x, \\
&\delta\ddot{y} + k_1\,\delta\dot{y} + \\
&\quad + \left[(1+k_2)\left(\frac{\mu}{r^3} - \omega_x^2\right) - \omega_z^2 - k_2\frac{a}{r}u_z\omega_z\right]\delta y + \\
&\quad + \left[(1+k_2)\,\omega_y\omega_x + \dot{\omega}_z + k_1\left(\omega_z - u_z\frac{a}{r}\right) - \right. \\
&\quad \left. - k_2 u_z\frac{a\dot{r}}{r^2} + k_2\frac{a}{r}(\omega_y u_x - \omega_x u_y)\right]\delta x + \\
&\qquad\qquad + \left(2\omega_z + k_2\frac{a}{r}u_z\right)\delta\dot{x} = f_y.
\end{aligned}\right\} \qquad (5.215)$$

The functions f_x, f_y on the right in (5.215) are the projections of the right-hand sides of (5.201) on the co-moving axes x, y. Using (5.188a), we write

$$\left.\begin{aligned}
f_x &= (1+k_2)(\Delta n_x - r\omega_x\,\Delta m_z) - \\
&\quad - (\omega_x\omega_z + \dot{\omega}_y + k_1\omega_y)\,\Delta r - (2+k_2)\,\omega_y\,\Delta\dot{r} - \\
&\quad - (2+k_2)\,\dot{r}\,\Delta m_y - r\,\Delta\dot{m}_y - r\omega_z\,\Delta m_x + \\
&\quad + k_1[\Delta V_{Dx} - r\,\Delta m_y + a(u_z\theta_x - u_x\theta_z)] - \\
&\quad - k_2\left\{\frac{d}{dt}[\Delta V_{Dx} + a(u_z\theta_x - u_x\theta_z)] - \right. \\
&\qquad\qquad \left. - \omega_z[\Delta V_{Dy} + a(u_z\theta_y - u_y\theta_z)]\right\}, \\
f_y &= (1+k_2)(\Delta n_y - r\omega_y\,\Delta m_z) + \\
&\quad + (\dot{\omega}_x - \omega_y\omega_z + k_1\omega_x)\,\Delta r + (2+k_2)\,\omega_x\,\Delta\dot{r} + \\
&\quad + (2+k_2)\,\dot{r}\,\Delta m_x + r\,\Delta\dot{m}_x - r\omega_z\,\Delta m_y + \\
&\quad + k_1[\Delta V_{Dy} + r\,\Delta m_x + a(u_z\theta_y - u_y\theta_z)] - \\
&\quad - k_2\left\{\frac{d}{dt}[\Delta V_{Dy} + a(u_z\theta_y - u_y\theta_z)] + \right. \\
&\qquad\qquad \left. + \omega_z[\Delta V_{Dx} + a(u_z\theta_x - u_x\theta_z)]\right\}.
\end{aligned}\right\} \qquad (5.216)$$

Clearly, for $k_1 = 0$ Eqs. (5.215) coincide with Eqs. (5.190), and for $k_2 = 0$ they reduce to Eqs. (5.44). If both k_1 and k_2 are zero, Eqs. (5.215) reduce to Eqs. (2.28).

Eqs. (5.215) correspond to the vector equation (5.201). The error equations for a two-accelerometer system corresponding to the vector equation (5.208) are obtained in the same way.

5.5.2. Orthodromic coordinates

Before proceeding with further analysis of the error equations of damped inertial systems with changed frequencies, let us consider the particular case of an orthodromic two-accelerometer system. We will write the ideal operation equations for this system and take their variations to obtain the error equations. We will only consider the case of Doppler correction using Eqs. (5.196). The case corresponding to Eqs. (5.198) can be treated analogously.

The ideal operation equations of an orthodromic two-accelerometer system without Doppler correction are Eqs. (2.23). In accordance with Eqs. (5.196), the first four equations in (2.23) are replaced by

$$\left.\begin{aligned}
v_x &= \int (n_x + v_y\omega_z - \dot{r}\omega_y + g_x - k_1\Delta V_x)\,dt + v_x^0,\\
v_y &= \int (n_y + \dot{r}\omega_x - v_x\omega_z + g_y - k_1\Delta V_y)\,dt + v_y^0;\\
\omega_y &= \frac{v_x - k_2\Delta V_x}{r}, \quad \omega_x = -\frac{v_y + k_2\Delta V_y}{r};\\
\Delta V_x &= V_x - V_{Dx}, \quad \Delta V_y = V_y - V_{Dy};\\
V_x &= v_x - \\
&- \rho u(-\beta_{31}\sin z\cos S - \beta_{32}\sin z\sin S + \beta_{33}\cos z),\\
V_y &= v_y + \rho u(-\beta_{31}\sin S + \beta_{32}\cos S).
\end{aligned}\right| \qquad (5.217)$$

The other eleven relations in (2.23) clearly remain unchanged.

Let us now derive the error equations of a system functioning in accordance with (5.217) and the last eleven equalities in (2.23). The results of this alternative derivation should clearly lead back to Eqs. (5.215).

Varying the first four relations in (5.217) and dropping the variations δg_x, δg_y of the nonspherical gravitation component, we obtain the set of equations

$$\left.\begin{aligned}
\delta\dot{v}_x &= \delta n_x + \delta v_y\omega_z + v_y\,\delta\omega_z - \omega_y\,\Delta\dot{r} - \dot{r}\,\delta\omega_y - k_1\delta\,\Delta V_x,\\
\delta\dot{v}_y &= \delta n_y + \omega_x\,\Delta\dot{r} + \dot{r}\,\delta\omega_x - \omega_z\,\delta v_x - v_x\,\delta\omega_z - k_1\delta\,\Delta V_y,\\
r\,\delta\omega_y + \omega_y\,\Delta r &= \delta v_x + k_2\delta\,\Delta V_x,\\
-r\,\delta\omega_x - \omega_x\,\Delta r &= \delta v_y + k_2\delta\,\Delta V_y\\
(\Delta r &= \Delta h).
\end{aligned}\right| \qquad (5.218)$$

Let $x_0y_0z_0$ denote the unperturbed position of the axes fixed to the inertial platform. The axis z_0 points along the radius-vector $\boldsymbol{r}$.

The axes x_0, y_0 lie in the plane of the geocentric horizon, with the accelerometers n_{x_0}, n_{y_0} directed along them. The perturbed position of these axes

will be designated xyz. The relative orientation of the axes $x_0y_0z_0$ and xyz is determined by the small angles α, β, γ from the matrix of direction cosines (4.22).

Using this definition of the angles α, β, γ, we obtain the previous expressions (5.58) for the variations $\delta\Delta V_x$, $\delta\Delta V_y$ of the last four equalities in (5.217). In these equalities, ΔV_{Dx_0}, ΔV_{Dy_0} are the Doppler errors, the angles θ_{x_0}, θ_{y_0}, θ_{z_0} are defined by Eqs. (4.39), where the angles α_2, β_2, γ_2 in their turn are related to the total errors δz, δS of the orthodromic coordinates that the system measures. This relation is expressed by the equalities

$$\alpha_2 = -\delta z, \quad \beta_2 = \delta S \cos z, \quad \gamma_2 = \delta S \sin z, \tag{5.219}$$

which follow from Eqs. (4.37) taking $\varepsilon = 0$.

In (5.218) the subscripts x, y, z can be replaced by x_0, y_0, z_0, respectively. We thus obtain the following expressions for δv_{x_0}, δv_{y_0}:

$$\left.\begin{aligned} \delta v_{x_0} &= r\,\delta\omega_{y_0} + \omega_{y_0}\Delta r - k_2\delta\Delta V_{x_0}, \\ \delta v_{y_0} &= -r\,\delta\omega_{x_0} - \omega_{x_0}\Delta r - k_2\delta\Delta V_{y_0}. \end{aligned}\right\} \tag{5.220}$$

Inserting these expressions in Eqs. (5.58) and solving them for $\delta\Delta V_{x_0}$, $\delta\Delta V_{y_0}$ we obtain (writing ρ for a as before)

$$\left.\begin{aligned} \delta\Delta V_{x_0} &= \frac{1}{1+k_2}[r\,\delta\omega_{y_0} + \omega_{y_0}\Delta r + \\ &+ a\,(u_{x_0}\theta_{z_0} - u_{z_0}\theta_{x_0}) - au_{z_0}\alpha + r\omega_{x_0}\gamma + \dot{r}\beta - \Delta V_{Dx_0}], \\ \delta\Delta V_{y_0} &= \frac{1}{1+k_2}[-r\,\delta\omega_{x_0} - \omega_{x_0}\Delta r + \\ &+ a\,(u_{y_0}\theta_{z_0} - u_{z_0}\theta_{y_0}) - au_{z_0}\beta + r\omega_{y_0}\gamma - \dot{r}\alpha - \Delta V_{Dy_0}]. \end{aligned}\right\} \tag{5.221}$$

To obtain the error equations, we should now proceed as follows. For the projections δv_{x_0}, δv_{y_0} in the first two equalities in (5.218) we insert their expressions from the last two equalities, replacing v_{x_0}, v_{y_0} with $-r\omega_{y_0}$, $r\omega_{x_0}$. In the resulting expressions, $\delta\Delta V_{x_0}$, $\delta\Delta V_{y_0}$ are expressed according to (5.221). After that, $\delta\omega_{x_0}$, $\delta\omega_{y_0}$, $\delta\omega_{z_0}$, δn_{x_0}, δn_{y_0} are replaced with their expressions from (4.24), (4.26), (4.7), (4.27), (4.33), (4.8), (4.12). Finally, the substitution of variables (4.32) is used. In the result we obtain equations differing from Eqs. (5.215) only in that the right-hand sides contain not Δm_{x_0}, Δm_{y_0}, Δm_{z_0} but the equivalent terms $-\Delta\omega_{x_0}$, $-\Delta\omega_{y_0}$, $-\Delta\omega_{z_0}$.

The above method is fairly tedious. Our problem is simplified, however, since not all the stages are absolutely essential. Some of them have already been carried out, and we can use ready-made results. Indeed, we have shown in 5.3.2 that the direct derivation of the error equations of a damped inertial system (by the same method as outlined above) leads to Eqs. (5.44) with $-\Delta\omega_{x_0}$, $-\Delta\omega_{y_0}$, $-\Delta\omega_{z_0}$ substituted for Δm_{x_0}, Δm_{y_0}, Δm_{z_0}. Now, Eqs. (5.44) coincide with Eqs. (5.215) if we take $k_2 = 0$. Hence, to prove that Eqs. (5.218), (5.221), (4.24), (4.26), (4.7), (4.27), (4.33), (4.8), (4.12), (4.32) indeed lead to Eqs. (5.215), we should only verify that the long string of equations gives the same terms with k_2 which are actually contained in Eqs. (5.215). In what follows, we will show that this is indeed so.

From (5.221), (5.218) we have

$$\left.\begin{aligned}
&\delta x + \ldots - k_2 \frac{d}{dt}\,[a\,(u_{x_0}\theta_{z_1} - u_{z_0}\theta_{x_0}) - a u_{z_1}\alpha + \\
&\quad + r\omega_{x_1}\gamma + \dot r\beta - \Delta V_{Dx_1}] = \\
&\quad = \ldots + k_2\,\delta n_{x_1} - k_2\omega_{z_1}\,[a\,(u_{y_1}\theta_{z_1} - u_{z_0}\theta_{y_1}) - \\
&\quad - a u_{z_0}\beta + r\omega_{y_0}\gamma - \dot r\alpha - \Delta V_{Dy_1}] + \\
&\qquad + k_2\,(v_{y_1}\delta\omega_{z_0} - \omega_{y_1}\Delta\dot r - \dot r\,\delta\omega_{y_1}), \\
&\delta\ddot y + \ldots - k_2 \frac{d}{dt}\,[a\,(u_{y_1}\theta_{z_1} - u_{z_1}\theta_{y_1}) - \\
&\quad - a u_{z_1}\beta + r\omega_{y_1}\gamma - \dot r\alpha - \Delta V_{Dy_1}] = \\
&\quad = \ldots + k_2\,\delta n_{y_1} + k_2\omega_{z_1}\,[a\,(u_{x_1}\theta_{z_1} - u_{z_1}\theta_{x_1}) - \\
&\quad - a u_{z_1}\alpha + r\omega_{x_1}\gamma + \dot r\beta - \Delta V_{Dx_1}] - \\
&\qquad - k_2\,(v_{x_1}\delta\omega_{z_1} - \omega_{x_1}\Delta\dot r - \dot r\,\delta\omega_{x_1}).
\end{aligned}\right\} \qquad (5.222)$$

Here triple dots stand for terms not containing k_2 (except $\delta\ddot x$, $\delta\ddot y$, which are written out explicitly).

On the other hand, retaining in Eqs. (5.215) only terms with k_2 and replacing the subscripts x, y, z with x_0, y_0, z_0, we write these equations in the form

$$\left.\begin{aligned}
&\delta\ddot x + \ldots + k_2\left(\frac{u}{r^3} - \omega_y^2 - \frac{a}{r}u_z\omega_{z_1}\right)\delta x + \\
&+ k_2\left[\omega_{x_0}\omega_{y_1} - u_{z_1}\frac{a\dot r}{r^2} + \frac{a}{r}(\omega_{x_1}u_{y_1} - \omega_{y_1}u_{x_1})\right]\delta y - \\
&- k_2\frac{a}{r}u_{z_1}\delta\dot y = \ldots + k_2(\Delta n_{x_1} - r\omega_{x_1}\Delta m_{z_1}) - \\
&- k_2(\omega_{y_1}\Delta\dot r + \dot r\,\Delta m_{y_1}) - k_2\left\{\frac{d}{dt}\,[\Delta V_{Dx_0} + \right. \\
&\left. + a\,u_{z_1}\theta_{x_0} - u_{x_1}\theta_{z_0})] - \omega_{z_1}\,[\Delta V_{Dy_1} + a\,(u_{z_1}\theta_{y_1} - u_{y_1}\theta_{z_1})]\right\}, \\
&\delta\ddot y + \ldots + k_2\left(\frac{u}{r^3} - \omega_{x_1}^2 - \frac{a}{r}u_{z_1}\omega_{z_1}\right)\delta y + k_2\left[\omega_{x_1}\omega_{y_1} - \right. \\
&\left. - u_{z_1}\frac{a\dot r}{r^2} + \frac{a}{r}(\omega_{y_1}u_{x_1} - \omega_{x_1}u_{y_1})\right]\delta x + k_2\frac{a}{r}u_{z_1}\delta\dot x = \\
&= \ldots + k_2(\Delta n_{y_1} - r\omega_{y_1}\Delta m_{z_1}) + k_2(\omega_{x_1}\Delta\dot r + \dot r\,\Delta m_{x_1}) - \\
&- k_2\left\{\frac{d}{dt}\,[\Delta V_{Dy_1} + a\,(u_{z_0}\theta_{y_0} - u_{y_1}\theta_{z_0})] + \right. \\
&\left. + \omega_{z_1}\,[\Delta V_{Dx_1} + a\,(u_{z_0}\theta_{x_1} - u_{x_1}\theta_{z_1})]\right\}.
\end{aligned}\right\} \qquad (5.223)$$

It now remains to show that (5.222) identically coincide with (5.223). Now it is evident that the terms with ΔV_{Dx_1}, ΔV_{Dy_1}, θ_{x_1}, θ_{y_1}, θ_{z_0} in (5.222), (5.223) are identical. The problem thus reduces to a comparison of the equalities

$$\left.\begin{aligned}
&\delta\ddot x + \ldots + k_2\frac{d}{dt}\,(a u_{z_0}\alpha - r\omega_{x_1}\gamma - \dot r\beta) = \\
&\quad = \ldots + k_2\,[\delta n_{x_0} + v_{y_0}\delta\omega_{z_1} - \omega_{y_0}\Delta\dot r - \\
&\qquad - \omega_{z_1}(-a u_{z_1}\beta + r\omega_{y_0}\gamma - \dot r\alpha) - \dot r\,\delta\omega_{y_0}], \\
&\delta\ddot y - \ldots + k_2\frac{d}{dt}\,(a u_{z_1}\beta - r\omega_{y_0}\gamma + \dot r\alpha) = \\
&\quad = \ldots + k_2\,[\delta n_{y_0} - v_{x_1}\delta\omega_{z_0} + \omega_{x_1}\Delta\dot r + \\
&\qquad + \omega_{z_0}(-a u_{z_1}\alpha + r\omega_{x_0}\gamma + \dot r\beta) + \dot r\,\delta\omega_{x_0}].
\end{aligned}\right\} \qquad (5.224)$$

which are obtained from (5.222) when the terms with ΔV_{Dx_0}, ΔV_{Dy_0}, θ_{x_0}, θ_{y_0}, θ_{z_0} are omitted, and the equalities

$$\left.\begin{aligned}
&\delta\ddot{x} + \ldots + k_2\left(\frac{\mu}{r^3} - \omega_{y_0}^2 - \frac{a}{r}u_{z_0}\omega_{z_0}\right)\delta x + \\
&\quad + k_2\left[\omega_{x_0}\omega_{y_0} - u_{z_0}\frac{a\dot{r}}{r^2} + \frac{a}{r}(\omega_{x_0}u_{y_0} - \omega_{y_0}u_{x_0})\right]\delta y - \\
&-k_2\frac{a}{r}u_{z_0}\delta\dot{y} = \ldots + k_2(\Delta n_{x_0} - r\omega_{x_0}\Delta m_{z_0} - \omega_{y_0}\Delta\dot{r} - \dot{r}\Delta m_{y_0}), \\
&\delta\ddot{y} + \ldots + k_2\left(\frac{\mu}{r^3} - \omega_{x_0}^2 - \frac{a}{r}u_{z_0}\omega_{z_0}\right)\delta y + \\
&\quad + k_2\left[\omega_{x_0}\omega_{y_0} - u_{z_0}\frac{a\dot{r}}{r^2} + \frac{a}{r}(\omega_{y_0}u_{x_0} - \omega_{x_0}u_{y_0})\right]\delta x + \\
&+k_2\frac{a}{r}u_{z_0}\delta\dot{x} = \ldots + k_2(\Delta n_{y_0} - r\omega_{y_0}\Delta m_{z_0} + \omega_{x_0}\Delta\dot{r} + \dot{r}\Delta m_{x_0}),
\end{aligned}\right\} \tag{5.225}$$

which are obtained from (5.223) when the terms dependent on ΔV_{Dx_0}, ΔV_{Dy_0}, θ_{x_0}, θ_{y_0}, θ_{z_0} are omitted on the right.

To check that Eqs. (5.224) and (5.225) coincide, let us compare the first pair. In the process we will use Eqs. (4.24), (4.26), (4.7), (4.27), (4.33), (4.8), (4.12), (4.32).

By (4.26), (4.7), (4.27), we obtain for the variations δn_{x_0}, δn_{y_0} in (5.224)

$$\left.\begin{aligned}
\delta n_{x_0} &= (w_{y_0} - g_{y_0})\gamma - (w_{z_0} - g_{z_0})\beta + \Delta n_{x_0}, \\
\delta n_{y_0} &= -(w_{x_0} - g_{x_0})\gamma + (w_{z_0} - g_{z_0})\alpha + \Delta n_{y_0}.
\end{aligned}\right\} \tag{5.226}$$

Ignoring the second-order terms $g_{y_0}\gamma$ and $g_{x_0}\gamma$ on the right and inserting for g_{z_0} its expression from (4.33), we get

$$\left.\begin{aligned}
\delta n_{x_0} &= w_{y_0}\gamma - \left(w_{z_0} + \frac{\mu}{r^2}\right)\beta + \Delta n_{x_0}, \\
\delta n_{y_0} &= -w_{x_0}\gamma + \left(w_{z_0} + \frac{\mu}{r^2}\right)\alpha + \Delta n_{y_0}.
\end{aligned}\right\} \tag{5.227}$$

Inserting for w_{x_0}, w_{y_0}, w_{z_0} their expressions from (4.8), (4.12), we write (5.227) in the form

$$\left.\begin{aligned}
\delta n_{x_0} &= [r(\omega_{y_0}\omega_{z_0} - \dot{\omega}_{x_0}) - 2\dot{r}\omega_{x_0}]\gamma - \\
&\quad - \left[\ddot{r} + r\left(\frac{\mu}{r^3} - \omega_{x_0}^2 - \omega_{y_0}^2\right)\right]\beta + \Delta n_{x_0}, \\
\delta n_{y_0} &= -[r(\omega_{x_0}\omega_{z_0} + \dot{\omega}_{y_0}) + 2\dot{r}\omega_{y_0}]\gamma + \\
&\quad + \left[\ddot{r} + r\left(\frac{\mu}{r^3} - \omega_{x_0}^2 - \omega_{y_0}^2\right)\right]\alpha + \Delta n_{y_0}.
\end{aligned}\right\} \tag{5.228}$$

For δn_{x_0}, δn_{y_0} in (5.224) we now insert their expressions from (5.228). At the same time, v_{x_0}, v_{y_0}, $\delta\omega_{x_0}$, $\delta\omega_{y_0}$, $\delta\omega_{z_0}$ are expressed by (4.12), (4.24). Collecting similar terms in (5.224), we see that the angle γ drops out completely. Eqs. (5.224) thus take the form

$$\left.\begin{aligned}
&\delta\ddot{x} + \ldots + k_2\left(\frac{\mu}{r^3} - \omega_{y_0}^2 - k_2\frac{a}{r}u_{z_0}\omega_{z_0}\right)r\beta - k_2\left(\omega_{x_0}\omega_{y_0} - \frac{a}{r}\dot{u}_{z_0}\right)r\alpha + k_2 a u_{z_0}\dot{\alpha} = \\
&\qquad = \ldots + k_2(\Delta n_{x_0} + r\omega_{x_0}\Delta\omega_{z_0} - \omega_{y_0}\Delta\dot{r} + \dot{r}\Delta\omega_{y_0}), \\
&\delta\ddot{y} + \ldots - k_2\left(\frac{\mu}{r^3} - \omega_{x_0}^2 - \frac{a}{r}u_{z_0}\omega_{z_0}\right)r\alpha + k_2\left(\omega_{x_0}\omega_{y_0} + \frac{a}{r}\dot{u}_{z_0}\right)r\beta + k_2 a u_{z_0}\dot{\beta} = \\
&\qquad = \ldots + k_2(\Delta n_{y_0} + r\omega_{y_0}\Delta\omega_{z_0} + \omega_{x_0}\Delta\dot{r} - \dot{r}\Delta\omega_{x_0}).
\end{aligned}\right\} \tag{5.229}$$

For α and β we now substitute δx, δy according to (4.32). This gives

$$\left.\begin{aligned}
&\delta\ddot{x} + \ldots + k_2\left(\frac{\mu}{r^3} - \omega_{y_0}^2 - k_2\frac{a}{r}u_z\omega_{z_0}\right)\delta x + \\
&+ k_2\left(\omega_{x_0}\omega_{y_0} - u_{z_0}\frac{a\dot{r}}{r^2} - \frac{a}{r}\dot{u}_{z_0}\right)\delta y - k_2\frac{a}{r}u_{z_0}\delta\dot{y} = \\
&= \ldots + k_2(\Delta n_{x_0} + r\omega_{x_0}\Delta\omega_{z_0} - \omega_{y_0}\Delta\dot{r} + \dot{r}\Delta\omega_{y_0}), \\
&\delta\ddot{y} + \ldots + k_2\left(\frac{\mu}{r^3} - \omega_{x_0}^2 - \frac{a}{r}u_{z_0}\omega_{z_0}\right)\delta y + \\
&+ k_2\left(\omega_{x_0}\omega_{y_0} - u_{z_0}\frac{a\dot{r}}{r^2} + \frac{a}{r}\dot{u}_{z_0}\right)\delta x + k_2\frac{a}{r}u_{z_0}\delta\dot{x} = \\
&= \ldots + k_2(\Delta n_{y_0} + r\omega_{y_0}\Delta\omega_{z_0} + \omega_{x_0}\Delta\dot{r} - \dot{r}\Delta\omega_{x_0}).
\end{aligned}\right\} \qquad (5.230)$$

Comparing the left-hand sides of (5.230) and (5.225) we see that they differ in one respect only: in (5.230) the coefficients before δx and δy contain $\dot{u}_{z_0}$, whereas the corresponding coefficients in (5.225) contain the expression $(\omega_{y_0}u_{x_0} - \omega_{x_0}u_{y_0})$. Now, the angular velocity vector of the Earth, $\boldsymbol{u}$, is constant in the axes $O_1\xi_0\eta_0\zeta_0$. Therefore, from

$$\frac{d\boldsymbol{u}}{dt} = \dot{\boldsymbol{u}} + \boldsymbol{\omega} \times \boldsymbol{u} = 0 \qquad (5.231)$$

we have

$$\dot{u}_{z_0} = -\omega_{x_0}u_{y_0} + \omega_{y_0}u_{x_0}. \qquad (5.232)$$

Thus, using (5.232), we see that the left-hand sides of (5.230) and (5.225) are identical. The right-hand sides are also identical, since as we know

$$\Delta m_{x_0} = -\Delta\omega_{x_0}, \quad \Delta m_{y_0} = -\Delta\omega_{y_0}, \quad \Delta m_{z_0} = -\Delta\omega_{z_0}. \qquad (5.233)$$

Comparison of Eqs. (5.225) and (5.230) shows that Eqs. (5.222) and (5.223) coincide. This means that the error equations of an orthodromic two-accelerometer system obtained by a direct variational technique from the ideal operation equations (2.23), (5.217) are exactly identical to the equations obtained from the vector equation (5.201) of a general inertial system.

5.5.3. Stability analysis

Consider the vector differential equations (5.201), (5.208). As we have seen in 5.5.1, these are the error equations of a three-accelerometer inertial system with Doppler correction according to the ideal operation equations (5.196)—(5.199).

The main purpose of Doppler correction, as we have seen in the preceding, is to ensure asymptotic stability of the state corresponding to the ideal operation equations of the system. This requires that the solutions of the error equations (5.201), (5.208) go to zero with time, i.e., the trivial solutions of the homogeneous equations (5.201), (5.208) must be stable.

Let us investigate the stability of the homogeneous equation (5.201)

$$\frac{d^2\delta\boldsymbol{r}}{dt} + k_1\frac{d\,\delta\boldsymbol{r}}{dt} + \frac{\mu}{r^3}(1 + k_2)\delta\boldsymbol{r} + k_2\frac{d}{dt}\left(\boldsymbol{u} \times \delta\boldsymbol{r}\,\frac{a}{r}\right) - k_1\boldsymbol{u} \times \delta\boldsymbol{r}\,\frac{a}{r} = 0 \qquad (5.234)$$

for $r = \text{const}$.

Taking its projections on the fixed-orientation axes $O_1\xi_*\eta_*\zeta_*$ (the axis $O_1\zeta_*$ points along the spin vector u of the Earth) and noting that

$$u_{\xi_*} = u_{\eta_*} = 0, \quad u_{\zeta_*} = u = \text{const}, \tag{5.235}$$

we obtain the scalar equations

$$\left.\begin{aligned} \delta\ddot{\xi}_* + k_1\delta\dot{\xi}_* + \omega_0^2(1+k_2)\delta\xi_* - \\ - k_2\frac{au}{r}\delta\dot{\eta}_* + k_1\frac{au}{r}\delta\eta_* = 0, \\ \delta\ddot{\eta}_* + k_1\delta\dot{\eta}_* + \omega_0^2(1+k_2)\delta\eta_* + \\ + k_2\frac{au}{r}\delta\dot{\xi}_* - k_1\frac{au}{r}\delta\xi_* = 0, \\ \delta\ddot{\zeta}_* + k_1\delta\dot{\zeta}_* + \frac{\mu}{r^3}(1+k_2)\delta\zeta_* = 0. \end{aligned}\right\} \tag{5.236}$$

If $r = \text{const}$, the coefficients in Eqs. (5.236) are constant. The third equation in (5.236) is separated. Its solution is obvious. For $k_1 > 0$ and $1 + k_2 > 0$, the solution of this equation goes to zero with time. The characteristic equation of the remaining fourth-order system (the first and the second equation in (5.236)) is

$$\begin{aligned} p^4 + 2k_1p^3 + \left[2\omega_0^2(1+k_2) + k_1^2 + k_2^2\frac{a^2u^2}{r^2}\right]p^2 + \\ + 2k_1\left[\omega_0^2(1+k_2) - k_2\frac{a^2u^2}{r^2}\right]p + \omega_0^4(1+k_2)^2 + k_1^2\frac{a^2u^2}{r^2} = 0. \end{aligned} \tag{5.237}$$

The stability conditions are provided by Hurwitz's inequalities:

$$\left.\begin{aligned} 2k_1 > 0, \\ 2k_1\left[\omega_0^2(1+k_2) + k_1^2 + k_2(1+k_2)\frac{a^2u^2}{r^2}\right] > 0, \\ 4k_1^2(1+k_2)\left(\omega_0^2 - \frac{a^2u^2}{r^2}\right)\left(k_1^2 + k_2^2\cdot\frac{a^2u^2}{r^2}\right) > 0, \\ \omega_0^4(1+k_2)^2 + k_1^2\frac{a^2u^2}{r^2} > 0. \end{aligned}\right\} \tag{5.238}$$

Since

$$\omega_0^2 \gg \frac{a^2u^2}{r^2}, \tag{5.238a}$$

wee see that for

$$k_1 > 0, \quad 1 + k_2 > 0 \tag{5.239}$$

conditions (5.238) are always satisfied.

The motion described by vector equation (5.234) is thus asymptotically stable for $r = \text{const}$. Doppler correction of a three-accelerometer system using Eqs. (5.196), (5.197) thus ensures asymptotic stability for any motion of the object at constant distance from the Earth's center. This result is also applicable to the motion of a satellite in a circular orbit.

Let us consider Eq. (5.208). The corresponding homogeneous equation is

$$\begin{aligned} \frac{d^2\delta r}{dt^2} + k_1'\frac{\mu}{r^3}\frac{d\,\delta r}{dt} + \frac{\mu}{r^3}(1+k_2)\left(1 - 3k_1'\frac{\dot{r}}{r}\right)\delta r + \\ + k_2\frac{d}{dt}\left(u\times\delta r\frac{a}{r}\right) + k_1'\frac{d^2}{dt^2}\left(u\times\delta r\frac{a}{r}\right) = 0. \end{aligned} \tag{5.240}$$

If the object moves at constant distance from the Earth's center, we have $\dot{r} = 0$, $\mu/r^3 = \omega_0^2 = \text{const}$, and projecting Eqs. (5.240) onto the fixed orientation axes ξ_*, η_*, ζ_* (the axis ζ_* points along the Earth's spin axis) we obtain the following set of differential equations with constant coefficients:

$$\left.\begin{aligned} \delta\ddot{\xi}_* + k_1'\omega_0^2\,\delta\dot{\xi}_* + \omega_0^2(1+k_2)\,\delta\xi_* - k_2 u\frac{a}{r}\,\delta\dot{\eta}_* - k_1' u\frac{a}{r}\,\delta\ddot{\eta}_* = 0,\\ \delta\ddot{\eta}_* + k_1'\omega_0^2\,\delta\dot{\eta}_* + \omega_0^2(1+k_2)\,\delta\eta_* + k_2 u\frac{a}{r}\,\delta\dot{\xi}_* + k_1' u\frac{a}{r}\,\delta\ddot{\xi}_* = 0,\\ \delta\ddot{\zeta}_* + k_1'\omega_0^2\,\delta\dot{\zeta}_* + \omega_0^2(1+k_2)\,\delta\zeta_* = 0. \end{aligned}\right\} \tag{5.241}$$

The stability of the third equation in (5.241) for

$$k_1' > 0, \quad 1 + k_2 > 0 \tag{5.242}$$

is self-evident. The characteristic equation of the first two equations in (5.241) is a fourth-degree equation

$$\left(1 + k_1'^2\frac{u^2a^2}{r^2}\right)p^4 + 2k_1'\left(\omega_0^2 + k_2\frac{a^2u^2}{r^2}\right)p^3 + \\ + \left[2\omega_0^2(1+k_2) + k_1'^2\omega_0^4 + k_2^2\frac{a^2u^2}{r^2}\right]p^2 + 2k_1'\omega_0^4(1+k_2)\,p + \omega_0^4(1+k_2)^2 = 0. \tag{5.243}$$

Applying Hurwitz's criterion to this equation we conclude that, in virtue of inequality (5.238a), the real parts of the roots of the characteristic equation are negative when $k_1' > 0$ and $1 + k_2 > 0$. Inertial systems with Doppler correction using Eqs. (5.198), (5.199) are thus asymptotically stable when conditions (5.242) are satisfied. Conditions (5.242) are clearly completely analogous to conditions (5.239).

Let us now consider the stability of two-accelerometer Doppler aided systems. The corresponding error equations are obtained by projecting the vector equations (2.201), (2.208) on the co-moving axes x, y with the z axis directed along the radius-vector r. We will only consider Eqs. (2.201). The stability of equations which follow from the vector equation (2.208) can be analyzed along the same lines.

In accordance with (5.215), the homogeneous error equations of a two-accelerometer inertial system in this case are

$$\left.\begin{aligned} \delta\ddot{x} + k_1\delta\dot{x} + \left[(1+k_2)\left(\frac{\mu}{r^3} - \omega_y^2\right) - \omega_z^2 - \right.\\ \left. - k_2\frac{a}{r}u_z\omega_z\right]\delta x + \left[(1+k_2)\omega_x\omega_y - \dot{\omega}_z - \right.\\ \left. - k_1\left(\omega_z - u_z\frac{a}{r}\right) - k_2u_z\frac{a\dot{r}}{r^2} + \right.\\ \left. + k_2\frac{a}{r}(\omega_x u_y - \omega_y u_x)\right]\delta y - \left(2\omega_z + k_2\frac{a}{r}u_z\right)\delta\dot{y} = 0,\\ \delta\ddot{y} + k_1\delta\dot{y} + \left[(1+k_2)\left(\frac{\mu}{r^3} - \omega_x^2\right) - \omega_z^2 - \right.\\ \left. - k_2\frac{a}{r}u_z\omega_z\right]\delta y + \left[(1+k_2)\omega_x\omega_y + \dot{\omega}_z + \right.\\ \left. + k_1\left(\omega_z - u_z\frac{a}{r}\right) - k_2\frac{a\dot{r}}{r^2} - \right.\\ \left. - k_2\frac{a}{r}(\omega_x u_y - \omega_y u_x)\right]\delta x + \left(2\omega_z + k_2\frac{a}{r}u_z\right)\delta\dot{x} = 0. \end{aligned}\right\} \tag{5.244}$$

If the motion of the object using the inertial system is quite general, the coefficients in Eqs. (5.244) are general functions of time. Exact stability

analysis of Eqs. (5.244) in this case involves insurmountable mathematical difficulties, since the modern theory of differential equations has no suitable methods to cope with the complex situation. Exact stability analysis of Eqs. (5.244) is therefore possible only for motion with constant velocity along a parallel, when the coefficients of these equations are constant. The sufficient conditions of stability can also be determined for slow motion close to the Earth's surface (at velocities much less than the circular velocity).

For motion with constant velocity along a parallel, we have

$$\left.\begin{aligned} u_x = \omega_x = 0, \quad u_y = \text{const}, \quad \omega_y = \text{const}, \\ u_z = \text{const}, \quad \omega_z = \text{const}, \quad r = \text{const}, \end{aligned}\right\} \tag{5.245}$$

and Eqs. (5.244) take the form $\left(\frac{\mu}{r^3} = \omega_0^2\right)$:

$$\left.\begin{aligned} \delta\ddot{x} + k_1\delta\dot{x} + \left[(1+k_2)(\omega_0^2 - \omega_y^2) - \right. \\ \left. - \omega_z^2 - k_2\frac{a}{r}u_z\omega_z\right]\delta x - k_1\left(\omega_z - \frac{a}{r}u_z\right)\delta y - \\ - \left(2\omega_z + k_2\frac{a}{r}u_z\right)\delta\dot{y} = 0, \\ \delta\ddot{y} + k_1\delta\dot{y} + \left[(1+k_2)\omega_0^2 - \omega_z^2 - k_2\frac{a}{r}u_z\omega_z\right]\delta y + \\ + k_1\left(\omega_z - \frac{a}{r}u_z\right)\delta x + \left(2\omega_z + k_2\frac{a}{r}u_z\right)\delta\dot{x} = 0. \end{aligned}\right\} \tag{5.246}$$

Before proceeding with the stability analysis of these equations, let us consider the particular case of equatorial motion. Then

$$u_z = 0, \quad \omega_z = 0 \tag{5.247}$$

and Eqs. (5.246) separate into two second-order differential equations:

$$\left.\begin{aligned} \delta\ddot{x} + k_1\delta\dot{x} + (1+k_2)(\omega_0^2 - \omega_y^2)\,\delta x = 0, \\ \delta\ddot{y} + k_1\delta\dot{y} + \omega_0^2(1+k_2)\,\delta y = 0. \end{aligned}\right\} \tag{5.248}$$

The first of these equations gives the position-error in the direction of motion, and the second the error at right angles to the direction of motion. We see from Eqs. (5.248) that for $k_1 > 0$, $1 + k_2 > 0$, the aided inertial system is stable if

$$\omega_0^2 - \omega_y^2 > 0, \tag{5.249}$$

i.e., for velocities less than the circular orbit velocity.

We now return to Eqs. (5.246), i.e., the general case of motion along the parallel. We will use Hurwitz's criteria for stability analysis.

Let

$$\left.\begin{aligned} a_1 = \omega_0^2(1+k_2) - \omega_z^2 - k_2\frac{a}{r}u_z\omega_z, \\ a_2 = \omega_y^2(1+k_2), \\ a_3 = \omega_z - \frac{a}{r}u_z, \quad a_4 = 2\omega_z + k_2\frac{a}{r}u_z. \end{aligned}\right\} \tag{5.250}$$

Using these symbols, we write (5.246) in the form

$$\left.\begin{aligned} \delta\ddot{x} + k_1\delta\dot{x} + (a_1 - a_2)\delta x - k_1 a_3\delta y - a_4\delta\dot{y} = 0, \\ \delta\ddot{y} + k_1\delta\dot{y} + a_1\delta y + k_1 a_3\delta x + a_4\delta\dot{x} = 0. \end{aligned}\right\} \tag{5.251}$$

The characteristic equation of this system is

$$p^4 + b_1 p^3 + b_2 p^2 + b_3 p + b_4 = 0, \tag{5.252}$$

where

$$\left.\begin{aligned} b_1 &= 2k_1, \quad b_2 = k_1^2 + 2a_2 - a_3^2 - a_4^2, \\ b_3 &= 2k_1 a_1 - k_1 a_2 - 2k_1 a_3 a_4, \\ b_4 &= a_1^2 - a_1 a_2 + k_1^2 a_3^2. \end{aligned}\right\} \tag{5.253}$$

Hurwitz's conditions for Eq. (5.252) are

$$\left.\begin{aligned} b_1 > 0, \quad b_1 b_2 - b_3 > 0, \\ b_1 b_2 b_3 - b_3^2 - b_1^2 b_4 > 0, \\ b_4 > 0. \end{aligned}\right\} \tag{5.254}$$

When the first, third, and fourth inequalities are satisfied, the second condition is satisfied automatically, and it is therefore of no significance.

The first condition in (5.254) implies that

$$k_1 > 0, \tag{5.255}$$

and is self-evident.

The third condition in (5.254), with the coefficients b_i expressed from (5.253) in terms of a_i, takes the form

$$k_1^2\{2(k_1^2 + a_3^2)[2a_1 - a_2 + 2a_3(a_4 - a_3)] + a_3^2\} > 0. \tag{5.256}$$

Inserting the expressions for a_1, a_2, a_3, a_4 from (5.250), we get

$$(1 + k_2)\left\{2\left[k_1^2 + \left(2\omega_z + k_2\frac{a}{r}u_z\right)^2\right]\left(2\omega_0^2 - \omega_y^2 - 2\frac{a^2}{r^2}u_z^2\right) + (1 + k_2)\omega_y^4\right\} > 0. \tag{5.257}$$

Finally, the third condition in (5.254) gives the inequality

$$\begin{aligned} &(1 + k_2)^2(\omega_0^2 - \omega_z^2)(\omega_0^2 - \omega_y^2 - \omega_z^2) + \\ &\quad + k_2(1 + k_2)\omega_z\left(\omega_z - \frac{a}{r}u_z\right)(2\omega_0^2 - 2\omega_z^2 - \omega_y^2) + \\ &\quad\quad + (k_1^2 + k_2^2\omega_z^2)\left(\omega_z - \frac{a}{r}u_z\right)^2 > 0. \end{aligned} \tag{5.258}$$

Inequalities (5.255), (5.257), (5.258) in fact define (together with the second inequality in (5.254)) the regions of asymptotic stability of the aided inertial system for the case of motion along a parallel. For $k_2 = 0$ these inequalities reduce to inequality (5.107); if together with $k_2 = 0$ we also put $u_z = 0$, these inequalities coincide with (5.11).

Since u_z is small compared to ω_0, the stability regions defined by (5.255), (5.257), (5.258) are close to the stability regions determined assuming $u_z = 0$.

Under this assumption, taking $k_1 > 0$ and $1 + k_2 > 0$, we write inequality (5.257) in the form

$$\omega_z^2 > -\frac{k_1^2}{4} + \frac{1+k_2}{8}\frac{\omega_y^4}{\omega_y^2 - 2\omega_0^2}. \tag{5.259}$$

Inequality (5.258) is similarly written in the form

$$\omega_y^2 < \frac{1}{1+k_2}\left[(1+k_2)\,\omega_0^2 - \omega_z^2 + \frac{k_1^2\omega_z^2}{\omega_0^2(1+k_2) - \omega_z^2}\right]. \tag{5.260}$$

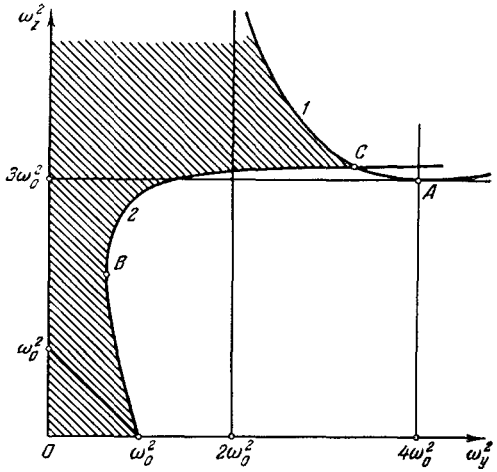

FIGURE 5.4.

Inequality (5.259) partitions the ω_y^2, ω_z^2 plane into three regions. The boundary curve of these regions (Figure 5.4) is the hyperbola 1 (only one branch is shown in the figure)

$$\omega_z^2 = -\frac{k_1^2}{4} + \frac{1+k_2}{8}\frac{\omega_y^4}{\omega_y^2 - 2\omega_0^2}. \tag{5.261}$$

The asymptotes of this hyperbola are the lines

$$\omega_y^2 = 2\omega_0^2, \quad \omega_z^2 = \frac{1+k_2}{8}\omega_y^2 - \frac{k_1^2}{4} + \frac{\omega_0^2}{4}(1+k_2). \tag{5.262}$$

Inequality (5.259) corresponds to the region between the branches of the hyperbola (5.261). Since one of the branches lies entirely in the lower half-plane $\omega_z^2 < 0$, we are left only with the region between the branch 1 and the axes $\omega_y^2 = 0$, $\omega_z^2 = 0$. The physical branch of the hyperbola is closest to the axis $\omega_z^2 = 0$ at the point $A\,[\omega_y^2 = 4\omega_0^2,\ \omega_z^2 = \omega_0^2(1+k_2)]$. In Figure 5.4 the hyperbola

(5.261) is shown for the case

$$k_2 = 2, \quad k_1^2 = \frac{\omega_0^2}{4}. \tag{5.263}$$

Inequality (5.260) is also satisfied by another hyperbola (2 in Figure 5.4)

$$\omega_y^2 = \frac{1}{1+k_2}\left[(1+k_2)\omega_0^2 - \omega_z^2 + \frac{k_1\omega_z^2}{\omega_0^2(1+k_2)-\omega_z^2}\right], \tag{5.264}$$

with the asymptotes

$$\left.\begin{aligned} &\omega_z^2 = (1+k_2)\omega_0^2, \\ &(1+k_2)\omega_y^2 = -\omega_z^2 + (1+k_2)\omega_0^2 - k_1^2. \end{aligned}\right\} \tag{5.265}$$

The region between the branch of the hyperbola (5.264) lying in the first quadrant and the axes $\omega_y^2 = 0$, $\omega_z^2 = 0$ satisfies inequality (5.260). This branch passes through the point ($\omega_z^2 = 0$, $\omega_y^2 = \omega_0^2$) and it is closest to the axes $\omega_y^2 = 0$ at the point B with the coordinates

$$\left.\begin{aligned} &\omega_z^2 = \omega_0^2(1+k_2) - k_1\omega_0\sqrt{1+k_2}, \\ &\omega_y^2 = \frac{k_1}{1+k_2}(2\omega_0\sqrt{1+k_2} - k_1). \end{aligned}\right\} \tag{5.266}$$

It meets the hyperbola (5.261) at the point C $[\tilde{\omega}_y^2 < 4\omega_0^2,\ \tilde{\omega}_z^2 < \omega_0^2(1+k_2)]$. In Figure 5.4 this branch is drawn for the coefficients k_1 and k_2 from (5.263).

The stability region of Eqs. (5.246) corresponding to conditions (5.259), (5.260) is thus the region delimited by the segment of the straight line $\omega_z^2 = 0$ between the points $\omega_y^2 = 0$, $\omega_y^2 = \omega_0^2$, the part of the hyperbola (5.264) between the point ($\omega_z^2 = 0$, $\omega_y^2 = \omega_0^2$) and the point $C(\tilde{\omega}_z^2, \tilde{\omega}_y^2)$, the part of the hyperbola (5.261) from the point $C(\tilde{\omega}_z^2, \tilde{\omega}_y^2)$ to the asymptote $\omega_y^2 = 2\omega_0^2$ and the axis $\omega_y^2 = 0$. For $k_1 = k_2 = 0$ this region coincides with the region of nonasymptotic stability in Figure 2.1. For a given k_1, the stability region is the largest for

$$k_2 = \frac{k_1^2}{\omega_0^2} - 1, \tag{5.267}$$

when the tangent to the hyperbola (5.264) at the point ($\omega_y^2 = \omega_0^2$, $\omega_z^2 = 0$) is parallel to the axis $\omega_y^2 = 0$.

We have so far considered the stability of a two-accelerometer Doppler-aided system moving with constant velocity along a parallel. This is one of the possible modes of motion, but it is highly restrictive. In general the object moves at an arbitrary angle to the parallel with a variable velocity.

In the general case the coefficients in Eqs. (5.244) are variable and, as we have noted at the beginning of this section, exact stability analysis meets with prodigious mathematical difficulties.

However, if we assume that the velocity of the object is small compared to the circular orbit velocity, we can obtain a highly powerful approximate solution of the problem, which leads to the sufficient conditions of stability of Eqs. (5.244).

Consider Eqs. (5.244) in projection onto the azimuthally free axes. To this end we take $\omega_z = 0$. This gives

$$\left.\begin{aligned}
&\delta\ddot{x} + k_1\delta\dot{x} + \left(\frac{\mu}{r^3} - \omega_y^2\right)(1+k_2)\delta x + \\
&\quad + \left[(1+k_2)\omega_x\omega_y + k_1\frac{a}{r}u_z - k_2u_z\frac{a\dot{r}}{r^2} + \right. \\
&\quad \left. + k_2\frac{a}{r}(\omega_x u_y - \omega_y u_x)\right]\delta y - k_2\frac{a}{r}u_z\delta\dot{y} = 0, \\
&\delta\ddot{y} + k_1\delta\dot{y} + \left(\frac{\mu}{r^3} - \omega_x^2\right)(1+k_2)\delta y + \\
&\quad + \left[(1+k_2)\omega_x\omega_y - k_1u_z\frac{a}{r} - k_2\frac{a\dot{r}}{r^2} - \right. \\
&\quad \left. - k_2\frac{a}{r}(\omega_x u_y - \omega_y u_x)\right]\delta x + k_2\frac{a}{r}u_z\delta\dot{x} = 0.
\end{aligned}\right\} \tag{5.268}$$

To simplify the mathematics, we take $r = \text{const}$. Let $\omega_0^2 = \frac{\mu}{r^3}$ and substituting the variables

$$\delta\dot{x} = x_1, \quad \delta x = x_2, \quad \delta\dot{y} = y_1, \quad \delta y = y_2 \tag{5.268a}$$

we write Eqs. (5.268) in Cauchy form:

$$\left.\begin{aligned}
\dot{x}_1 &= -k_1x_1 - \omega_0^2(1+k_2)x_2 + \varphi_1, \\
\dot{x}_2 &= x_1, \\
\dot{y}_1 &= -k_1y_1 - \omega_0^2(1+k_2)y_2 + \varphi_2, \\
\dot{y}_2 &= y_1.
\end{aligned}\right\} \tag{5.269}$$

Here

$$\left.\begin{aligned}
\varphi_1 &= (1+k_2)\omega_y^2x_2 - \left[(1+k_2)\omega_x\omega_y + k_1u_z\frac{a}{r} + \right. \\
&\quad \left. + k_2\frac{a}{r}(\omega_x u_y - \omega_y u_x)\right]y_2 + k_2\frac{a}{r}u_zy_1, \\
\varphi_2 &= (1+k_2)\omega_x^2y_2 - \left[(1+k_2)\omega_x\omega_y - k_1u_z\frac{a}{r} - \right. \\
&\quad \left. - k_2\frac{a}{r}(\omega_x u_y - \omega_y u_x)\right]x_2 - k_2\frac{a}{r}u_zx_1.
\end{aligned}\right\} \tag{5.270}$$

In Eqs. (5.269) we change over to the normal coordinates b_1, u_1, b_2, u_2 using the relations

$$\left.\begin{aligned}
x_1 &= b_1(\varepsilon\cos u_1 - \nu\sin u_1), \quad x_2 = b_1\cos u_1, \\
y_1 &= b_2(\varepsilon\cos u_2 - \nu\sin u_2), \quad y_2 = b_2\cos u_2,
\end{aligned}\right\} \tag{5.271}$$

where

$$\varepsilon = -\frac{k_1}{2}, \quad \nu = \sqrt{\omega_0^2(1+k_2) - \frac{k_1^2}{4}}. \tag{5.272}$$

Eqs. (5.269) in normal coordinates take the form

$$\left.\begin{aligned}
\frac{db_1}{dt} &= \varepsilon b_1 - \frac{1}{\nu}\tilde{\varphi}_1\sin u_1, \\
\frac{du_1}{dt} &= \nu - \frac{1}{b_1\nu}\tilde{\varphi}_1\cos u_1, \\
\frac{db_2}{dt} &= \varepsilon b_2 - \frac{1}{\nu}\tilde{\varphi}_1\sin u_2, \\
\frac{du_2}{dt} &= \nu - \frac{1}{b_2\nu}\tilde{\varphi}_2\cos u_2.
\end{aligned}\right\} \tag{5.273}$$

Eqs. (5.273) on the right contain the functions $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$. By (5.270), (5.271), we have

$$\left.\begin{aligned}\tilde{\varphi}_1 &= (1+k_2)\omega_y^2 b_1 \cos u_1 - \Big[(1+k_2)\omega_x\omega_y + k_1\frac{a}{r}u_z + \\ &\quad + k_2\frac{a}{r}(\omega_x u_y - \omega_y u_x)\Big] b_2 \cos u_2 + k_2\frac{a}{r}u_z b_2(\varepsilon\cos u_2 - \nu\sin u_2), \\ \tilde{\varphi}_2 &= (1+k_2)\omega_x^2 b_2 \cos u_2 - \Big[(1+k_2)\omega_x\omega_y - k_1\frac{a}{r}u_z - \\ &\quad - k_2\frac{a}{r}(\omega_x u_y - \omega_y u_x)\Big] b_1 \cos u_1 - k_2\frac{a}{r}u_z b_1(\varepsilon\cos u_1 - \nu\sin u_1).\end{aligned}\right\} \tag{5.274}$$

Consider the quadratic form

$$U = b_1^2 + b_2^2. \tag{5.275}$$

The total derivative of the function U, in virtue of (5.273), is

$$\frac{dU}{dt} = 2\Big[\varepsilon U - \frac{1}{\nu}(b_1\tilde{\varphi}_1\sin u_1 + b_2\tilde{\varphi}_2\sin u_2)\Big]. \tag{5.276}$$

Inserting $\tilde{\varphi}_1, \tilde{\varphi}_2$ from (5.274), we get

$$\begin{aligned}\frac{dU}{dt} = 2\Big\{\varepsilon U - \frac{1}{\nu}\Big[\frac{1+k_2}{2}\big(b_1^2\omega_y^2\sin 2u_1 + \\ + b_2^2\omega_x^2\sin 2u_2 - 2\omega_x\omega_y b_1 b_2\sin(u_1+u_2)\big) - \\ - \frac{a}{r}b_1 b_2\big(k_1 u_z + k_2(\omega_x u_y - \omega_y u_x) - k_2 u_z\varepsilon\big)\sin(u_1-u_2)\Big]\Big\}.\end{aligned} \tag{5.277}$$

The normal coordinates b_1, b_2 can be written in the form

$$b_1 = b\sin\psi_1, \quad b_2 = b\cos\psi_1, \quad b^2 = U. \tag{5.278}$$

Similarly

$$\omega_x = \Omega\sin\psi_2, \quad \omega_y = \Omega\cos\psi_2, \quad \Omega^2 = \omega_x^2 + \omega_y^2. \tag{5.279}$$

If v is the absolute velocity of the object, we clearly have $\Omega = v/r$, since $\dot{r} = 0$.

Finally, for u_x and u_y we have

$$u_x = u_0\sin\chi, \quad u_y = u_0\cos\chi, \quad u_0 = u_x^2 + u_y^2, \tag{5.280}$$

where clearly u_0 is the projection of the Earth's spin vector $\boldsymbol{u}$ on the horizontal plane.

Using (5.278), (5.279), (5.280), we write Eq. (5.277) for the derivative dU/dt in the form

$$\begin{aligned}\frac{dU}{dt} = 2U\Big\{\varepsilon - \frac{1}{\nu}\Big[\frac{1+k_2}{2}\Omega^2\Big(\sin^2\psi_1\cos^2\psi_2\sin 2u_1 + \\ + \cos^2\psi_1\sin^2\psi_2\sin 2u_2 - \frac{1}{2}\sin 2\psi_1\sin 2\psi_2\sin(u_1+u_2)\Big) + \\ + \frac{a}{2r}\big((k_1 - \varepsilon k_2)u_z + k_2\Omega u_0\sin(\psi_2-\chi)\big)\sin(u_2-u_1)\sin 2\psi_1\Big]\Big\}.\end{aligned} \tag{5.281}$$

Writing

$$\mu^* = -\varepsilon + \frac{1}{\nu}\left\{\frac{1+k_2}{2}\Omega^2\left[\sin^2\psi_1\cos^2\psi_2\sin 2u_1 + \right.\right.$$
$$\left. + \cos^2\psi_1\sin^2\psi_2\sin 2u_2 - \frac{1}{2}\sin 2\psi_1\sin 2\psi_2\sin(u_1+u_2)\right] +$$
$$\left. + \frac{a}{2r}\left[(k_1 - \varepsilon k_2)u_z + k_2\Omega u_0\sin(\psi_2 - \chi)\right]\sin(u_2-u_1)\sin 2\psi_1\right\}, \qquad (5.282)$$

we reduce Eq. (5.281) to the form

$$\frac{dU}{dt} = -2U\mu^*. \qquad (5.283)$$

A sufficient condition of asymptotic stability of Eqs. (5.268) for $r = \text{const}$ is that μ^* be always positive. Indeed, from Eq. (5.283) we then see that U goes to zero with time. It is clear from the definition of this function that both b_1 and b_2 also go to zero, and then by (5.271), x_1, x_2, y_1, y_2 and hence by (5.268a) δx, δy, $\delta\dot{x}$, $\delta\dot{y}$ all go to zero. The sufficient condition of stability is thus

$$\mu^* > 0. \qquad (5.284)$$

Inserting for ε and ν in Eq. (5.282) for μ^* their expressions in terms of k_1, k_2, ω_0, we get

$$\mu^* = \frac{k_1}{2} + \frac{1}{\sqrt{\omega_0^2(1+k_2) - \frac{k_1^2}{4}}}\left\{\frac{1+k_2}{2}\Omega^2\left[\sin^2\psi_1\cos^2\psi_2\sin 2u_1 + \right.\right.$$
$$\left. + \cos^2\psi_1\sin^2\psi_2\sin 2u_2 - \frac{1}{2}\sin 2\psi_1\sin 2\psi_2\sin(u_1+u_2)\right] +$$
$$\left. + \frac{a}{2r}\left[k_1\left(1+\frac{k_2}{2}\right)u_z + k_2\Omega u_0\sin(\psi_2-\chi)\right]\times\right.$$
$$\left.\times\sin(u_2-u_1)\sin 2\psi_1\right\} > 0. \qquad (5.285)$$

Clearly,

$$\left.\begin{aligned}&\left|\sin^2\psi_1\cos^2\psi_2\sin 2u_1 + \cos^2\psi_2\sin^2\psi_2\sin 2u_2 - \right.\\ &\qquad\left. - \frac{1}{2}\sin 2\psi_1\sin 2\psi_2\sin(u_1-u_2)\right| \leqslant \frac{3}{2},\\ &|\sin(u_2-u_1)\sin 2\psi_1| \leqslant 1.\\ &|\sin(\psi_2-\chi)\sin(u_2-u_1)\sin 2\psi_1| \leqslant 1.\end{aligned}\right\} \qquad (5.286)$$

Therefore if $k_1 > 0$, $1 + k_2 > 0$, we have from (5.285)

$$\min\mu^* \geqslant \tilde{\mu}^* = \frac{k_1}{2} - \frac{1}{\sqrt{\omega_0^2(1+k_2) - \frac{k_1^2}{4}}}\left\{\frac{3(1+k_2)}{4}\Omega^{*2} + \right.$$
$$\left. + \frac{a}{2r}\left[k_1\left(1+\frac{k_2}{2}\right)u + k_2\Omega^* u\right]\right\}, \qquad (5.287)$$

where u is the Earth's angular velocity and $\Omega^* = \max|\Omega|$.

A sufficient condition of stability is that $\tilde{\mu}^*$ be positive. This gives the following condition of asymptotic stability:

$$k_1 - \frac{1}{\sqrt{\omega_0^2(1+k_2) - \frac{k_1^2}{4}}}\left\{\frac{3}{2}(1+k_2)\Omega^{*2} + \right.$$
$$\left. + \frac{a}{r}\left[k_1\left(1+\frac{k_2}{2}\right)u + k_2\Omega^* u\right]\right\} > 0, \qquad (5.288)$$

which for $k_2 = 0$ reduces to condition (5.128) if we note that $\frac{a}{r} \approx 1$ and write in (5.128) $c = 0$.

Condition (5.288) for given r and Ω^* enables us to choose the parameters k_1 and k_2 so as to ensure asymptotic stability of the inertial system for any law of variation of the velocity. A measure of damping of the initial errors is provided by the factor $\tilde{\mu}^*$. Indeed, by (5.283),

$$U = U^0 e^{-2\int_0^t \mu^* dt} \leqslant U^0 e^{-2\tilde{\mu}^* t}, \tag{5.289}$$

where U^0 is the initial value of the function U.

From the definition (5.275) of U, we get

$$h = \sqrt{b_1^2 + b_2^2} \leqslant b^0 e^{-\tilde{\mu}^* t}. \tag{5.290}$$

In conclusion note that the stability condition (5.288) was derived for the case $r = \text{const}$, i.e., general motion at constant distance from the Earth's center. It is readily seen that the same method can be used to find the sufficient condition of asymptotic stability of an object with a vertical displacement component. Then, this condition will include the maximum values of $|\dot{r}|$ and $|r - r^0|$, where r^0 is the constant value of r near which the object moves. A similar remark applies to the stability of three-accelerometer Doppler-aided inertial systems, which were previously treated for the case $r = \text{const}$.

5.5.4. The solution of error equations. The effect of instrumental errors. Choice of correction coefficients

Consider the homogeneous equation corresponding to the vector equation (5.234). When projected onto the fixed-orientation axes $O_1\xi_1\eta_1\zeta_1$ (the axis ζ_1 pointing along the Earth's spin axis), Eq. (5.234) gives three scalar equations (5.236). For $r = \text{const}$ the coefficients of the linear differential equations (5.236) are constant. The general solution of the last equation in the set is obvious. The solution of the corresponding inhomogeneous equation can be found by the method of variation of the constants in the general solution of the homogeneous equation.

To construct the general solution of the homogeneous fourth-order system comprising the first and the second equations in (5.236), we need the roots of the characteristic equation (5.237).

Since $\omega_0^2 \gg \frac{u^2 a^2}{r^2}$, sufficiently accurate approximations to the roots of Eq. (5.237) can be obtained by a method similar to that applied to determine the roots (5.143) of Eq. (5.68). In our case the auxiliary cubic equation (5.137) is

$$\begin{aligned} 2y^3 - \left[2\omega_0^2(1+k_2) + k_1^2 + k_2^2 \frac{a^2u^2}{r^2}\right] y^2 + \\ + 2\left\{\omega_0^2(1+k_2)\left[k_1^2 - \omega_0^2(1+k_2)\right] - k_1^2(1+k_2)\frac{a^2u^2}{r^2}\right\} y + \\ + \omega_0^4(1+k_2)^2\left[2\omega_0^2(1+k_2) - k_1^2\right] + \\ + \omega_0^2(1+k_2)^2(2k_1^2 + \omega_0^2 k_2^2)\frac{a^2u^2}{r^2} = 0. \end{aligned} \tag{5.291}$$

The following approximate expression for the root y_1' of this equation is readily obtained:

$$y_1' = \frac{k_1^2}{2} - \omega_0^2(1+k_2) - \frac{k_1^2 a^2 u^2}{2\sigma^2 r^2}\left[k_2^2 + 4(1+k_2)\right], \tag{5.292}$$

where $\sigma^2 = 4\omega_0^2(1+k_2) - k_1^2$.

This expression constitutes the first terms in the expansion of the exact root in powers of $\frac{a^2u^2}{r^2\sigma^2}$.

We will only use the zero and first terms of this expansion. Since usually

$$\sigma^2 = 4\omega_0^2(1+k_2) - k_1^2 \gg \frac{a^2u^2}{r^2}. \tag{5.293}$$

Eq. (5.292) ensures satisfactory accuracy in the calculation of y_1.

Thus, to obtain an approximate solution of Eq. (5.237), we require the roots of two quadratic equations

$$p^2 + b_1 p + c_1 = 0, \quad p^2 + b_2 p + c_2 = 0, \tag{5.294}$$

whose coefficients are expressible in terms of the coefficients of Eq. (5.237) and the approximate root y_1'. By (5.135), (5.136), these coefficients can be written in the form

$$b_{1,2} = k_1 \pm j\sigma\left(1 + q_1\frac{a^2u^2}{r^2\sigma^2}\right),$$
$$c_{1,2} = \frac{k_1^2}{2} - \omega_0^2(1+k_2) \pm j\frac{k_1\sigma}{2} + (q_2 \pm jq_3)\frac{a^2u^2}{r^2\sigma^2},$$

where

$$q_1 = \frac{2}{\sigma^2}(1+k_2)(k_1^2 + \omega_0^2 k_2^2), \quad q_2 = -\frac{k_1^2}{2}\left[4(1+k_2) + k_2^2\right],$$
$$q_3 = \frac{k_1}{2\sigma}\left[k_1^2(2+4k_2+k_2^2) - 2\omega_0^2 k_2(1+k_2)(k_2+4)\right].$$

Solving the first of the two equations in (5.294), we obtain two roots of the characteristic equation (5.237):

$$p_{1,2} = -\frac{k_1}{2} - j\frac{\sigma}{2} \pm \frac{au}{r\sigma}\sqrt{-\frac{q_1\sigma^2}{2} - q_2 + j\left(\frac{k_1 q_1 \sigma}{2} - q_3\right)} - \frac{j\sigma q_1 a^2 u^2}{2r^2\sigma^2}. \tag{5.295}$$

Similarly, from the second equation in (5.294) we get

$$p_{3,4} = -\frac{k_1}{2} + j\frac{\sigma}{2} \pm \frac{au}{r}\sqrt{-\frac{q_1\sigma^2}{2} - q_2 + j\left(\frac{k_1 q_1 \sigma}{2} - q_3\right)} + \frac{j\sigma q_1 a^2 u^2}{2r^2\sigma^2}. \tag{5.295a}$$

For $k_2 = 0$ the roots (5.295), (5.295a) of the characteristic equation coincide with the corresponding expressions in (5.143).

Once the roots of the characteristic equation have been determined, we proceed in the usual way to construct the general solution of the homogeneous system comprising the first two equations in (5.236) and thence the general solution of the corresponding inhomogeneous system. The result, however, is too unwieldy for practical purposes.

The solution can be simplified in the following way.

It follows from the particular form of the roots (5.295), (5.295a) of the characteristic equation (5.237) that when inequality (5.293) is satisfied, these roots are close to the expressions

$$p_{1,2} = p_{3,4} = -\varepsilon \pm j\nu, \quad \varepsilon = \frac{k_1}{2}, \quad \nu = \frac{1}{2}\sqrt{4\omega_0^2(1+k_2) - k_1^2}, \tag{5.295b}$$

i.e., they are close to the roots of the characteristic equation

$$p^2 + k_1 p + \omega_0^2(1+k_2) = 0.$$

which corresponds to a second-order vector equation

$$\frac{d^2\delta r}{dt^2} + k_1\frac{d\delta r}{dt} + \omega_0^2(1+k_2)\delta r = 0. \tag{5.296}$$

This means that, $\frac{ua}{r}$ being small compared to k_1 and ω_0, the coupling between the first and the second equation in (5.236) is weak. In a damped system, the coupling effects cannot develop to a considerable extent during the damping time of the initial errors.

Thus, when inequality (5.293) is satisfied, Eq. (5.201) can be replaced to first approximation with the equation

$$\frac{d^2\delta r}{dt^2} + k_1\frac{d\delta r}{dt} + \omega_0^2(1+k_2)\delta r = f(t), \tag{5.297}$$

where $f(t)$ stands for the right-hand side of Eq. (5.201).

The general solution of Eq. (5.297) can be written in the form

$$\delta r = e^{-\varepsilon t}\left[\delta r^0 \cos\nu t + \frac{1}{\nu}\left(\frac{d\delta r^0}{dt} + \varepsilon\delta r^0\right)\sin\nu t\right] + \frac{1}{\nu}\int_0^t f(\tau)\, e^{-\varepsilon(t-\tau)}\sin\nu(t-\tau)\, d\tau. \tag{5.298}$$

where ε and ν are given by the last two equalities in (5.295b), and differentiation and integration are carried out in the $O_1\xi\eta\zeta$ axes.

In (5.298), $f(t)$ stands for the right-hand side of Eq. (5.201). To first approximation, neglecting on the right in Eq. (5.201) terms which contain the factor u, we may write

$$f(t) = (1+k_2)\left(\Delta n - \Delta m \times \frac{dr}{dt} + \frac{3ur\,\Delta r}{r^4}\right) - \\ - \frac{a}{r}(\Delta m \times r) - k_1(\Delta m \times r) - k_2\frac{d\Delta V_D}{dt} + k_1\Delta V_D. \tag{5.299}$$

The admissibility of this simplification of the function $f(t)$ is evident from the following considerations. The second group of error equations of an inertial system with Doppler correction is the same as for an autonomous system. According to (1.94), the errors $\theta_x, \theta_y, \theta_z$ determined by the equations of the second group lead to position errors $\delta r = \theta_1 \times r$. If the position is to be determined to within 1 km, we should have $|\theta| \leqslant 1.5 \cdot 10^{-4}$ rad. Then $\left|(\theta \times u) \times \frac{ra}{r}\right| < 7.4 \cdot 10^{-2}$ m/sec. $|u \times r|\frac{a\,\Delta r}{r^2}$ is of the same order of magnitude

for $\Delta r < 1$ km. The error $(\theta \times u) \times \frac{ra}{r} + u \times r \frac{a\,\Delta r}{r^2}$ is added to the instrumental Doppler error ΔV_D in the right-hand side of Eq. (2.201). We have $\left|(\theta \times u) \times \right.$ $\left. \times r \frac{a}{r} + u \times r \frac{a\,\Delta r}{r^2}\right| < 0.15$ m/sec, whereas the Doppler error is generally of the order of 1 m/sec. Hence it follows that terms with u on the right in Eq. (5.201) are indeed ignorable.

Consider the solution (5.298). If the function f constitutes a constant vector b, the steady-state error is

$$\delta r = \frac{b}{\nu^2 + \varepsilon^2} = \frac{b}{\omega_0^2 (1 + k_2)}. \tag{5.300}$$

Let us compare this error with the steady-state error (5.152) of a damped system whose frequency has not been changed. If

$$\Delta n - \Delta m \times \frac{dr}{dt} + 3\,\frac{\mu r\,\Delta r}{r^4} = a_1, \tag{5.301}$$

where a_1 is a constant vector, we have $b_1 = a_1(1 + k_2)$. Hence, by (5.300), (5.152),

$$\delta r = \frac{a_1}{\omega_0^2} = \frac{b_1}{\omega_0^2 (1 + k_2)}. \tag{5.301a}$$

The steady-state error due to a constant instrumental error (5.301) is thus the same in both cases and is independent of the choice of the correction coefficients k_1, k_2.

The right-hand side (5.150) of the error equation of a damped system with unchanged frequency has one additional term

$$k_1 (\Delta V_D - \Delta m \times r) - \frac{d}{dt} (\Delta m \times r), \tag{5.302}$$

and the right-hand side (5.299) of Eq. (5.297) contains the additional term

$$-\frac{d}{dt} (\Delta m \times r) + k_1 (\Delta V_D - \Delta m \times r) - k_2 \frac{d\,\Delta V_D}{dt}. \tag{5.303}$$

Let

$$-\frac{d}{dt} (\Delta m \times r) - \Delta m \times r = a_2 = b_2. \tag{5.304}$$

where $a_2 = b_2$ are constant vectors. From (5.152) and (5.300) we then respectively have

$$\delta r = \frac{a_2}{\omega_0^2}, \quad \delta r = \frac{a_2}{\omega_0^2 (1 + k_2)}, \tag{5.305}$$

i.e., the steady-state error from instrumental error (5.304) is less by a factor of $(1 + k_2)$ in the system with changed frequency. A similar conclusion applies to the instrumental error $k_1 \Delta V_D$.

In a frequency-changing system, the right-hand side also contains a term $k_2 \frac{d\Delta V_D}{dt}$. If $\frac{d\Delta V_D}{dt} = b_3$, the steady-state error in this case is

$$\delta r = \frac{k_2 b_3}{\omega_0^2 (1 + k_2)}. \tag{5.306}$$

Since usually $\frac{k_2}{1+k_2} \approx 1$, we have

$$\delta r \approx \frac{b_3}{\omega_0^2}. \tag{5.307}$$

Let us now consider Eq. (5.208). The corresponding homogeneous equation (5.240) projected onto the axes ξ, η, ζ, gives for $r = \text{const}$ the linear differential equations (5.241). The solution of the last of these equations is obvious. The roots of the characteristic equation (5.243) of the system comprising the first two equations in (5.241) can be found by the same method as we have applied to determine the roots of Eqs. (5.68), (5.237). Since u is small compared to ω_0, the roots are also found to be close to the expressions (5.295b), where k_1 should be replaced with $k_1'\omega_0^2$. The coupling between the first two equations in (5.241) is seen to be insignificant in virtue of the smallness of u, as in the case of Eq. (5.236). Therefore, Eq. (5.240) can be replaced to first approximation with Eq. (5.296), where k_1 should be replaced with $k_1'\omega_0^2$. Correspondingly, instead of Eq. (5.208) we may take (putting $u = 0$, $r = \text{const}$)

$$\frac{d^2 \delta r}{dt^2} + k_1' \omega_0^2 \frac{d\,\delta r}{dt} + \omega_0^2 (1 + k_2)\,\delta r = f'(t), \tag{5.308}$$

where now

$$f'(t) = (1 + k_2)\left(\Delta n - \Delta m \times \frac{dr}{dt} + 3\frac{\mu r\,\Delta r}{r^4}\right) + \\ + k_1' \frac{d}{dt}\left(\Delta n - \Delta m \times \frac{dr}{dt} + 3\frac{\mu r\,\Delta r}{r^4}\right) - \frac{d}{dt}(\Delta m \times r) - \\ - k_2 \frac{d\,\Delta V_D}{dt} - k_1' \frac{d^2\,\Delta V_D}{dt^2}. \tag{5.309}$$

For $r = \text{const}$ and $k_1'\omega_0^2 = k_1$ the left-hand sides of Eqs. (5.297) and (5.308) coincide. Thus, to first approximation, the four alternatives (5.196)—(5.199) are equivalent in the error-free case.

The general solution of the inhomogeneous equation (5.308) for $r = \text{const}$ is evidently Eq. (5.298) with $f'(t)$ substituted for $f(t)$. The right-hand sides of Eqs. (5.297) and (5.308), i.e., the vector functions $f(t)$ and $f'(t)$ are significantly different from each other. The main distinction between systems (5.196), (5.197) and systems (5.198), (5.199) is therefore the different effects of the instrumental errors and external information errors ΔV_D and Δr on the operational accuracy. This point is naturally of decisive significance in practice. Thus, for instance, if the main error is a constant Doppler deviation ΔV_D, it is advisable to prefer systems (5.198), (5.199), since the right-hand sides (5.309) of the corresponding error equations contain only the time derivatives of ΔV_D. The constant component thus does not lead to any system errors.

Eqs. (2.201), (2.208) whose solutions have been discussed so far are the error equations of three-accelerometer aided systems. The corresponding equations for two-accelerometer systems are obtained by taking the projections of the vector equations (2.201), (2.208) on the co-moving axes x, y, with the z axis directed along r. From Eq. (2.201) we then obtain Eqs. (5.215). The corresponding homogeneous system is (5.244).

The structure of these equations is fairly complex. As we have already noted, the coefficients of (5.244) in the general case are some functions of time.

The complete exact solution of the error equations can be found only for motion with constant velocity along a parallel, when Eqs. (5.244) reduce to a system of differential equations with constant coefficients, Eqs. (5.246). If ω_x, ω_y, ω_z are small compared to ω_0, the approximate solution of the characteristic equation (5.252) of Eqs. (5.246) can be obtained by the same method as above, in connection with Eqs. (5.68), (5.237). If ω_x, ω_y, ω_z are sufficiently small, the roots of the characteristic equation (5.252) are close to expressions (5.295b). The coupling between the first and the second equation in (5.246) is weak, and we can thus consider in the first approximation two independent second-order equations which are obtained from Eqs. (5.246) when we take

$$\omega_y = \omega_z = u_z = 0. \tag{5.310}$$

Note that for motion along a parallel with velocities substantially less than the circular orbit velocity, we always have $|\omega_y| \ll \omega_0$. At the same time $|\omega_z|$ may prove to be close to ω_0 or it may even exceed ω_0 if the latitude of the corresponding parallel is sufficiently high. This is a consequence of the fact that Eqs. (5.246) are written in projections on the xyz axes, with the axis y pointing northward. If we project the error equations on the co-moving axes xyz with the axis z directed along r and zero spin around this axis ($\omega_z = 0$, i.e., azimuthally free axes), this difficulty can be avoided.

The error equations of the system projected onto the azimuthally free axes take the form (5.268). Here ω_z is omitted from the coefficients. The components ω_x, ω_y, u_x, u_y, u_z (in this case they are variable even for motion along a parallel) are substantially less in magnitude than ω_0 for slow motion (1000—2000 m/sec). Therefore, to first approximation, they can be taken equal to zero. Eqs. (5.215) projected onto the azimuthally free axes in the case of slow motion can thus be written in the greatly simplified form

$$\left.\begin{aligned} \delta\ddot{x} + k_1\delta\dot{x} + (1+k_2)\omega_0^2\delta x &= f_x(t), \\ \delta\ddot{y} + k_1\delta\dot{y} + (1+k_2)\omega_0^2\delta y &= f_y(t), \end{aligned}\right\} \tag{5.311}$$

where to first approximations we may take

$$\left.\begin{aligned} f_x(t) &= (1+k_2)\Delta n_x - r\Delta\dot{m}_y + k_1(\Delta V_{Dx} - r\Delta m_y) - k_2\Delta\dot{V}_{Dx}, \\ f_y(t) &= (1+k_2)\Delta n_y + r\Delta\dot{m}_x + k_1(\Delta V_{Dy} + r\Delta m_x) - k_2\Delta\dot{V}_{Dy}; \end{aligned}\right\} \tag{5.312}$$

these functions are obtained from the right-hand sides (5.216) of Eqs. (5.215) if we put $\omega_x = \omega_y = u_x = u_y = u_z = 0$.

Eqs. (5.311) are the simplified form of Eqs. (5.215), which in their turn are the projections of the vector equation (5.201) on the horizontal axes x, y.

We can similarly obtain the simplified equations corresponding to the vector equation (5.208) of a two-accelerometer inertial system. They have the form

$$\left.\begin{aligned}\delta\ddot{x}+k_1'\omega_0^2\delta\dot{x}+\omega_0^2(1+k_2)\delta x=f_x',\\ \delta\ddot{y}+k_1'\omega_0^2\delta\dot{y}+\omega_0^2(1+k_2)\delta y=f_y',\end{aligned}\right\}\qquad(5.313)$$

where

$$\left.\begin{aligned}f_x'=\Delta n_x(1+k_2)+k_1'\Delta\dot{n}_x-r\,\Delta\dot{m}_y-k_2\,\Delta\dot{V}_{Dx}-k_1'\,\Delta\ddot{V}_{Dx},\\ f_y'=\Delta n_y(1+k_2)+k_1'\Delta\dot{n}_y+r\,\Delta\dot{m}_x-k_2\,\Delta\dot{V}_{Dy}-k_1'\,\Delta\ddot{V}_{Dy},\end{aligned}\right\}\qquad(5.314)$$

For $k_1'\omega_0^2=k_1$, the homogeneous equations (5.311) and (5.313) coincide. The roots of the corresponding characteristic equations are (5.295b). The solution of Eqs. (5.311), (5.313) is written in the form (5.298), where δr is replaced with δx or δy, respectively, and the vector function $f(\tau)$ is represented by the scalar functions $f_x(\tau)$, $f_y(\tau)$, $f_x'(\tau)$, $f_y'(\tau)$.

Note that the functions f_x, f_y and f_x', f_y' differ only in terms which contain the factors k_1 and k_1'. The terms with k_2 are identical in both cases. The difference in the solutions of Eqs. (5.311) and (5.313) is associated with this dissimilarity in the functions f_x, f_y, and f_x', f_y'.

Eqs. (5.297), (5.308), (5.311), (5.313) all have the same characteristic equation

$$p^2+k_1p+(1+k_2)\omega_0^2=0\qquad(5.315)$$

with the roots

$$p_{1,2}=-\varepsilon\pm j\nu,\quad \varepsilon=\frac{k_1}{2},\quad \nu=\frac{1}{2}\sqrt{4\omega_0^2(1+k_2)-k_1^2}.\qquad(5.316)$$

The roots (5.316) of the characteristic equation (5.315) constitute one of the main computational parameters of an inertial system with Doppler correction. The following table characterizes the values of these roots as

	r, m n →	10	20	30	40	50	60
t_c, sec	k_1	k_2	k_2	k_2	k_2	k_2	k_2
1	2	649070 1.000	649017 1.000	649007 1.000	649004 1.000	649002 1.000	649001 1.000
10	0.2	6560.3 0.995	6507.0 0.999	6497.0 1.000	6494.0 1.000	6492.0 1.000	6491.0 1.000
100	0.02	135.2 0.691	81.87 0.886	76.86 0.914	68.39 0.968	66.78 0.979	65.92 0.985
200	0.01	86.5 0.431	33.17 0.689	23.16 0.819	19.69 0.886	18.08 0.922	17.21 0.944
300	0.0067	77.6 0.304	24.23 0.538	14.21 0.692	10.74 0.787	9.14 0.849	8.27 0.888
400	0.0050	74.32 0.233	21.00 0.431	10.98 0.534	7.51 0.692	5.91 0.768	5.04 0.821
500	0.0040	72.86 0.187	19.54 0.355	9.52 0.497	6.05 0.605	4.45 0.691	3.58 0.752
600	0.0033	72.03 0.156	18.71 0.300	8.69 0.428	5.22 0.534	3.62 0.619	2.75 0.686
700	0.0029	71.59 0.137	18.27 0.267	8.25 0.385	4.78 0.488	3.18 0.574	2.31 0.643
800	0.0025	71.28 0.119	17.95 0.232	7.94 0.338	4.47 0.432	2.86 0.515	2.00 0.584
900	0.0022	71.07 0.105	17.74 0.206	7.72 0.302	4.26 0.387	2.65 0.466	1.79 0.533
1000	0.002	70.59 0.016	17.60 0.188	7.57 0.276	4.11 0.358	2.50 0.433	1.63 0.500

a function of the correction coefficients k_1 and k_2. The table lists the values of the time constant

$$t_c = \frac{1}{\varepsilon} = \frac{2}{k_1} \tag{5.317}$$

and the period of oscillation

$$T = \frac{2\pi}{\nu} = \frac{4\pi}{\sqrt{4\omega_0^2(1+k_2) - k_1^2}}. \tag{5.318}$$

The second row under k_2 gives the degree of damping

$$\zeta = \frac{k_1}{2\omega_0\sqrt{1+k_2}}. \tag{5.319}$$

The choice of the coefficients k_1 and k_2 is intended to ensure sufficiently effective damping, i.e., a sufficiently small time constant t_c with minimum contribution from instrumental errors. The period T generally should be made as large as possible, so as to preserve the low-frequency filtering properties of the system.

In conclusion note that the choice of the coefficients k_1 and k_2 using the above table is practicable only in the region where the simplified solutions (5.297), (5.308), (5.311), (5.313) are valid, i.e., in a region where conditions of the form (5.293) are satisfied. Otherwise, for a three-accelerometer system we have to consider Eqs. (5.237), (5.243), and not the simplified characteristic equation (5.315). In two-accelerometer systems we should introduce additional stability conditions of the form (5.288) for the total error equations.

The actual choice of the correction coefficients k_1 and k_2 is generally governed by the random character of the instrumental errors and primarily the random character of the Doppler errors. The effect of random errors is considered in Chapter 7.

§5.6. SOME ADDITIONAL REMARKS ON DOPPLER CORRECTION

5.6.1. Correction techniques using a gyroscopically stabilized platform

Consider a two-accelerometer inertial system using a gyroscopically stabilized platform (or a controlled gyroframe). The unperturbed axis z of the platform is directed along the radius-vector r, and the accelerometers are set along the axes x and y, which in the unperturbed state lie in the plane of the geocentric horizon. We will consider the case of an orthodromic system, without loss of generality. The ideal operation equations of the system without Doppler correction are Eqs. (2.23).

The alternative Doppler correction techniques considered in the previous sections were based on Eqs. (5.196)—(5.199) derived for an arbitrary orientation of the inertial platform. In application to an orthodromic system, Eqs. (5.196)—(5.199) affected only the first four equations in (2.23). The

other equations did not change, as we see, e.g., from (5.217). In particular, the fifth and the sixth equation in (2.23) used in determining the coordinates S and z from known ω_x and ω_y were not affected. The eleventh, twelfth, and thirteenth equations in (2.23) did not change either; these equations are used in forming the control torques of the stable platform (gyroframe).

The fifth, sixth, eleventh, twelfth, and thirteenth equations in (2.23) nevertheless contain the projections ω_x, ω_y of the absolute angular velocity of the platform axes xyz. Since xyz are co-moving axes, i.e., the unperturbed axis z points along r, ω_x and ω_y can be expressed in terms of V_x, V_y, u_x, u_y. Clearly, V_x and V_y are provided by the Doppler velocity meter. We thus have a possibility of Doppler correction if we introduce additional terms with ω_x, ω_y in the equations for S and z and in the equations for the control torques. Let us examine this possibility in some detail. For ω_x and ω_y we take

$$\omega_x - k_2 \frac{\Delta V_y}{r}, \quad \omega_y + k_2 \frac{\Delta V_x}{r}, \tag{5.320}$$

where ΔV_x and ΔV_y are given by the last four equalities in (5.217). Inserting these expressions for ω_x and ω_y in the equations for the control torques, we obtain

$$\left.\begin{aligned} M_y^1 &= -H_2\left(\omega_x - k_2 \frac{\Delta V_y}{r}\right), \\ M_x^1 &= H_1\left(\omega_y + k_2 \frac{\Delta V_x}{r}\right), \\ M_x^2 &= H_3\left[\left(\omega_y + \frac{k_2 \Delta V_x}{r}\right) \operatorname{tg} z + \right. \\ &\qquad \left. + \frac{u}{\cos z}(\beta_{31} \cos S + \beta_{32} \sin S)\right]. \end{aligned}\right\} \tag{5.321}$$

The first two equations in (2.23) are replaced with Eqs. (5.49) using (5.50). The other equations in (2.23) are not changed.

Let us derive the error equations for this case. Taking the variations of Eq. (5.49) together with the third and fourth equations in (2.23), dropping the small variations δg_x, δg_y, and putting $\delta r = \Delta r$, we get

$$\left.\begin{aligned} 2\omega_y \Delta \dot{r} + \dot{r}\,\delta\omega_y + (\dot{\omega}_y + \omega_x\omega_z)\Delta r + \frac{d}{dt}(r\,\delta\omega_y) &= \\ = \delta n_x - r\omega_z \delta\omega_x - r\omega_x \delta\omega_z - k_1 \delta\,\Delta V_x, & \\ -2\omega_x \Delta \dot{r} - \dot{r}\,\delta\omega_x + (-\dot{\omega}_x + \omega_y\omega_z)\Delta r - \frac{d}{dt}(r\,\delta\omega_x) &= \\ = \delta n_y - r\omega_y \delta\omega_z - r\omega_z \delta\omega_y - k_1 \delta\,\Delta V_y & \end{aligned}\right\} \tag{5.322}$$

The unperturbed position of the axes xyz fixed to the platform is designated as before by $x_0 y_0 z_0$. The deviations of the platform from its unperturbed position are specified by the small angles α, β, γ according to the matrix of direction cosines (4.22). Then δn_{x_0}, δn_{y_0} are given by Eqs. (4.26), (4.27), (4.7), (4.8), (4.12).

Eqs. (5.322) contain the variations $\delta\omega_{x_0}$, $\delta\omega_{y_0}$, $\delta\omega_{z_0}$. They should be expressed in terms of $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$, α, β, γ. Relations (4.24) previously derived for this purpose are inapplicable in this case, since we changed the expressions for the control torques (replacing (2.23) with (5.321)). From (5.321) and (4.23) we thus obtain the new relations

$$\left.\delta\omega_{x_0} = \dot{\alpha} + \omega_{y_0}\gamma - \omega_{z_0}\beta - \Delta\omega_{x_0} + k_2 \frac{\delta\,\Delta V_{y_0}}{r}, \right\} \tag{5.323}$$

$$\left.\begin{aligned}\delta\omega_{y_0} &= \dot{\beta} - \omega_{x_0}\gamma + \omega_{z_0}\alpha - \Delta\omega_{y_0} - k_2\frac{\delta\Delta V_{x_0}}{r},\\ \delta\omega_{z_0} &= \dot{\gamma} + \omega_{x_0}\beta - \omega_{y_0}\alpha - \Delta\omega_{z_0} - k_2\frac{\delta\Delta V_{x_0}}{r}\operatorname{tg} z.\end{aligned}\right\} \tag{5.323}$$

Here, as in (5.322), $\delta\Delta V_{x_0}$, $\delta\Delta V_{y_0}$ are given by Eqs. (5.58).

From the third and the fourth equations in (2.23) we have

$$\left.\begin{aligned}\delta v_{y_0} &= -\Delta r\omega_{x_0} - r\,\delta\omega_{x_0},\\ \delta v_{x_0} &= \Delta r\omega_{y_0} + r\,\delta\omega_{y_0}.\end{aligned}\right\} \tag{5.324}$$

Inserting these expressions for δv_{x_0} and δv_{y_0} in the right-hand sides of Eqs. (5.58) and expressing θ_{x_0}, θ_{y_0}, θ_{z_0} in terms of α, β, γ and α_2, β_2, γ_2 from (4.39), we obtain

$$\left.\begin{aligned}\delta\Delta V_{x_0} &= r\,\delta\omega_{y_0} + \omega_{y_0}\Delta r + \rho\left(-u_{z_0}\alpha_2 + u_{x_0}\gamma_2\right) +\\ &\quad + r\left(\omega_{x_0} - \frac{\rho}{r}u_{x_0}\right)\gamma + \dot{r}\beta - \Delta V_{Dx_0},\\ \delta\Delta V_{y_0} &= -r\,\delta\omega_{x_0} - \omega_{x_0}\Delta r - \rho\left(u_{z_0}\beta_2 - u_{y_0}\gamma_2\right) +\\ &\quad + r\left(\omega_{y_0} - \frac{\rho}{r}u_{y_0}\right)\gamma - \dot{r}\alpha - \Delta V_{Dy_0}.\end{aligned}\right\} \tag{5.325}$$

From (5.323) and (5.325) we obtain the following expressions for $\delta\omega_{x_0}$, $\delta\omega_{y_0}$, $\delta\omega_{z_0}$:

$$\left.\begin{aligned}\delta\omega_{x_0} &= \frac{1}{1+k_2}\left\{\dot{\alpha} + \omega_{y_0}\gamma - \omega_{z_0}\beta - \Delta\omega_{x_0} +\right.\\ &\quad + \frac{k_2}{r}\left[-\omega_{x_0}\Delta r - \rho\left(u_{z_0}\beta_2 - u_{y_0}\gamma_2\right) +\right.\\ &\quad \left.\left. + \left(r\omega_{y_0} - \rho u_{y_0}\right)\gamma - \dot{r}\alpha - \Delta V_{Dy_0}\right]\right\},\\ \delta\omega_{y_0} &= \frac{1}{1+k_2}\left\{\dot{\beta} - \omega_{x_0}\gamma + \omega_{z_0}\alpha - \Delta\omega_{y_0} -\right.\\ &\quad - \frac{k_2}{r}\left[\omega_{y_0}\Delta r + \rho\left(-u_{z_0}\alpha_2 + u_{x_0}\gamma_2\right) +\right.\\ &\quad \left.\left. + \left(r\omega_{x_0} - \rho u_{x_0}\right)\gamma + \dot{r}\beta - \Delta V_{Dx_0}\right]\right\},\\ \delta\omega_{z_0} &= \dot{\gamma} + \omega_{z_0}\beta - \omega_{y_0}\alpha - \Delta\omega_{z_0} -\\ &\quad - \frac{k_2\operatorname{tg} z}{1+k_2}\left\{\dot{\beta} + \omega_{z_0}\alpha - \Delta\omega_{y_0} + \frac{\Delta r}{r}\omega_{y_0} +\right.\\ &\quad \left. + \frac{\rho}{r}\left(-u_{z_0}\alpha_2 + u_{x_0}\gamma_2\right) - \frac{\rho}{r}u_{x_0}\gamma + \frac{\dot{r}}{r}\beta - \frac{\Delta V_{Dx_0}}{r}\right\}.\end{aligned}\right\} \tag{5.326}$$

These expressions should now be inserted for $\delta\omega_{x_0}$, $\delta\omega_{y_0}$, $\delta\omega_{z_0}$ in Eqs. (5.322). For $\delta\Delta V_{x_0}$, $\delta\Delta V_{y_0}$ in these equations we insert (5.325), having first changed $\delta\omega_{x_0}$, $\delta\omega_{y_0}$ in accordance with (5.326). In Eqs. (5.322) we should also introduce δn_{x_0}, δn_{y_0} from Eqs. (4.26), (4.27), (4.7), (4.8), (4.12). The final result is the first group of error equations of the system being considered.

Let us now turn to the second group of error equations. We should stress that in all the preceding cases of correction that we have so far treated in this book, the second group of error equations did not change, retaining its original form as derived for autonomous inertial systems. In this case, the situation is different. This is a consequence of our additional interference in the formation of the control torques of the stable platform.

Varying the fifth and the sixth equations in (2.23), we get

$$\left.\begin{aligned} \delta\dot{z} &= -\delta\omega_x - u(-\beta_{11}\cos S + \beta_{12}\sin S)\delta S, \\ \delta\dot{S}\cos z &= \delta\omega_y + \left[\omega_y \operatorname{tg} z + \frac{u}{\cos z}(\beta_{11}\cos S + \beta_{12}\sin S)\right]\delta z + \\ &\quad + u\sin z(-\beta_{11}\sin S + \beta_{12}\cos S)\delta S, \end{aligned}\right\} \tag{5.327}$$

where the variations $\delta\omega_x$, $\delta\omega_y$ are expressed by (5.326). Making the substitution

$$\alpha_2 = -\delta z, \quad \beta_2 = \delta S\cos z, \quad \gamma_2 = \delta S\sin z \tag{5.328}$$

and using the seventh equality in (2.23) and Eqs. (4.15), (4.16), we get

$$\left.\begin{aligned} \dot{\alpha}_2 + \omega_y\gamma_2 - \omega_z\beta_2 &= \delta\omega_x, \\ \dot{\beta}_2 + \omega_z\alpha_2 - \omega_x\gamma_2 &= \delta\omega_y, \\ \dot{\gamma}_2 + \omega_x\beta_2 - \omega_y\alpha_2 &= \delta\omega_z. \end{aligned}\right\} \tag{5.329}$$

Together with (5.326), Eqs. (5.329) constitute the second group of error equations.

After the necessary manipulations, we can show that the substitution of variables (4.39) does not reduce Eqs. (5.329), (5.326) to Eqs. (4.40). The second group of error equations is thus radically different in this case from the corresponding equations of an autonomous inertial system. The second group of error equations in general cannot be separated from the first group (5.322), (5.325), (5.326). In our case, the first and second group equations constitute a seventh-order coupled system of differential equations.

In the following we will only consider the case of small ω_{x_0}, ω_{y_0}, ω_{z_0}. To simplify the treatment, we take $\rho = r = \text{const}$ and $z = 0$. Dropping the terms with ω_x, ω_y, ω_z, u_x, u_y, u_z and their derivatives, we obtain instead of Eqs. (5.325)

$$\left.\begin{aligned} \delta\Delta V_x &= r\,\delta\omega_y - \Delta V_{Dx}, \\ \delta\Delta V_y &= -r\,\delta\omega_x - \Delta V_{Dy}. \end{aligned}\right\} \tag{5.330}$$

Eqs. (5.326) are now replaced by the simpler equalities

$$\left.\begin{aligned} \delta\omega_x &= \frac{1}{1+k_2}(\dot{\alpha} - \Delta\omega_x) - \frac{k_2}{r(1+k_2)}\Delta V_{Dy}, \\ \delta\omega_y &= \frac{1}{1+k_2}(\dot{\beta} - \Delta\omega_y) + \frac{k_2}{r(1+k_2)}\Delta V_{Dx}, \\ \delta\omega_z &= \dot{\gamma} - \Delta\omega_z. \end{aligned}\right\} \tag{5.331}$$

Eqs. (5.329) are similarly simplified:

$$\dot{\alpha}_2 = \delta\omega_x, \quad \dot{\beta}_2 = \delta\omega_y, \quad \dot{\gamma}_2 = \delta\omega_z. \tag{5.332}$$

Eqs. (5.322) can be written in the form

$$\left.\begin{aligned} \frac{d}{dt}(r\,\delta\omega_y) &= \delta n_x - k_1\delta\Delta V_x, \\ -\frac{d}{dt}(r\,\delta\omega_x) &= \delta n_y - k_1\delta\Delta V_y, \end{aligned}\right\} \tag{5.333}$$

where, by (4.26),

$$\delta n_{x_0} = -\omega_0^2 r\beta + \Delta n_{x_0}, \quad \delta n_{y_0} = \omega_0^2 r\alpha + \Delta n_{y_0}, \tag{5.334}$$

writing $\omega_0^2 = \mu/r^3$, as usually.

Inserting for δn_{x_0}, δn_{y_0}, $\delta\omega_{x_0}$, $\delta\omega_{y_0}$, $\delta\Delta V_{x_0}$, $\delta\Delta V_{y_0}$ in Eqs. (5.333) their expressions from (5.334), (5.331), (5.330), and making the substitution $\delta x = r\beta$, $\delta y = -r\alpha$, we obtain

$$\left.\begin{aligned} &\delta\ddot{x} + k_1\delta\dot{x} + (1+k_2)\omega_0^2\delta x = \\ &\quad = (1+k_2)\Delta n_{x_0} + r\,\Delta\dot{\omega}_{y_0} - k_2\Delta\dot{V}_{Dx_0} + \\ &\qquad + k_1(r\,\Delta\omega_{y_0} + \Delta V_{Dx_0}), \\ &\delta\ddot{y} + k_1\delta\dot{y} + (1+k_2)\omega_0^2\delta y = \\ &\quad = (1+k_2)\Delta n_{y_0} - r\,\Delta\dot{\omega}_{x_0} - k_2\Delta\dot{V}_{Dy_0} + \\ &\qquad + k_1(-r\,\Delta\omega_{x_0} + \Delta V_{Dy_0}). \end{aligned}\right\} \tag{5.335}$$

These equations clearly coincide with Eqs. (5.311), since $\Delta\omega_{x_0} = -\Delta m_{x_0}$, $\Delta\omega_{y_0} = -\Delta m_{y_0}$, $\Delta\omega_{z_0} = -\Delta m_{z_0}$.

Let us now turn to the second group of equations, Eqs. (5.332). Substituting

$$\left.\begin{aligned} \vartheta_{x_0} &= (1+k_2)\alpha_2 - \alpha, \quad \vartheta_{y_0} = (1+k_2)\beta_2 - \beta, \\ \vartheta_{z_0} &= \gamma_2 - \gamma, \end{aligned}\right\} \tag{5.336}$$

we get

$$\left.\begin{aligned} \dot{\vartheta}_{x_0} &= -\Delta\omega_{x_0} - k_2\frac{\Delta V_{Dy_0}}{r}, \\ \dot{\vartheta}_{y_0} &= -\Delta\omega_{y_0} + k_2\frac{\Delta V_{Dx_0}}{r}, \\ \dot{\vartheta}_{z_0} &= -\Delta\omega_{z_0}. \end{aligned}\right\} \tag{5.337}$$

In all the previous cases, the second-group error equations (after the relevant simplifications) had the form

$$\left.\begin{aligned} \dot{\theta}_{x_0} &= -\Delta\omega_{z_0}, \quad \theta_{y_0} = -\Delta\omega_{y_0}, \\ \dot{\theta}_{z_0} &= -\Delta\omega_{z_0}. \end{aligned}\right\} \tag{5.338}$$

Eqs. (5.337) differ from Eqs. (5.338) not only in form but primarily in the meaning of the variables. In Eqs. (5.338) the total position errors δx_2, δy_2 are calculated from the equalities

$$\delta x_2 = \delta x + \theta_{y_0} r, \quad \delta y_2 = \delta y - \theta_{x_0} r, \tag{5.339}$$

whereas in Eqs. (5.337) the total errors are given by

$$\left.\begin{aligned} \delta x_2 &= r\beta_2 = (r\vartheta_{y_0} + \delta x)\frac{1}{1+k_2}, \\ \delta y_2 &= -r\alpha_2 = (-r\vartheta_{x_0} + \delta y)\frac{1}{1+k_2}. \end{aligned}\right\} \tag{5.340}$$

The above difference suggests a comparison of this system with the principal Doppler correction technique, that based on Eqs. (5.217).

In case (5.217), the complete system of error equations includes Eqs. (5.311), (5.338), and (5.339), which are supplemented by defining

equations for platform orientation errors

$$\theta_{1x} = -\frac{\delta y}{r}, \quad \theta_{1y} = \frac{\delta x}{r}, \quad \theta_{1z} = -\theta_z. \tag{5.341}$$

The error equations of the system being considered, on the other hand, are Eqs. (5.335), (5.337), (5.340) and the equalities

$$\theta_{2x} = -\frac{\delta y}{r}, \quad \theta_{2y} = \frac{\delta x}{r}, \quad \theta_{2z} = -\theta_{z_2}. \tag{5.342}$$

From the viewpoint of orientation errors (deviation of the platform z axis from the geocentric vertical and azimuthal errors), the two systems are equivalent.

Indeed, Eqs. (5.335) coincide with Eqs. (5.311), and (5.341) and (5.342) also coincide, since

$$\theta_{z_2} = \theta_z. \tag{5.343}$$

The last equality is apparent from a comparison of the third equation in (5.338) with the third equation in (5.337).*

The total position errors, on the other hand, are different for the two systems, even if we assume equal instrumental errors in both cases.

Indeed, let $\Delta\omega_x, \Delta\omega_{y_2}, \Delta\omega_{z_2}, \Delta V_{Dx_2}, \Delta V_{Dy_2}$ be constant. From Eqs. (5.337), assuming zero initial conditions, we then get

$$\left.\begin{aligned} \theta_{x_2} &= -\left(\Delta\omega_{x_2} + k_2\frac{\Delta V_{Dy_2}}{r}\right)t, \\ \theta_{y_2} &= \left(-\Delta\omega_{y_2} + k_2\frac{\Delta V_{Dx_2}}{r}\right)t. \end{aligned}\right\} \tag{5.344}$$

From (5.338) we have

$$\theta_{x_2} = -\Delta\omega_x t, \quad \theta_{y_2} = -\Delta\omega_y t, \quad \theta_{z_2} = -\Delta\omega_z t. \tag{5.345}$$

On the other hand, if the accelerometer errors $\Delta n_{x_2}, \Delta n_y$ are also constant, we have from (5.335) the steady-state position errors

$$\left.\begin{aligned} \delta x &= \frac{\Delta n_{x_2}}{\omega_0^2} + \frac{k_1}{\omega_0^2(1+k_2)}(r\,\Delta\omega_{y_2} + \Delta V_{Dx_2}), \\ \delta y &= \frac{\Delta n_{y_2}}{\omega_0^2} + \frac{k_1}{\omega_0^2(1+k_2)}(-r\,\Delta\omega_{x_2} + \Delta V_{Dy_2}). \end{aligned}\right\} \tag{5.346}$$

From Eqs. (5.339), (5.345), (5.346) we now have

$$\left.\begin{aligned} \delta x_2 &= \frac{\Delta n_{x_2}}{\omega_0^2} + \frac{k_1}{\omega_0^2(1+k_2)}(r\,\Delta\omega_{y_2} + \Delta V_{Dx_2}) - r\,\Delta\omega_{y_2}t, \\ \delta y_2 &= \frac{\Delta n_y}{\omega_0^2} + \frac{k_1}{\omega_0^2(1+k_2)}(-r\,\Delta\omega_{x_2} + \Delta V_{Dy_2}) + r\,\Delta\omega_{x_2}t. \end{aligned}\right\} \tag{5.347}$$

* This is a comparison between differential equations. Strictly speaking, the initial conditions should also be compared. In our case they are readily seen to coincide.

From Eqs. (5.340), (5.344), (5.346) we correspondingly get

$$\left.\begin{aligned}
\delta x_2 &= \frac{1}{1+k_2}(-r\,\Delta\omega_{y_0}+k_2\,\Delta V_{Dx_0})\,t + \\
&+\frac{1}{1+k_2}\left[\frac{\Delta n_{x_0}}{\omega_0^2}+\frac{k_1}{\omega_0^2(1+k_2)}(r\,\Delta\omega_{y_0}+\Delta V_{Dx_0})\right]. \\
\delta y_2 &= \frac{1}{1+k_2}(r\,\Delta\omega_{x_0}+k_2\,\Delta V_{Dy_0})\,t + \\
&+\frac{1}{1+k_2}\left[\frac{\Delta n_{y_0}}{\omega_0^2}+\frac{k_1}{\omega_0^2(1+k_2)}(-r\,\Delta\omega_{x_0}+\Delta V_{Dy_0})\right].
\end{aligned}\right\} \tag{5.348}$$

Comparison of the right-hand sides of Eqs. (5.347) and (5.348) shows that the constant error components in Eqs. (5.348) are a factor of $(1+k_2)$ less than those in Eqs. (5.347). The time-proportional error components are also substantially different. In case (5.347) we have

$$\delta x_2' = -r\,\Delta\omega_{y_0}t, \quad \delta y_2' = r\,\Delta\omega_{x_0}t, \tag{5.349}$$

and in case (5.348)

$$\left.\begin{aligned}
\delta x_2' &= \frac{1}{1+k_2}(-r\,\Delta\omega_{y_0}+k_2\,\Delta V_{Dx_0})\,t, \\
\delta y_2' &= \frac{1}{1+k_2}(r\,\Delta\omega_{x_0}+k_2\,\Delta V_{Dy_0})\,t.
\end{aligned}\right\} \tag{5.350}$$

Since usually $k_2 \gg 1$, so that $k_2/(1+k_2) \approx 1$, and furthermore

$$\left.\begin{aligned}
\frac{1}{1+k_2}|r\,\Delta\omega_{y_0}| &\ll |\Delta V_{Dx_0}|, \\
\frac{1}{1+k_2}|r\,\Delta\omega_{x_0}| &\ll |\Delta V_{Dy_0}|,
\end{aligned}\right\} \tag{5.351}$$

Eqs. (5.350) are correspondingly replaced by the approximate equalities

$$\delta x_2' = \Delta V_{Dx_0}t, \quad \delta y_2' = \Delta V_{Dy_0}t. \tag{5.352}$$

Comparing (5.349) and (5.352) we see that for

$$|r\,\Delta\omega_{y_0}| > |\Delta V_{Dx_0}|, \quad |r\,\Delta\omega_{x_0}| > |\Delta V_{Dy_0}| \tag{5.353}$$

the Doppler correction technique using torques (5.321) gives smaller errors. Otherwise, the system using correction (5.217) is more advantageous. Note that this comparison is carried out for constant instrumental errors. Random errors naturally require a special analysis, which will be made later on.

Note that for $k_2 \gg 1$ the Doppler correction technique changing the control torques is fundamentally equivalent (as we see, in particular, from (5.352)) to integration of the vector V_D measured by the Doppler velocity meter. The role of the inertial system here is mainly to maintain the platform orientation in the plane of the horizon and relative to the meridian, so that the Doppler indeed measures the velocity we expect it to measure.

The above case of Doppler correction is based, as we remember, on the following alteration of Eqs. (2.23). The first two equations are replaced by Eqs. (5.49), (5.50), and the control torques are formed according to (5.321). The expressions for the torques differ from the corresponding expressions in (2.23) in that ω_x, ω_y are replaced by (5.320).

However, the projections ω_x, ω_y of the absolute angular velocity of the platform axes xyz on the corresponding directions enter other equations in (2.23), as well as those from which the control torques are formed. They also enter the fifth and the sixth equations in (2.23), which give the coordinates S and z.

In these equations ω_x, ω_y can also be replaced using (5.320). The control torques are either left in the same form as in Eqs. (2.23) or they are altered in accordance with (5.321). It is readily seen that the first alternative does not affect the natural frequencies of the inertial system and is therefore of no significance. Much more interesting is the second alternative, in which ω_x, ω_y are expressed according to (5.320) in the equations for z and S, as well as in the expressions for the control torques.

The fifth and the sixth equations in (2.23) thus give

$$\left.\begin{aligned} S &= \int_0^t \left[\frac{1}{\cos z}\left(\omega_{y_2} + k_2\frac{\Delta V_x}{r}\right) - \frac{u}{\cos z}(\beta_{31}\cos z - \right. \\ &\qquad \left. - \beta_{11}\sin z\cos S - \beta_{32}\sin z\sin S)\right] dt + S^0, \\ z &= \int_0^t \left[-\omega_x + k_2\frac{\Delta V_{y_2}}{r} + \right. \\ &\qquad \left. + u(-\beta_{31}\sin S + \beta_{32}\cos S)\right] dt + z^0. \end{aligned}\right\} \tag{5.354}$$

Let us consider the error equations. Eqs. (5.322), (5.325), (5.326) clearly retain their original form. Eqs. (5.329) are changed as follows:

$$\left.\begin{aligned} \dot{\alpha}_2 - \omega_y\gamma_2 - \omega_z\beta_2 &= \delta\omega_x - \frac{k_2}{r}\delta\Delta V_{y_2}, \\ \dot{\beta}_2 + \omega_z\alpha_2 - \omega_x\gamma_2 &= \delta\omega_{y_2} + \frac{k_2}{r}\delta\Delta V_{x_2}, \\ \dot{\gamma}_2 + \omega_x\beta_2 - \omega_y\gamma_2 &= \delta\omega_{z_2} - \frac{k_2}{r}\delta\Delta V_x\operatorname{tg} z. \end{aligned}\right\} \tag{5.355}$$

If we take $r = \text{const}$, $z = 0$, $\omega_{x_2} = \omega_{y_2} = \omega_{z_2} = 0$, we obtain Eqs. (5.335) for δx and δy and Eqs. (5.355) change over to Eqs. (5.338) (if we use (4.43) and (4.44)). This correction technique is thus dynamically equivalent to the previous alternative, described by Eqs. (5.217).

5.6.2. Doppler correction increasing the order of the error equations. Linear correction with variable coefficients. Nonlinear correction

Consider an inertial system which receives the Doppler information in the following way. The first two equations in (2.23) are replaced by Eqs. (5.49), the control torques are formed using the equalities (k_3 is some constant coefficient)

$$\left.\begin{aligned} M_y^1 &= -H_2\left(\omega_{x_2} - \frac{k_2\Delta V_{y_2}}{r} - k_2\int_0^t \frac{\Delta V_{y_2}}{r}\,dt\right), \\ M_x^1 &= H_1\left(\omega_{y_2} + \frac{k_2}{r}\Delta V_{x_2} + k_3\int_0^t \frac{\Delta V_{x_2}}{r}\,dt\right), \end{aligned}\right\} \tag{5.356}$$

$$\left.\begin{aligned} M_x^3 = H_3\Big[\omega_{y_0} + \frac{k_2}{r}\Delta V_{x_0} + \\ + k_3\int_0^t \frac{\Delta V_{x_0}}{r}\,dt + \frac{u}{\cos z}(\beta_{31}\cos S + \beta_{32}\sin S)\Big], \end{aligned}\right\} \tag{5.356}$$

and the other equations in (2.23) remain without change.

This is a further development of the correction technique considered in the previous subsection. Eqs. (5.356) for the control torques differ from Eqs. (5.321) only in that the right-hand sides in (5.356) contain integrals over ΔV_{x_0} and ΔV_{y_0}.

Let us derive the error equations for this system. For simplicity we take ω_{x_0}, ω_{y_0}, ω_{z_0}, u_{x_0}, u_{y_0}, u_{z_0} to be zero, $r = \text{const}$, $z = 0$. From (5.49) we then have

$$\left.\begin{aligned} \frac{d}{dt}(r\,\delta\omega_{y_0}) = \delta n_{x_0} - k_1\,\delta\Delta V_{x_0}, \\ -\frac{d}{dt}(r\,\delta\omega_{x_0}) = \delta n_{y_0} - k_1\,\delta\Delta V_{y_0}, \end{aligned}\right\} \tag{5.357}$$

and these equations coincide with Eqs. (5.333).

Eqs. (5.323) are replaced by

$$\left.\begin{aligned} \delta\omega_{x_0} = \dot\alpha - \Delta\omega_{x_0} + \frac{k_2}{r}\delta\Delta V_{y_0} + \frac{k_3}{r}\int_0^t \delta\,\Delta V_{y_0}\,dt, \\ \delta\omega_{y_0} = \dot\beta - \Delta\omega_{y_0} - \frac{k_2}{r}\,\delta\,\Delta V_{x_0} - \frac{k_3}{r}\int_0^t \delta\,\Delta V_{x_0}\,dt, \\ \delta\omega_z = \dot\gamma - \Delta\omega_{z_0}. \end{aligned}\right\} \tag{5.358}$$

For $\delta\Delta V_{x_0}$, $\delta\Delta V_{y_0}$, δn_{x_0}, and δn_{y_0}, Eqs. (5.330), (5.334) remain in force.

Inserting (5.330), (5.334) in Eqs. (5.357) and solving for $\delta\dot\omega_{x_0}$, $\delta\dot\omega_{y_0}$, we obtain

$$\left.\begin{aligned} \delta\dot\omega_{x_0} = -\omega_0^2\alpha - \frac{\Delta n_{y_0}}{r} + k_1\Big(\delta\omega_{x_0} + \frac{\Delta V_{Dy_0}}{r}\Big), \\ \delta\dot\omega_{y_0} = -\omega_0^2\beta + \frac{\Delta n_{x_0}}{r} - k_1\Big(\delta\omega_{y_0} - \frac{\Delta V_{Dx_0}}{r}\Big). \end{aligned}\right\} \tag{5.359}$$

On the other hand, inserting $\delta\Delta V_{x_0}$, $\delta\Delta V_{y_0}$ from (5.330) in the first two equalities in (5.358) and differentiating, we obtain

$$\left.\begin{aligned} \delta\dot\omega_{x_0} = \frac{1}{1+k_2}\Big[\ddot\alpha - \Delta\dot\omega_{x_0} - \frac{k_2}{r}\Delta\dot V_{Dy_0} + \frac{k_3}{r}(-r\,\delta\omega_{x_0} - \Delta V_{Dy_0})\Big], \\ \delta\dot\omega_{y_0} = \frac{1}{1+k_2}\Big[\ddot\beta - \Delta\dot\omega_{y_0} + \frac{k_2}{r}\Delta\dot V_{Dx_0} - \frac{k_3}{r}(r\,\delta\omega_{y_0} - \Delta V_{Dx_0})\Big]. \end{aligned}\right\} \tag{5.360}$$

Equating the right-hand sides of (5.359) and (5.360) we obtain for $\delta\omega_{x_0}$ and $\delta\omega_{y_0}$

$$\left.\begin{aligned} \delta\omega_{x_0} = \frac{1+k_2}{k_1(1+k_2)-k_3}\Big(-\omega_0^2\alpha - \frac{\Delta n_{y_0}}{r}\Big) - \\ - \frac{1}{k_1(1+k_2)-k_3}\Big(\ddot\alpha - \Delta\dot\omega_{x_0} - \frac{k_2}{r}\Delta\dot V_{Dy_0}\Big) - \frac{\Delta V_{Dy_0}}{r}, \\ \delta\omega_{y_0} = \frac{1+k_2}{k_1(1+k_2)-k_3}\Big(-\omega_0^2\beta + \frac{\Delta n_{x_0}}{r}\Big) - \\ - \frac{1}{k_1(1+k_2)-k_3}\Big(\ddot\beta - \Delta\dot\omega_{y_0} + \frac{k_2}{r}\Delta\dot V_{Dx_0}\Big) + \frac{\Delta V_{Dx_0}}{r}. \end{aligned}\right\} \tag{5.361}$$

Inserting (5.361) in Eqs. (5.359) or (5.360), we make the substitution $\delta x = r\beta$, $\delta y = -r\alpha$. After obvious simplifications, we get

$$\left.\begin{aligned}
&\delta\dddot{x} + k_1\,\delta\ddot{x} + (1+k_2)\,\omega_0^2\,\delta\dot{x} + \omega_0^2 k_1\,\delta x = \\
&\quad = (1+k_2)\,\Delta\dot{n}_{x_2} + k_3\,\Delta n_x + r\,\Delta\ddot{\omega}_y + rk_1\,\Delta\dot{\omega}_y - \\
&\qquad - k_2\,\Delta\dot{V}_{Dx} + (k_1 - k_3)\,\Delta V_{Dx}, \\
&\delta\dddot{y} + k_1\,\delta\ddot{y} + (1+k_2)\,\omega_0^2\,\delta\dot{y} + \omega_0^2 k_3\,\delta y = \\
&\quad = (1+k_2)\,\Delta\dot{n}_{y_2} + k_3\,\Delta n_y - r\,\Delta\ddot{\omega}_x - \\
&\qquad - rk_1\,\Delta\dot{\omega}_{x_2} - k_2\,\Delta\dot{V}_{Dy_2} + (k_1 - k_3)\,\Delta V_{Dy}.
\end{aligned}\right\} \tag{5.362}$$

Eqs. (5.362) describe $\left(\alpha = -\frac{\delta y}{r},\ \beta = \frac{\delta x}{r}\right)$ oscillations of the platform z axis about the geocentric vertical. Eqs. (5.362) are third-order equations. In the previous treatment, analogous simplifications have invariably led to second-order equations: e.g., Eqs. (5.335), (5.311), (5.313). The increase in the order of the differential equations is associated with the introduction of the integrals over ΔV_x, ΔV_y in the expressions for the control torques. It is readily seen that for $k_3 = 0$ Eqs. (5.362) are reduced by integration to Eqs. (5.335). Similarly, Eqs. (5.361) for $k_3 = 0$ reduce to Eqs. (5.331). To show this, insert $\dot{\alpha}$ and $\dot{\beta}$ from (5.335) in Eqs. (5.361).

The characteristic equation corresponding to each of the two equations in (5.362) is a complete cubic equation

$$p^3 + k_1 p^2 + (1+k_2)\,\omega_0^2 p + \omega_0^2 k_3 = 0. \tag{5.363}$$

Since the coefficients k_1, k_2, k_3 are independent, they can always be chosen so that the characteristic equation has roots with negative real parts. The free oscillations of the platform are then damped, and only the forced part of the solution remains.

Comparing the right-hand sides of Eqs. (5.335) and (5.362) we see that they contain the same instrumental errors. The only difference is that Eqs. (5.362) contain higher order derivatives than Eqs. (5.335). If the instrumental errors Δn_{x_2}, Δn_{y_2}, $\Delta\omega_{x_2}$, $\Delta\omega_{y_2}$, ΔV_{Dx_2}, ΔV_{Dy_2} are constant, the steady-state errors δx and δy from Eqs. (5.362) are given by

$$\delta x = \frac{\Delta n_{x_2}}{\omega_0^2},\quad \delta y = \frac{\Delta n_{y_2}}{\omega_0^2}. \tag{5.364}$$

They depend only on the accelerometer errors. Unlike the solutions (5.346) of Eqs. (5.335), constant gyro errors (gyro drift) and Doppler errors do not lead to steady-state orientation errors of the z axis.

Let us examine the situation concerning the total position errors δx_2, δy_2. As before, we have

$$\delta x_2 = r\beta_2,\ \delta y_2 = -r\alpha_2, \tag{5.365}$$

where α_2 and β_2 are given by

$$\dot{\alpha}_2 = \delta\omega_{x_2},\ \dot{\beta}_2 = \delta\omega_{y_2}. \tag{5.366}$$

Thus,

$$\left.\begin{aligned}\delta x_2 &= r\int_0^t \delta\omega_{y_0}\,dt + \delta x_2^0,\\ \delta y_2 &= -r\int_0^t \delta\omega_{y_0}\,dt + \delta y_2^0.\end{aligned}\right\} \tag{5.367}$$

To evaluate the integrals on the right in (5.367), we transform Eqs. (5.361) for $\delta\omega_{x_0}$, $\delta\omega_{y_0}$. Inserting for α and β their expressions in terms of $\delta\dddot{x}$, $\delta\ddot{x}$, $\delta\dot{x}$, $\delta\dddot{y}$, $\delta\ddot{y}$, $\delta\dot{y}$ and expressing the instrumental errors from Eqs. (5.362), we obtain for constant instrumental errors

$$\left.\begin{aligned}\delta\omega_{x_0} &= \frac{1}{r[k_3 - k_1(1+k_2)]}\left[-\frac{1+k_2}{k_3}\delta\dddot{y} + \right.\\ &\left. + \frac{k_3 - k_1(1+k_2)}{k_3}\delta\ddot{y} - \frac{\omega_0^2(1+k_2)^2}{k_3}\delta\dot{y}\right] - \frac{\Delta V_{Dy_0}}{r},\\ \delta\omega_{y_0} &= \frac{1}{r[k_3 - k_1(1+k_2)]}\left[-\frac{1+k_2}{k_3}\delta\dddot{x} + \right.\\ &\left. + \frac{k_3 - k_1(1+k_2)}{k_3}\delta\ddot{x} - \frac{\omega_0^2(1+k_2)^2}{k_3}\delta\dot{x}\right] + \frac{\Delta V_{Dx_0}}{r}.\end{aligned}\right\} \tag{5.368}$$

Inserting these expressions in Eqs. (5.367) and integrating, we note that when the transients are damped

$$\delta x = \frac{\Delta n_{x_0}}{\omega_0^2} = \text{const}, \quad \delta y = \frac{\Delta n_{y_0}}{\omega_0^2} = \text{const}, \tag{5.369}$$

so that

$$\delta\dot{x} = \delta\ddot{x} = 0, \quad \delta\dot{y} = \delta\ddot{y} = 0, \tag{5.370}$$

and the following steady-state total position errors are obtained:

$$\left.\begin{aligned}\delta x_2 &= \frac{1}{k_3 - k_1(1+k_2)}\left[\frac{1+k_2}{k_3}\delta\ddot{x}^0 - \right.\\ &\left. - \frac{k_3 - k_1(1+k_2)}{k_3}\delta\dot{x}^0 - \frac{\omega_0^2(1+k_2)^2}{k_3}\left(\frac{\Delta n_{x_0}}{\omega_0^2} - \delta x^0\right)\right] + \delta x_2^0 + \Delta V_{Dx_0}t,\\ \delta y_2 &= \frac{1}{k_3 - k_1(1+k_2)}\left[\frac{1+k_2}{k_3}\delta\ddot{y}^0 - \frac{k_3 - k_1(1+k_2)}{k_3}\delta\dot{y}^0 - \right.\\ &\left. - \frac{(1+k_2)^2}{k_3}\left(\frac{\Delta n_{y_0}}{\omega_0^2} - \delta y^0\right)\right] + \delta y_2^0 + \Delta V_{Dy_0}t.\end{aligned}\right\} \tag{5.371}$$

Unlike Eqs. (5.348), these expressions for δx_2, δy_2 do not contain terms of the form $r\Delta\omega_{x_0}t$, $r\Delta\omega_{y_0}t$; the terms proportional to ΔV_{Dx_0}, ΔV_{Dy_0} for $k_2 \gg 1$ are virtually identical both in solutions (5.371) and in solutions (5.348).

Note that the approach of this subsection can be further extended. Adding to the right-hand sides of Eqs. (5.356) for the control torques second integrals over ΔV_{x_0}, ΔV_{y_0}, we end up with fourth-order error equations. This increase in the order of the error equations relative to fixed instrumental errors may, however, lead to an increase in the random errors of the aided systems (this will be proved in the following).

We have so far considered correction with constant coefficients. The coefficients k_1, k_2, k_3, etc., clearly can be made known functions of time.

The equations corresponding to this case may be derived by the same methods as in the case of constant coefficients. However, we must not simply replace in the previous equations the constant coefficients by the given functions of time, since in the derivation of the error equations we had to differentiate expressions containing these coefficients. The coefficients being constant in the previous case, the corresponding derivatives dropped out, which is no longer true.

Interesting results are obtained if, say, the coefficients are represented by functions of the form*

$$k_i(t) = k_i' + k_{i0}e^{-\alpha t}, \tag{5.372}$$

where k_i' and k_{i0} are some constants.

For $t = 0$ we have

$$k_i = k_i' + k_{i0},$$

which may be quite large, whereas $k_i(t=\infty) = k_i'$ should be made sufficiently small.

Finally, nonlinear correction systems can be considered, where k_1, k_2, k_3 are functions of ΔV. The corresponding error equations can be derived without much difficulty. However, no regular methods are available for analyzing the resulting equations even for the simplest functions $k_1(\Delta V)$, $k_2(\Delta V)$, $k_3(\Delta V)$.

This analysis generally requires numerical solution of the corresponding equations, and we will not go into this problem in any detail.

* McClure, C.L. Theory of Inertial Guidance. — Prentice Hall. 1960.

Chapter 6

ASTRONOMICAL CORRECTION OF INERTIAL NAVIGATION SYSTEMS. COMBINED ASTRO-DOPPLER CORRECTION

§6.1. INTRODUCTORY REMARKS

The inertial navigation systems with altimeter or Doppler correction considered in the previous two chapters only change the structure of the first group of error equations (1.95) and do not affect the second group of error equations (1.96) (with the exception of the case of Doppler correction changing the control torques, which was considered in the last section of the previous chapter).

The right-hand sides of the second group of error equations include the instrumental errors of the gyroscopic sensitive elements of the inertial system. The analysis of the error equations in the previous chapters shows that the gyro errors (errors of the absolute angular velocity meters, platform drift) may lead to growing errors of position and orientation. In two-accelerometer inertial systems the instrumental gyro errors primarily limit the time of continuous operation of the system during which the desired accuracy is ensured. Note that if the inertial system is free from gyroscopes, and the only sensitive elements are accelerometer, the situation is essentially the same. In this case the instrumental errors of the accelerometers used to determine the platform orientation also lead to growing errors in navigation parameters.*

To maintain the desired accuracy of inertial systems, the gyro elements (or the substituting accelerometers) should be appropriately corrected. The correction in this case involves comparison of the actual and the calculated orientation of the sensitive elements relative to known directions in the axes $O_1\xi_*\eta_*\zeta_*$ and $O_1\xi\eta\zeta$.

The known directions are generally provided by celestial markers: stars, Sun, the planets and the Moon, cosmic radio sources. Artificial Earth satellites and ground markers can also be used for these purposes.

Optical correction using the positions of the fixed stars is most commonly used (astronomical correction or astrocorrection). The correction systems in this case are very simple for two reasons. First, when an object moves near the Earth, the direction from the current position point to the distant stars can be regarded as fixed in the axes $O_1\xi_*\eta_*\zeta_*$, i.e., the parallactic effects and the proper motions of stars are ignored. Second, the directions to the stars can be determined ignoring their angular dimensions.

* See my "Theory of Inertial Navigation: Autonomous Systems." Nauka. 1966.

In the following we will mainly consider celestial correction using stars (astrocorrection). The main results can be easily extended to other cases of celestial correction. In the following we will be only concerned with the correction of gyro elements. However, in view of the above remarks, the results are equally applicable to gyro-less systems.

Let us briefly discuss some problems which arise in the analysis of astrocorrection systems.

The first problem is the calculation of the direction to celestial objects at any given time. This requires knowledge of the Earth's motion relative to the stars. The astronomical methods of specifying directions to celestial bodies in various axes are also applied. The second problem is associated with the information supplied by telescopic observations of stars (or other celestial bodies). Here the deviation of the apparent position of the celestial bodies from their true position due to aberration (Earth's orbital motion) and atmospheric refraction should be taken into consideration. Moreover, daytime correction in lower atmospheric layers should allow for interference from scattered atmospheric radiation. The third principal problem is the analysis of the dynamics of astrocorrection systems, the changes in the error equations of the inertial system, and the dependence of navigation errors on instrumental errors.

§6.2. EARTH'S MOTION. DETERMINATION OF DIRECTIONS TO CELESTIAL BODIES IN AXES FIXED TO THE EARTH

6.2.1. Simplified description of the Earth's motion

If we ignore the effect of the planets and the Moon on the Earth's orbital motion, regard the Sun as a fixed star, and reduce the Sun's attraction on the Earth to a resultant force directed along the line joining the Earth's center to the Sun's center, we conclude that the Earth moves like a point mass in a central (spherical) field.*

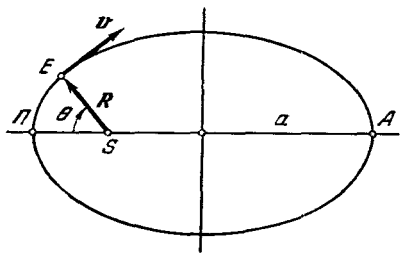

FIGURE 6.1.

In accordance with the laws of this motion, the center of mass of the Earth will move in a plane through the center of the Sun; the orientation of this plane relative to the fixed stars remains invariable. It is known as the plane of the ecliptic in astronomy.** The Earth's orbit in the plane of the ecliptic is an ellipse (Figure 6.1) with the Sun in one of its foci S. The point Π on the major axis of the ellipse (the line of apsides) is the perihelion of the Earth's orbit (the point of maximum approach to the Sun), and the point A is the aphelion,

* See, e.g., Appell, P. Traité de Mécanique Rationelle.—Gauthier-Villars, 1953.

** Exact definition of the plane of the ecliptic should take into consideration the effect of the planets and the Moon on the motion of the Earth's center of mass.

the most distant point from the Sun. If R is the radius vector from point S to the current position of the Earth E and θ is the angle between this radius-vector and the radius-vector of the point Π, the equation of the ellipse is

$$R = \frac{a(1-e^2)}{1+e\cos\theta}, \tag{6.1}$$

where a is the semimajor axis of the Earth's orbit, e is the eccentricity. Numerical values:*

$$a = 149.5 \cdot 10^6 \text{ km}, \quad e = 0.01675. \tag{6.2}$$

Let σ be the area of the sector $ES\Pi$, P the orbital period of the Earth around the Sun, t the time. The areal integral of motion in a central field (Kepler's law) is written in the form

$$\frac{\sigma}{\pi a^2 \sqrt{1-e^2}} = \frac{t}{P\pi}. \tag{6.3}$$

By (6.1) and (6.3) we have

$$\frac{d\theta}{dt} = \frac{2\pi(1+e\cos\theta)^2}{P(1-e^2)^{3/2}}, \quad \frac{dR}{dt} = \frac{2\pi ae\sin\theta}{P(1-e^2)^{1/2}}. \tag{6.4}$$

The velocity v of the Earth's center of mass is therefore expressed as a function of the angle θ by the following equation:

$$v = \frac{2\pi a}{P\sqrt{1-e^2}}\sqrt{1+e^2+2e\cos\theta}. \tag{6.5}$$

Since $P = 31.558 \cdot 10^6$ sec, the average velocity of motion is

$$v_0 = \frac{2\pi a}{P\sqrt{1-e^2}} = 29.770 \text{ km/sec}. \tag{6.6}$$

If in the first equation in (6.4) we ignore the terms with e^2 and terms of higher order, we readily find

$$\theta = \frac{2\pi}{P}t + 2e\sin\frac{2\pi}{P}t. \tag{6.7}$$

The motion of the Earth about the center of mass in navigation problems can be considered to first approximation* as uniform spin with angular velocity $u = 7.292116 \cdot 10^{-5}$ sec^{-1} about its axis (the minor axis of the Clairaut ellipsoid) in the sense of orbital revolution. The Earth's axis is inclined at an angle $\varepsilon = 23°27'$ to the normal to the plane of the ecliptic.

* Blazhko, S.N. Kurs sfericheskoi astronomii (A Course in Spherical Astronomy). — Gostekhizdat. 1954.

6.2.2. Determination of directions to celestial bodies

We will use the horizontal, equatorial, and ecliptic coordinate systems often applied in astronomy.

The horizontal coordinate system is defined by the axes $Ox_1y_1z_1$ introduced in 1.1.2. If we take $h = 0$ and ignore the difference between the direction of the normal to the Clairaut ellipsoid and the geodesic vertical (plumb line), the axes $Ox_1y_1z_1$ (see Figure 1.3), where the plane Ox_1y_1 is tangent to the geoid, will coincide with the horizontal coordinate system used in astronomy. An arbitrary direction with unit vector ρ is described in the $Ox_1y_1z_1$ axes by the angle H between this unit vector and the plane Ox_1y_1 and the angle A between the projection of the unit vector ρ onto the plane Ox_1y_1 and the axis Oy_1 (Figure 6.2). The angle A varying from 0 to 2π is the azimuth of the direction ρ. The positive angles A are reckoned from the axis Oy_1 to the axis Ox_1, i.e., from north to east. The angle H is the **height** of ρ above the horizon. It varies from 0 to $\pm\pi$; when the angle H is positive, the direction ρ lies above the plane of the horizon. If x_1, y_1, z_1 are the unit vectors of the axes Ox_1, Oy_1, Oz_1, we have

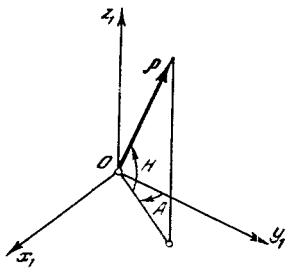

FIGURE 6.2.

$$\rho = x_1 \cos H \sin A + y_1 \cos H \cos A + z_1 \sin H. \tag{6.8}$$

The angle H is often replaced by the angle z between the vector ρ and the axis Oz_1; since the axis Oz_1 is directed to the zenith, this angle is known as the **zenith distance** of the direction ρ. Since $z = \pi/2 - H$, we have

$$\rho = x_1 \sin z \sin A + y_1 \sin z \cos A + z_1 \cos z. \tag{6.9}$$

The first equatorial system $O\xi_1\eta_1\zeta_1$ (Figure 6.3) is defined as follows. The axis $O\zeta_1$ is directed parallel to the Earth's spin vector u (the celestial axis). The axes $O\xi_1$ and $O\eta_1$ lie in a plane normal to the vector u, i.e., parallel to the Earth's equatorial plane, with the axis $O\xi_1$ in the meridional plane through the point O. The arbitrary direction characterized by the unit vector ρ is described in this system by the angles δ and t, which are analogous to the angles H and A.

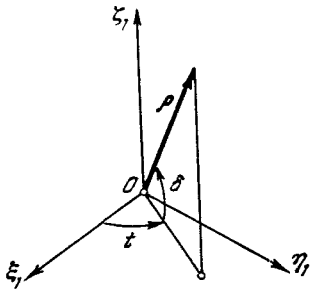

FIGURE 6.3.

The angle δ, which is the **declination** of the direction ρ, varies from 0 to $\pm\pi/2$. Positive angles correspond to vectors ρ making an acute angle with the axis $O\zeta_1$. The angle δ is often replaced by its complementary angle, the **polar distance** p.

The angle t is the **hour angle** of the direction ρ. It varies from 0 to 2π. Positive hour angles are reckoned against the Earth's spin, i.e., the angle t is reckoned from the axis $O\xi_1$ to the negative axis $O\eta_1$. The angle t is conveniently measured in time units from 0 to 24^h.

Thus in the $O\xi_1\eta_1\zeta_1$ axes

$$\rho = \xi_1 \cos\delta \cos t - \eta_1 \cos\delta \sin t + \zeta_1 \sin\delta \tag{6.10}$$

or

$$\rho = \xi_1 \sin p \cos t - \eta_1 \sin p \sin t + \zeta_1 \cos p, \tag{6.11}$$

where ξ_1, η_1, ζ_1 are the unit vectors of the respective axes.

Note that the orientation of the axes $O\xi_1\eta_1\zeta_1$ and $Ox_1y_1z_1$ relative to the distant stars changes both due to the Earth's rotation and the displacement of the point O near the Earth's surface. The orientation of the axes $Ox_1y_1z_1$ evidently changes with time irrespective of whether the two geographical coordinates φ' and λ of the point O are variable or constant. The orientation of the axes $O\xi_1\eta_1\zeta_1$, on the other hand, will change with time only if the longitude λ of the point O is variable.

The transformation from the axes $Ox_1y_1z_1$ to $O\xi_1\eta_1\zeta_1$ is described by the following matrix of direction cosines:

	ξ_1	η_1	ζ_1
x_1	0	1	0
y_1	$-\sin\varphi'$	0	$\cos\varphi'$
z_1	$\cos\varphi'$	0	$\sin\varphi'$.

(6.12)

Let us change over from the angles δ and t, say, to the angles H and A. From (6.12), using (6.10), we get

$$\rho = -x_1 \cos\delta \sin t + y_1 (-\sin\varphi' \cos\delta \cos t + \cos\varphi' \sin\delta) + z_1 (\cos\varphi' \cos\delta \cos t + \sin\varphi' \sin\delta). \tag{6.13}$$

Comparison with Eq. (6.8) gives the transformation formulas

$$\left.\begin{aligned} \cos H \sin A &= -\cos\delta \sin t, \\ \cos H \cos A &= -\sin\varphi' \cos\delta \cos t + \cos\varphi' \sin\delta, \\ \sin H &= \cos\varphi' \cos\delta \cos t + \sin\varphi' \sin\delta. \end{aligned}\right\} \tag{6.14}$$

To define the second equatorial system and the ecliptic system, we pass through the point O another plane, parallel to the plane of the ecliptic this time. If the Earth's orbital plane and the direction of its spin vector are assumed to maintain fixed orientation relative to the distant stars, the two planes parallel to the plane of the equator and the plane of the ecliptic respectively will intersect along a straight line through the point O, which also retains a fixed orientation in the celestial space. Let l be a unit vector normal to the orbital plane of the Earth, so that the Earth revolves around the Sun counterclockwise when viewed from the tip of the vector l. Then clearly the angular velocity $\dot{\theta}$ is directed along l, and the vector $u \times l$ points along the intersection line of the ecliptic plane with the equatorial plane.

The second equatorial system of coordinates $O\xi_2\eta_2\zeta_2$ (Figure 6.4) can now be introduced as follows. The axis $O\zeta_2$ is directed along the Earth's spin vector u, the axis $O\xi_2$ points along the vector $u \times l$, and the axis $O\eta_2$ naturally complements the two axes $O\xi_2$ and $O\zeta_2$ to a right orthogonal system. The plane $O\xi_2\eta_2$ of this system is parallel to the equatorial plane. The axis $O\xi_2$ is said to be directed to the point of **vernal equinox**. We shall correspondingly

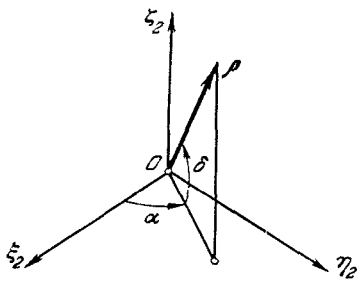

FIGURE 6.4.

speak of the **direction of the vernal equinox**. When the Earth in the course of its orbital motion reaches a point where the radius-vector R from the Sun to the Earth is directed against the axis $O\xi_2$, the Earth's axis is clearly normal to the radius-vector R and an astronomical change of season is observed in the Northern Hemisphere: winter gives way to spring. This explains the particular direction chosen for the vector $u \perp l$.

The general direction ρ is described in the second equatorial system of coordinates (Figure 6.4) by the angle δ between ρ and the plane $O\xi_2\eta_2$ and the angle α between the axis $O\xi_2$ and the projection of ρ onto the plane $O\xi_2\eta_2$. The declination angle δ is identical to that in the first equatorial system. The angle α is called the **right ascension**. It goes from 0 to 2π and it is reckoned from the axis $O\xi_2$ to $O\eta_2$. The angle α, like the hour angle t, is measured in time units (from 0 to 24^h), as well as in degrees or radians.

Writing ξ_2, η_2, ζ_2 for the unit vectors along the axes $O\xi_2$, $O\eta_2$, $O\zeta_2$, we get

$$\rho = \xi_2 \cos\delta\cos\alpha + \eta_2 \cos\delta\sin\alpha + \zeta_2 \sin\delta. \tag{6.15}$$

If s is the angle between the axis $O\xi_2$ and the axis $O\xi_1$ of the first equatorial system, we have

$$s = \alpha + t. \tag{6.16}$$

It will become clear from the following that s is numerically equal to the sidereal time on the meridian of the axis $O\xi_1$.

The relative orientation of the axes $O\xi_1\eta_1\zeta_1$ and $O\xi_2\eta_2\zeta_2$ is described by the following matrix of direction cosines:

	ξ_2	η_2	ζ_2
ξ_1	$\cos s$	$-\sin s$	0
η_1	$\sin s$	$\cos s$	0
ζ_1	0	0	1.

(6.17)

Matrix (6.17), used together with Eqs. (6.15), (6.16), enables us to change over from the coordinates t and δ to the coordinates α and δ, and vice versa. The transformation formulas are obtained along the same lines as Eqs. (6.14) for conversion from δ and t to H and A.

We can now define the ecliptic system of coordinates $O\xi_3\eta_3\zeta_3$. Its origin, as in the previous cases, is placed at the point O — the current position of the object; the axis $O\zeta_3$ is directed along the normal l to the plane of the ecliptic, and the axis $O\xi_3$ is made to coincide with the axis $O\xi_2$. The axis $O\eta_3$ complements the other two axes to a right orthogonal system. Since the angle between the equatorial plane and the plane of the ecliptic is ε, the direction cosines of the angles between the axes $O\xi_2\eta_2\zeta_2$ and $O\xi_3\eta_3\zeta_3$ are given by the following matrix:

	ξ_3	η_3	ζ_3
ξ_2	1	0	0
η_2	0	$\cos\varepsilon$	$-\sin\varepsilon$
ζ_2	0	$\sin\varepsilon$	$\cos\varepsilon$.

(6.18)

The general direction ρ is described in the coordinate system $O\xi_3\eta_3\zeta_3$ by the latitude B and the longitude L, which are analogous to the geocentric coordinates φ and λ. Thus

$$\rho = \xi_3 \cos B \cos L + \eta_3 \cos B \sin L + \zeta_3 \sin B. \qquad (6.19)$$

Unlike the axes $Ox_1y_1z_1$ and $O\xi_1\eta_1\zeta_1$, the axes $O\xi_2\eta_2\zeta_2$ and $O\xi_3\eta_3\zeta_3$ retain an invariable orientation relative to the stars. The directions to the distant stars in these axes are therefore fixed, i.e., the coordinates α, δ and B, L of these directions are constant.

In astronomical almanacs the directions to various celestial bodies are generally described by the coordinates α, δ or B, L for the beginning of the corresponding astronomical year. Given these coordinates, matrices (6.17), (6.12) can be used to change over to the coordinates of the celestial bodies in the first equatorial system and in the horizontal system; suitable matrices of direction cosines can be applied to transform to any other coordinate system whose orientation relative to the Earth at a given time is known, e.g., the system rigidly fixed to an inertial platform. In this way the astrocorrection telescopes can be properly aimed relative to the platform axes.

Note that we have so far assumed that the orientation of the second equatorial system and the ecliptic system was invariable relative to the directions to fixed stars. This is associated with an earlier assumption of the fixed orientation of the Earth's spin axis and its orbital plane (the plane of the ecliptic).

Actually these assumptions are valid only to first approximation. The orientation of the instantaneous spin axis of the Earth does change relative to the stars. The main reason is the noncentral character of the lunisolar attraction: the gravitational forces cannot be reduced to two resultant vectors directed along the lines which join the Earth's center with the respective centers of the Sun and the Moon, and resultant torques about the Earth's center of mass have to be considered. The origin of these torques is attributed to the Earth's flattening, i.e., the asymmetric distribution of the Earth's mass about the directions to the center of mass of the Sun and that of the Moon. The resultant lunisolar torque vectors lie in the plane of the Earth's equator and they work to shift this plane to coincide with the plane of the ecliptic and the lunar orbital plane. The effect of these torques is to produce a precession of the Earth's spin vector about the normal to the plane of the ecliptic on the surface of a cone with an opening angle of 2ε and with a period of about 26,000 years and a nutation with a principal period of about 18.6 years, which periodically changes the inclination of the plane of the ecliptic to the equatorial plane by an angle $\Delta\varepsilon$ ($\approx 10''$).

Planetary perturbations also change the orientation of the Earth's orbital plane in space. It rotates about an axis in the orbital plane at a rate of 47" per century (at the current epoch). This rotation of the line of apsides slowly changes (reduces at the current epoch) the angle ε. Moreover, the Earth and the Moon both move around a common center of mass, so that the Earth's orbit departs by about 1" from the plane of the ecliptic, near which the Earth—Moon center of mass moves.

The rotation of the plane of the ecliptic, precession and nutation of the instantaneous spin vector are the three main factors which change the orientation of the ecliptic and the second equatorial coordinate systems relative to

the fixed stars. The coordinates of the directions to celestial bodies defined assuming fixed orientation of the axes $O\xi_2\eta_2\zeta_2$ and $O\xi_3\eta_3\zeta_3$ will therefore contain certain errors. If the directions to these bodies are used for purposes of astrocorrection in an inertial system, the position determination will prove to be inaccurate.

The rotation of the plane of the ecliptic ($\approx 0''.5$ annually), the effect of the Earth's motion about the Moon—Earth center of mass ($\approx 0''.1$ annually) and nutation ($0''.5$ annually for the long-period component, and a similar contribution from the short-period component) are clearly ignorable as far as navigation is concerned, if we permit errors of about 1 km in position determination. This follows from the fact that the resulting position errors do not exceed the sum of the different contributions when the orientation of the axes $O\xi_2\eta_2\zeta_2$ and $O\xi_3\eta_3\zeta_3$ is taken corresponding to the beginning of the astronomical year (and not the particular time in question).

The precession of the Earth's spin axis, however, gives a substantial error, since it alters the orientation of the spin vector (and hence of the $O\zeta_2$ axis) by some 20" annually and rotates the direction of the vernal equinox, i.e., the $O\xi_2$ ($O\xi_3$) axis through 50" annually in the plane of the ecliptic. The position error will be of the same order of magnitude. The precession effect should be allowed for by taking the actual orientation of the axes $O\xi_2\eta_2\zeta_2$ and $O\xi_3\eta_3\zeta_3$ at the time the astrocorrection system begins to function or near that time. Since the period of continuous operation of the system is relatively short and mostly does not exceed one month, the error buildup will not exceed some 5". Note that when speaking of distant stars, the variation of the "fixed" directions in the axes $O\xi_2\eta_2\zeta_2$ and $O\xi_3\eta_3\zeta_3$ is associated with the definition of these coordinate systems and with the position of the stars in these particular systems: it has nothing to do with any change in the position of the stars in the absolute space. The angles between different directions to distant stars either do not change at all or change by negligible amounts ($\approx 1''$). These changes are associated with the proper motion of stars and their parallactic displacement.

In astrocorrection calculations we should distinguish between the true and the apparent position of the celestial objects. The two sets of coordinates differ by the amount of aberration due to the Earth's orbital motion. The annual aberration constant, determined by the ratio of the Earth's orbital velocity to the velocity of light, is $\approx 20''$. If no aberration correction is introduced and the apparent coordinates of the stars are used, their positions should be determined at the time the correction system begins to function. Indeed, if we measure the apparent directions to the stars at zero time (on the system's clock), the change in these directions due to aberration will not exceed $\approx 10''$ during the following month, since the direction of the Earth's velocity vector relative to the stars changes by about $\pi/6$ in one month. Somewhat smaller aberration is caused by the diurnal rotation of the Earth and by the displacement of the object relative to the Earth's surface. Even if the velocity of the object is close to the circular orbit velocity, the aberration remains less than 5".

In conclusion, let us consider the parallactic effects. In astrocorrection, the stellar parallax and the proper motions of stars can be ignored: the annual parallax of stars does not exceed 1" and the proper motions are vanishingly small. The diurnal parallaxes are meaningful only when the correction system uses the Sun, the Moon, the planets and, of course,

artificial Earth satellites. The diurnal parallax of the Sun is small enough to be ignored ($\approx 8''$). The diurnal parallaxes of the near planets, however, may reach 1', and the diurnal parallax of the Moon is as high as 2° and should be allowed for with due precision. In astrocorrection systems using the directions to artificial Earth satellites, the parallactic displacement is evidently one of the principal effects to be allowed for in direction determination.

6.2.3. Earth's motion and measurement of time

Eq. (6.16) defined the sidereal time s. This is a highly important concept in astrocorrection, and we will consider it in more detail.

If the Earth's spin relative to the fixed star is assumed to have constant magnitude, a natural time unit is provided by the **sidereal day** — the axial period of the Earth relative to the stars. The sidereal day begins at the time when the axes $O\xi_1$ and $O\xi_2$ of the first and second equatorial systems coincide, i.e., at the time when the direction of the vernal equinox passes through the observer's meridian. For this reason Eq. (6.16) indeed defines the sidereal time. Evidently, if s_1 and s_2 are the sidereal times of two different meridians λ_1 and λ_2 at the same (chronological) time, the difference of sidereal time is numerically equal to minus the difference in longitude:

$$s_1 - s_2 = \lambda_2 - \lambda_1. \tag{6.20}$$

In practical applications, solar time is more useful than the sidereal time. The unit of true solar time — the **solar day** — is defined as the length of time between two successive meetings of the radius-vector $\boldsymbol{R}$ (Figure 6.1) with the plane of a fixed terrestrial meridian. A measure of the true solar time at some point O is provided by the angle between the projection of the radius-vector $\boldsymbol{R}$ on the plane of the Earth's equator and the intersection line of the equatorial plane with the meridional plane through O. Since the vectors $\boldsymbol{l}$ and $\boldsymbol{u}$ make an acute angle $\pi/2-\varepsilon$, the solar day is longer than the sidereal day.

Let us consider the equation of the true solar time on a meridian which coincided with the direction $-\boldsymbol{R}$ at the time of perihelion passage. The origin of the two equatorial and one ecliptic systems defined in the preceding is moved from the point O to the Earth's center O_1, and the corresponding axes are respectively designated $O_1\xi_1\eta_1\zeta_1$, $O_1\xi_2\eta_2\zeta_2$, $O_1\xi_3\eta_3\zeta_3$. If ω is the longitude of the perihelion in the axes $O_1\xi_3\eta_3\zeta_3$, writing $\boldsymbol{R}_0$ for the unit vector along $\boldsymbol{R}$, we have

$$-\boldsymbol{R}_0 = -\xi_1 \cos(\omega+\theta) - \eta_1 \sin(\omega+\theta). \tag{6.21}$$

The matrix of direction cosines between the axes $O_1\xi_2\eta_2\zeta_2$ and $O_1\xi_3\eta_3\zeta_3$ takes the form:

	ξ_3	η_1	ζ_3
ξ_2	$\cos(\chi+ut)$	$\sin(\chi+ut)$	0
η_2	$-\sin(\chi+ut)$	$\cos(\chi+ut)$	0
ζ_2	0	0	1.

(6.22)

Here u is the Earth's spin relative to the stars, t is the sidereal time* reckoned from the time of perihelion passage, and the angle χ is defined by the condition that the plane of the meridian initially coincides with the direction $-R_0$:

$$\operatorname{tg}\chi = \cos\varepsilon \operatorname{tg}\omega. \tag{6.23}$$

Together with Eq. (6.21) for the unit vector $-R$ we have from (6.10) the following expression for this vector in the axes $O_1\xi_1\eta_1\zeta_1$:

$$-R_1 = \xi_1 \cos\delta_\odot \cos T - \eta_1 \cos\delta_\odot \sin T + \zeta_1 \sin\delta_\odot, \tag{6.24}$$

where $\delta_\odot$ is the declination of the Sun, and T is the hour angle of the Sun, i.e., the true solar time measured in angular units.

Equating the components of the vector $-R_1$ along the axes ξ_1, η_1, ζ_1 from Eq. (6.24) to its components from Eq. (6.19) (using the matrices (6.22), (6.17) to transform from the axes $O_1\xi_1\eta_1\zeta_1$ to $O_1\xi_1\eta_1\zeta_1$), we find

$$\left.\begin{aligned}
\cos\delta_\odot \cos T &= -\cos(\omega+\theta)\cos(\chi+ut) - \\
&\quad -\cos\varepsilon \sin(\chi+ut)\sin(\omega+\theta), \\
\cos\delta_\odot \sin T &= -\cos(\omega+\theta)\sin(\chi+ut) + \\
&\quad +\cos\varepsilon \cos(\chi+ut)\sin(\omega+\theta), \\
\sin\delta_\odot &= -\sin\varepsilon \sin(\omega+\theta).
\end{aligned}\right\} \tag{6.25}$$

Dividing the second equation in (6.25) through by the first, we get

$$\operatorname{tg} T = \frac{\operatorname{tg}(\chi+ut-\omega-\theta) + \dfrac{(1-\cos\varepsilon)\cos(\chi+ut)\sin(\omega+\theta)}{\cos(\chi+ut-\omega-\theta)}}{1 - \dfrac{(1-\cos\varepsilon)\sin(\chi+ut)\sin(\omega+\theta)}{\cos(\chi+ut-\omega-\theta)}}. \tag{6.26}$$

Retaining only terms of first order in $(1-\cos\varepsilon)$ and using Eq. (6.23) and the approximate expression (6.7) for $\theta(t)$, we obtain

$$T = \left(u - \frac{2\pi}{P}\right)t - 2e\sin\frac{2\pi}{P}t + \sin^2\frac{\varepsilon}{2}\sin 2\left(\omega + \frac{2\pi}{P}t\right) - \sin^2\frac{\varepsilon}{2}\sin 2\omega. \tag{6.27}$$

This relation is known as the **equation of time**,** and within the framework of our simplifying assumptions it relates the sidereal time to the true solar time.

By (6.27) we see that the true solar time is a nonmonotonic function. For this reason the so-called **mean solar time** is introduced, which is related to the sidereal time by the first term on the right in Eq. (6.27). If nonzero initial conditions are specified, the dependence of the **hour angle of the mean Sun** T on the sidereal time t is written in the form

$$T - T_0 = \left(u - \frac{2\pi}{P}\right)(t - t_0), \tag{6.28}$$

where T_0 is the hour angle of the mean Sun at the given meridian at the sidereal time t_0. Eq. (6.28) would give an exact definition of the solar

* t is proportional to s, and the proportionality coefficient is clearly the Earth's spin relative to the stars, u.

** Blazhko, S.N. Kurs sfericheskoi astronomii (A Course in Spherical Astronomy). — Gostekhizdat. 1954.

time if the Earth moved around the Sun with constant velocity in a circular orbit ($e = 0$), and the Earth's spin axis were normal to the orbital plane ($\varepsilon = 0$). All this explains why the T defined by Eq. (6.28) is called the hour angle of the mean Sun.

The unit of sidereal time t is the sidereal day, i.e., the duration of one complete axial rotation of the Earth relative to the stars. Then numerically $u = 2\pi$, and P is the number of sidereal days in a year. The unit of mean solar time τ is the mean solar day, so that

$$T - T_0 = 2\pi(\tau - \tau_0). \tag{6.29}$$

Inserting (6.29) in (6.28) and seeing that $u = 2\pi$, we get

$$\tau - \tau_0 = \frac{P-1}{P}(t - t_0), \tag{6.30}$$

where τ_0 and t_0 are the corresponding initial values.

Let Q be the number of mean solar days in a year. Since the Earth completes in a year one rotation more relative to the stars than relative to the Sun, we have

$$Q = P - 1 = 365.2422, \tag{6.31}$$

and Eq. (6.30) can be written in the form

$$\tau - \tau_0 = \frac{Q}{Q+1}(t - t_0). \tag{6.32}$$

Eq. (6.32) relates the sidereal and the mean solar time at some instant on a given meridian. For two different meridians $\lambda_2 > \lambda_1$ and two different (chronological) times (1) and (2) we have

$$\left.\begin{aligned} t^{\lambda_1}_{(1)} &= t^{\lambda_1}_{(1)\,0} + \frac{Q+1}{Q}\left[\tau^{\lambda_1}_{(1)} - \tau^{\lambda_1}_{(1)\,0}\right], \\ t^{\lambda_2}_{(2)} &= t^{\lambda_2}_{(2)\,0} + \frac{Q+1}{Q}\left[\tau^{\lambda_2}_{(2)} - \tau^{\lambda_2}_{(2)\,0}\right]. \end{aligned}\right\} \tag{6.33}$$

Since on one meridian

$$t^{\lambda_1,2}_{(2)} - t^{\lambda_1,2}_{(1)} = \frac{Q+1}{Q}\left[\tau^{\lambda_1,2}_{(2)} - \tau^{\lambda_1,2}_{(1)}\right], \tag{6.34}$$

and for the same time

$$t^{\lambda_1}_{(1),\,(2)} - t^{\lambda_1}_{(1),\,(2)} = \frac{\lambda_2 - \lambda_1}{2\pi}, \tag{6.35}$$

we have

$$t^{\lambda_2}_{(2)} = t^{\lambda_1}_{(1)\,0} - \frac{\lambda_2 - \lambda_1}{2\pi} + \frac{Q+1}{Q}\left[\tau^{\lambda_2}_{(2)} - \tau^{\lambda_1}_{(1)\,0}\right]. \tag{6.36}$$

The last expression defines the sidereal time $t^{\lambda_2}_{(2)}$ at some (chronological) time on the meridian λ_2 in terms of the mean solar time $\tau^{\lambda_2}_{(2)}$ at the same chronological time on the same meridian and the sidereal and mean solar time $t^{\lambda_1}_{(1)\,0}$ and $\tau^{\lambda_1}_{(1)\,0}$ at some other chronological time on another meridian λ_1.

The Greenwich meridian is generally chosen as λ_1. If $t_{0\,0}^{\circ}$ is the sidereal time on the Greenwich meridian at the mean midnight, we clearly have $\tau_{0\,0}^{\circ} = 0$.

6.2.4. Fixing the position on the Earth from astronomical observations

Determination of directions to selected celestial bodies relative to the axes $Ox_1y_1z_1$ at fixed times has been long used in navigation, geodesy, and cartography for measuring the coordinates of a point O on the Earth's surface.

The method for the determination of the latitude φ' is suggested by Eqs. (6.14). If we measure the height H of a celestial object at the time of upper culmination, i.e., when $t = 0$, and assume the declination δ at the time of observation to be known (for a distant star $\delta = \text{const}$), we have from the third equation in (6.14)

$$\sin H = \cos(\varphi' - \delta), \tag{6.37}$$

and for $t = 0$ the direction to the celestial body lies in the plane Oy_1z_1, since $\sin A = 0$.

The measurement of the longitude λ clearly amounts to comparing the local time (whether sidereal, true solar, or mean solar) with the time of some reference meridian at the same physical time, since the difference in time between two meridians at the same instant is equal to the difference of the longitudes of these meridians.

If we measure the heights H_1 and H_2 of two celestial bodies above the horizon and know the sidereal time s_1 and s_2 on the Greenwich meridian corresponding to the two measurements, we can simultaneously find both the latitude and the longitudes (Somner's method). Indeed, from the last equation in (6.14), from (6.16) relating the right ascension α and the hour angle t to the local sidereal time s, and from the equality of the difference in the local and Greenwich sidereal time to the longitude of the observation point, we have the set of equations

$$\left.\begin{aligned} \sin H_1 &= \sin\varphi' \sin\delta_1 + \cos\varphi' \cos\delta_1 \cos(s_1 - \alpha_1 + \lambda), \\ \sin H_2 &= \sin\varphi' \sin\delta_2 + \cos\varphi' \cos\delta_2 \cos(s_2 - \alpha_2 + \lambda). \end{aligned}\right\} \tag{6.38}$$

Eqs. (6.38) are equations in the coordinates φ' and λ of two lines of equal height, corresponding to the measured H_1 and H_2. It is readily seen that Eqs. (6.38) have two pairs of solutions. To choose between the two, we should have a rough idea of the position. When the difference between H_1 and H_2 is sufficiently large, this does not cause additional difficulty.

§6.3. ASTROCORRECTION USING TWO STARS

6.3.1. Information obtained from telescopic observations. Astrocorrection techniques

The principal element of an astrocorrection system is the telescope, which fixes the direction to the star. Let us first see what information can be obtained with a telescope mounted on an inertial platform.

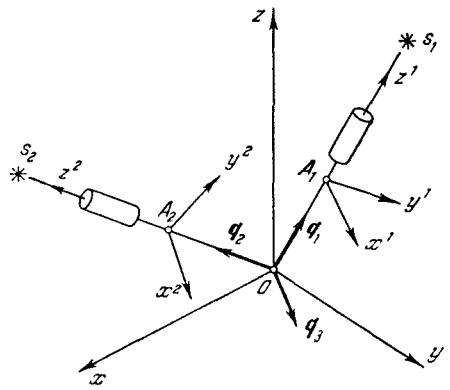

FIGURE 6.5.

Consider an inertial system using a free gyroscopically stabilized platform, with axes $Qxyz$ (Figure 6.5) rigidly fixed to it. Two telescopes are mounted on the platform, with their optical axes directed along the unit vectors q_1 and q_2, which maintain a fixed orientation relative to the axes $Oxyz$. We introduce the oblique axes $q_1q_2q_3$ with their origin at the point O, which are formed by the unit vectors q_1, q_2 pointing along the optical axes of the telescopes and a unit vector q_3 normal to these two directions:

$$q_3 = \frac{q_1 \times q_2}{\sin B}, \tag{6.39}$$

where B is the angle between q_1 and q_2, so that

$$\cos B = q_1 \cdot q_2. \tag{6.40}$$

The axes $q_1q_2q_3$ are the covariant (or fundamental) base vectors. We also define the contravariant base vectors $q^1q^2q^3$, so that

$$\left.\begin{aligned} q^1 &= \frac{1}{v}\, q_2 \times q_3, \quad q^2 = \frac{1}{v}\, q_3 \times q_1, \\ q^3 &= \frac{1}{v}\, q_1 \times q_2, \quad v = (q_1 \times q_2) \cdot q_3. \end{aligned}\right\} \tag{6.41}$$

Inserting the expression for q_3 from (6.39) in (6.41) and using (6.40), we find

$$\left.\begin{aligned} q^1 &= \frac{1}{\sin^2 B}(q_1 - q_2\cos B), \\ q^2 &= \frac{1}{\sin^2 B}(q_2 - q_1\cos B), \quad q^3 = q_3. \end{aligned}\right\} \tag{6.42}$$

From the definition of the vectors q_1, q_2, q_3 and q^1, q^2, q^3, we have

$$\left.\begin{aligned} q_s \cdot q^k &= 0 \quad \text{for} \quad s \neq k, \\ q_s \cdot q^s &= 1. \end{aligned}\right\} \tag{6.43}$$

We also introduce right orthogonal axes $A_1x^1y^1z^1$ and $A_2x^2y^2z^2$ fixed to the focal planes of the telescopes. The points of origin A_1 and A_2 lie on the telescope axes q_1 and q_2, the axes z^1 and z^2 are directed along the telescope axes, and the planes $A_1x^1y^1$ and $A_2x^2y^2$ are the focal planes of the first and the second telescope, respectively. The axes y^1 and y^2 are coplanar with the vectors q_1, q_2, so that the axes x^1, x^2 are normal to the plane spanned by these vectors.

If x^1, y^1, z^1 and x^2, y^2, z^2 are the unit vectors along the axes x^1, y^1, z^1 and x^2, y^2, z^2, we have the obvious equalities

$$\left.\begin{aligned} x^1 &= x^2 = q_1 = q^3, \\ y^1 &= -\sin B q^3, \quad y^2 = \sin B q^1, \\ z^1 &= q_1, \quad z^2 = q_2. \end{aligned}\right\} \tag{6.44}$$

Let the initial orientation of the platform and the directions q_1, q_2 be so chosen that the telescope axes q_1, q_2 are initially aimed exactly at the two stars s_1 and s_2. The images of these stars are then formed in the focal plane of the telescopes, at the points A_1 and A_2, i.e., on the optical axes of the two telescopes. If an ideal stable platform is used, which maintains a fixed orientation in space, the images of the stars will clearly remain all the time at the points A_1 and A_2.

If, however, there is gyro drift, the platform and its axes $Oxyz$ will change their orientation relative to the initial attitude, which is designated $Ox^*y^*z^*$. The telescope axes q_1, q_2 will no longer aim exactly at the stars s_1, s_2, and the image of these stars in the focal plane will shift to some point characterized by the coordinates x^1, y^1 and x^2, y^2.

The coordinates x^1, y^1 and x^2, y^2 can be measured using photosensitive elements in the focal plane of the telescopes. They constitute all the information that astronomical observations can provide for navigation purposes. These measurements clearly give the deviation of the platform from its initial attitude.

Let $(-\theta)$, as before, be a small rotation vector of the platform relative to its initial attitude. The same vector clearly characterizes the deviations of the axes $x^1y^1z^1$ and $x^2y^2z^2$ from the initial $x^{1*}y^{1*}z^{1*}$ and $x^{2*}y^{2*}z^{2*}$. The displacements x^1, y^1 and x^2, y^2 of the images of the stars s_1 and s_2 in the focal plane of the two telescopes are proportional to the projections $\theta_{y^1}, \theta_{x^1}, \theta_{y^2}, \theta_{x^2}$ of the vector θ on the corresponding axes:

$$\left.\begin{aligned} x^1 &= a\theta_{y^1}, \quad y^1 = -a\theta_{x^1}, \\ x^2 &= a\theta_{y^2}, \quad y^2 = -a\theta_{x^2}, \end{aligned}\right\} \tag{6.45}$$

where a is the proportionality coefficient.

From (6.45) we have

$$\left.\begin{aligned} \theta_{y^1} &= \theta \cdot q^2 \sin B = \theta^2 \sin B, \quad \theta_{x^1} = \theta \cdot q^3 = \theta^3, \\ \theta_{y^2} &= \theta \cdot q^1 \sin B = \theta^1 \sin B, \quad \theta_{x^2} = \theta \cdot q^3 = \theta^3. \end{aligned}\right\} \tag{6.46}$$

Here θ^1, θ^2, θ^3 are the contravariant components of the vector θ in the directions q_1, q_2, q_3, so that

$$\theta = q_s \theta^s, \tag{6.47}$$

where the summation over s goes from 1 to 3.

From (6.45), (6.46) we see that knowledge of the coordinates x^1, y^1, x^2, y^2 of the displaced star images in the focal plane of the two telescopes is equivalent to knowledge of the contravariant components of the gyro drift vector* θ along the vectors q_1, q_2, q_3:

$$\theta^1 = \frac{x^2}{a \sin B}, \quad \theta^2 = -\frac{x^1}{a \sin B}, \quad \theta^3 = -\frac{y^1}{a} = -\frac{y^2}{a}. \tag{6.48}$$

The drift vector θ is represented according to Eqs. (6.47), (6.48) by two equivalent expressions:

$$\left.\begin{aligned} \theta &= q_1 \frac{x^2}{a \sin B} - q_2 \frac{x^1}{a \sin B} - q_3 \frac{y^1}{a}, \\ \theta &= q_1 \frac{x^2}{a \sin B} - q_2 \frac{x^1}{a \sin B} - q_3 \frac{y^2}{a}. \end{aligned}\right\} \tag{6.49}$$

Hence it follows that simultaneous determination of the coordinates y^1 and y^2 is redundant: only one of them should be measured.

Representation (6.49) enables us to find from known x^1, y^1, x^2, y^2 the projections θ_x, θ_y, θ_z of the drift vector θ on the stable platform axes. From the first equation in (6.49) we get

$$\left.\begin{aligned} \theta_x &= \boldsymbol{x} \cdot q_1 \frac{x^2}{a \sin B} - \boldsymbol{x} \cdot q_2 \frac{x^1}{a \sin B} - \boldsymbol{x} \cdot q_3 \frac{y^1}{a}, \\ \theta_y &= \boldsymbol{y} \cdot q_1 \frac{x^2}{a \sin B} - \boldsymbol{y} \cdot q_2 \frac{x^1}{a \sin B} - \boldsymbol{y} \cdot q_3 \frac{y^1}{a}, \\ \theta_z &= \boldsymbol{z} \cdot q_1 \frac{x^2}{a \sin B} - \boldsymbol{z} \cdot q_2 \frac{x^1}{a \sin B} - \boldsymbol{z} \cdot q_3 \frac{y^1}{a}. \end{aligned}\right\} \tag{6.50}$$

Similar expressions are obtained from the second equation in (6.49).

Eqs. (6.50) include scalar products of the unit vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ of the stable platform axes with the unit vectors q_1, q_2, q_3, i.e., the direction cosines of the angles between the vectors q_1, q_2, q_3 and the axes x, y, z. These direction cosines are known. For a stable platform, when q_1, q_2, and therefore also q_3 have a fixed orientation relative to the platform, these direction cosines are constant.

Let us now consider an inertial system using a controlled (not free) gyroscopically stabilized platform (or gyroframe).

At a first glance, this case appears much more complex than the case of a free stable platform. After all, the orientation of the axes $Oxyz$ fixed to the controlled platform is variable relative to the stars. The directions

* Here and in what follows the vector θ is called the drift vector only for brevity; actually $(-\theta)$ is the drift vector.

q_1, q_2 therefore cannot remain rigidly fixed to the platform. The orientation of the telescope axes relative to the platform axes should thus be a function of the platform attitude relative to the axes ξ_*, η_*, ζ_* or, equivalently, relative to directions to the distant stars. Since the orientation of the platform relative to the axes ξ_*, η_*, ζ_* is a function of the coordinates that the inertial system measures, the attitude of the telescope axes relative to the platform is also dependent on these coordinates.

The inertial system measures the coordinates with certain errors, which are a function of the gyro errors as well as the errors of all the other system elements (accelerometers included).

At a first glance these errors seem to be carried over to telescope pointing errors, and the coordinates x^1, y^1, x^2, y^2 of the star images in the focal plane thus no longer depend only on the gyro errors.

A more detailed examination shows, however, that Eqs. (6.48)—(6.50) remain in force for a controlled platform as well. The vectors q_1, q_2, q_3, however, should be treated as variables and the angle θ is identified with the vector entering the second group of error equations (1.96). For a controlled gyroframe, this vector is determined only by the gyro errors Δm_x, Δm_y, Δm_z, just as for a stable platform.

To check the validity of this proposition, consider Eqs. (1.116). These equations give the projections of the orientation error vector θ_1 of some axes xyz, whose unperturbed position relative to the axes ξ_*, η_*, ζ_* is defined by the direction cosines $\alpha_{ij}(\xi_*, \eta_*, \zeta_*)$. The unperturbed orientation of these axes is thus a known function of the coordinates measured by the inertial system. The right-hand sides of Eqs. (1.116) therefore contain the errors θ_x, θ_y, θ_z associated only with gyro errors (in virtue of the second group of error equations), as well as terms due to the errors $\delta\xi_*$, $\delta\eta_*$, $\delta\zeta_*$ of position measurement.

We can now apply Eqs. (1.116) to the axes $Ox^1y^1z^1$ and $Ox^2y^2z^2$. In the unperturbed state these axes are caged to the directions q_1^*, q_2^* to the stars s_1, s_2, and their unperturbed orientation relative to the axes ξ_*, η_*, ζ_* is therefore invariable. In Eqs. (1.116) we may therefore take $\alpha_{ij} = \text{const}$. Thus, we get

$$\left.\begin{aligned} \theta_{1x^1} &= -\theta_{x^1}, \quad \theta_{1y^1} = -\theta_{y^1}, \\ \theta_{1y^2} &= -\theta_{y^2}, \quad \theta_{1x^2} = -\theta_{x^2}, \end{aligned}\right\} \tag{6.51}$$

as required. The same result can be obtained from Eq. (1.122).

Star observations thus fix the vector θ which enters the second group of error equations (1.96). Hence, if θ is known, we can find from Eqs. (1.96) the instrumental vector error Δm in the determination of the absolute angular velocity of the platform axes. In projection onto the platform axes x, y, z, we have

$$\left.\begin{aligned} \Delta m_x &= \dot\theta_x + \omega_y\theta_z - \omega_z\theta_y, \\ \Delta m_y &= \dot\theta_y + \omega_z\theta_x - \omega_x\theta_z, \\ \Delta m_z &= \dot\theta_z + \omega_x\theta_y - \omega_y\theta_x. \end{aligned}\right\} \tag{6.52}$$

The components (6.52) can be applied to correct the gyro readings. However, if Δm_x, Δm_y, Δm_z are formed from (6.52), the projections θ_x, θ_y, θ_z and hence x^1, y^1, x^2, y^2, should be differentiated. But the systems measuring the coordinates x^1, y^1, x^2, y^2 operate at high gain and high noise

level, since the luminous fluxes are exceedingly small. Differentiation of x^1, y^1, x^2, y^2 therefore involves considerable difficulties, and gyro correction systems are generally designed as servo systems which track the star and keep the angle θ at zero. The fundamental principles of these tracking systems are discussed in the following subsections.

6.3.2. Simultaneous linear correction using two stars

We will discuss in some detail the astrocorrection of gyro errors in an inertial system using a gyroscopically stabilized platform. Telescopic observations of two stars provide the coordinates x^1, y^1, x^2, y^2, so that from Eqs. (6.50) the components θ_x, θ_y, θ_z are calculated.

The equations of motion of a gyroscopically stabilized platform without astrocorrection constitute the second group of error equations (1.96). In projection onto the stable platform axes, these equations take the form

$$\dot{\theta}_x = \Delta m_x, \quad \dot{\theta}_y = \Delta m_y, \quad \dot{\theta}_z = \Delta m_z. \tag{6.53}$$

Eqs. (6.53) correspond to the vector equation (1.93):

$$\frac{d\theta}{dt} = \Delta \boldsymbol{m}. \tag{6.54}$$

Here the vector θ is the drift angle (a small rotation angle) of the platform, and Δm is in fact the drift velocity of the stable platform.*

The first group of error equations of an autonomous system reduces to the vector equation (1.92):

$$\frac{d^2\delta r}{dt^2} + \frac{\mu\,\delta r}{r^3} - \frac{\mu r}{r^5}\frac{3r\cdot\delta r}{r^2} = \Delta n - \Delta m \times \frac{dr}{dt} - \frac{d}{dt}(\Delta m \times r), \tag{6.55}$$

where $\delta r = \delta x x + \delta y y + \delta z z$, and x, y, z are the unit axes of the stable platform without drift.

For inertial systems with altimeter and Doppler correction, the first group of error equations has a different form, but at the present stage this is of no consequence.

Eqs. (6.54), (6.55) are supplemented in our case by the algebraic relations

$$\delta r_1 = \theta \times r, \quad \delta r_2 = \delta r + \delta r_1, \tag{6.56}$$

where

$$\delta r_1 = \delta x_1 x + \delta y_1 y + \delta z_1 z,$$
$$\delta r_2 = \delta x_2 x + \delta y_2 y + \delta z_2 z.$$

These relations give the total position error δr_2. Eqs. (1.116) in our case lead back to Eq. (6.54), and thus do not add anything new.

Examination of Eqs. (6.53)—(6.56) suggests the following methods for the application of the measured quantities θ_x, θ_y, θ_z to inertial system correction.

* More precisely, minus the drift velocity.

Using the first equation in (6.56), we can form the quantities

$$\delta x_1 = \theta_y z - \theta_z y, \quad \delta y_1 = \theta_z x - \theta_x z, \quad \delta z_1 = \theta_x y - \theta_y x \tag{6.57}$$

and subtract them from the coordinates measured by the inertial system. The coordinate errors are then reduced to the δx, δy, δz obtained from the first group of error equations, which follow from the vector equation (6.55). In this case we essentially do not interfere with the normal operation of the inertial system, and only correct its output. The information on θ_x, θ_y, θ_z in this case is incompletely utilized. The orientation of the gyroscopically stabilized platform is left uncorrected. The first group of error equations still contain the terms Δm_x, Δm_y, Δm_z on the right. They correspondingly enter the position errors δx, δy, δz, although they do not produce any error components which increase in proportion to time (at least not for constant Δm_x, Δm_y, Δm_z).

There is, however, an alternative approach to the problem. Differentiating the projections θ_x, θ_y, θ_z obtained from the telescopic measurements, we find Δm_x, Δm_y, Δm_z from Eqs. (6.53) and form suitable compensating torques, i.e., torques which when applied to the gyroscopes of the stable platform will make it precess at an angular velocity canceling the drifts Δm_x, Δm_y, Δm_z. Simultaneously using the initial values θ_x^0, θ_y^0, θ_z^0, we see that the equations of the second group (6.53) completely drop out from among the error equations, and the terms Δm_x, Δm_y, Δm_z disappear from the right-hand sides of the error equations of the first group, derived from the vector equation (6.55). This is the approach ensuring most comprehensive utilization of the information supplied by the telescopes. However, it involves differentiation of x^1, x^2, y^1, y^2, which as we have noted before is a difficult undertaking in practice.

The third approach is to use the components θ_x, θ_y, θ_z obtained from telescopic measurements in forming suitable control torques which, when applied to the platform gyros, will reduce x^1, y^1, x^2, y^2 and thus also θ_x, θ_y, θ_z to zero. In other words, the stable platform should be constantly maintained in an attitude with the optical axes of the telescopes aimed at the relevant stars.

The simplest method here is to make the correction torques M_y^2, M_x^1, M_x^2 of the stable platform proportional to θ_x, θ_y, θ_z, i.e., proportional to the right-hand sides of (6.50). Then

$$M_y^2 = -H_2 k\theta_x, \quad M_x^1 = H_1 k\theta_y, \quad M_x^2 = H_3 k\theta_z, \tag{6.58}$$

where H_1, H_2, H_3 are the angular momenta of the gyroscopes, and k is some constant factor.

The equations of motion of the platform (6.53) then take the form

$$\left.\begin{aligned} \dot\theta_x + k\theta_x = \Delta m_x, \ \dot\theta_y + k\theta_y = \Delta m_y, \\ \dot\theta_z + k\theta_z = \Delta m_z, \end{aligned}\right\} \tag{6.59}$$

which is equivalent to the vector equation

$$\frac{d\theta}{dt} + k_1\theta = \Delta m. \tag{6.60}$$

The solution of equation (6.60) is obviously

$$\theta = \int_0^t \Delta m(\tau)\, e^{-k(t-\tau)}\, d\tau + \theta^0 e^{-kt}. \tag{6.61}$$

Here the integral over the vector is a new vector whose projections on the axes x, y, z fixed to the stable platform are equal to the integrals over the corresponding projections of the integrand. The expression for θ_x therefore has the form

$$\theta_x = \int_0^t \Delta m_x(\tau)\, e^{-k(t-\tau)}\, d\tau + \theta_x^0 e^{-kt}. \tag{6.62}$$

The expressions for θ_y, θ_z are entirely analogous.

From Eq. (6.61) we see that for $k > 0$ and $\Delta m = 0$, the initial platform orientation error is eventually reduced to zero. If $\Delta m \neq 0$, but $|\Delta m|$ is bounded, $|\Delta\theta|$ is also bounded. Indeed, from (6.61) we have the estimate

$$|\theta| \leqslant \int_0^t |\Delta m|\, e^{-k(t-\tau)}\, d\tau + |\theta^0|\, e^{-kt}, \tag{6.63}$$

or

$$|\theta| \leqslant \frac{\max|\Delta m|}{k}(1 - e^{-kt}) + |\theta^0|\, e^{-kt}. \tag{6.64}$$

Thus, after some time (from the beginning of the correction routine), which depends on the factor k, the exponential terms on the right in inequality (6.64) are damped and we are left with the estimate

$$|\theta| \leqslant \frac{\max|\Delta m|}{k}. \tag{6.65}$$

Taking a sufficiently large k, we make $|\theta|$ so small that the errors δz_1, δx_1, δy_1 become ignorable. The inertial system errors are then determined only by the solutions of the first group of equations. From (6.60), (6.54) we see that in the equations of the first group Δm_x, Δm_y, Δm_z on the right should be replaced with

$$\left.\begin{aligned} \Delta m_x' = \Delta m_x - k\theta_x,\ \Delta m_y' = \Delta m_y - k\theta_y,\\ \Delta m_z' = \Delta m_z - k\theta_z. \end{aligned}\right\} \tag{6.66}$$

If Δm_x, Δm_y, Δm_z are constant, we see from (6.66) that some time after the beginning of the correction routine

$$\Delta m_x' = \Delta m_y' = \Delta m_z' = 0. \tag{6.67}$$

Note that the residual errors

$$\theta_x = \frac{\Delta m_x}{k},\quad \theta_y = \frac{\Delta m_y}{k},\quad \theta_z = \frac{\Delta m_z}{k} \tag{6.68}$$

obtaining for constant Δm_x, Δm_y, Δm_z can be applied to determine the constant drift components of the stable platform, and in principle the perturbing action of these constant components between successive applications of the correction routine can be compensated.

The residual errors can be also eliminated by introducing additional correcting torques proportional to the integrals over θ_x, θ_y, θ_z. Then Eqs. (6.58) are replaced by

$$\left.\begin{aligned} M_y^1 &= -H_2\left(k\theta_x + k_{-1}\int_0^t \theta_x\,dt\right), \\ M_x^1 &= H_1\left(k\theta_y - k_{-1}\int_0^t \theta_y\,dt\right), \\ M_z^1 &= H_3\left(k\theta_z + k_{-1}\int_0^t \theta_z\,dt\right), \end{aligned}\right\} \tag{6.69}$$

and Eqs. (6.60) give

$$\frac{d\theta}{dt} + k\theta + k_{-1}\int_0^t \theta\,dt = \Delta m. \tag{6.70}$$

The characteristic equation corresponding to the linear integro-differential equation (6.70) is

$$p^2 + kp + k_{-1} = 0. \tag{6.71}$$

For

$$k^2 - 4k_{-1} > 0 \tag{6.72}$$

the characteristic equation has two negative roots:

$$p_{1,2} = -\lambda_{1,2} = -\frac{k}{2} \pm \sqrt{\frac{k^2}{4} - k_{-1}}. \tag{6.73}$$

The solution of Eq. (6.70) has the form

$$\theta = \frac{1}{\lambda_2 - \lambda_1}\left\{\int_0^t \Delta\dot{m}(\tau)\left[e^{-\lambda_1(t-\tau)} - e^{-\lambda_2(t-\tau)}\right]d\tau + \right.$$
$$\left. + \left[\theta^0(\lambda_2 + k) - \Delta m^0\right]e^{-\lambda_1 t} - \left[\theta^0(\lambda_1 + k) - \Delta m^0\right]e^{-\lambda_2 t}\right\}. \tag{6.74}$$

Integration by parts gives

$$\theta = \frac{1}{\lambda_2 - \lambda_1}\left\{\int_0^t \Delta m(\tau)\left[\lambda_1 e^{-\lambda_1(t-\tau)} - \lambda_2 e^{-\lambda_2(t-\tau)}\right]d\tau + \right.$$
$$\left. + \theta^0\left[(\lambda_2 + k)e^{-\lambda_1 t} - (\lambda_1 + k)e^{-\lambda_2 t}\right]\right\}. \tag{6.75}$$

If $|\Delta m|$ is bounded, $|\theta|$ is also bounded, as in the preceding case.

For constant Δm, the angles θ and $\dot{\theta}$ can be taken constant after some time t_1 from the beginning of the correction routine, when the transients have passed. From (6.70) we have in this case

$$k_{-1}\int_0^{t_1} \theta\,dt = \Delta m, \tag{6.76}$$

i.e., the constant component of the vector Δm is continuously compensated

by the built-up integral. It naturally stays compensated when the astrocorrection routine is terminated.

Comparison of Eqs. (6.54) and (6.70) shows that the vector Δm on the right in error equation (6.55) should be replaced by the vector

$$\Delta m' = \Delta m - k\theta - k_{-1}\int_0^t \theta\, dt. \tag{6.77}$$

From (6.76) we see that after the transients

$$\Delta m' = 0 \tag{6.78}$$

and the gyro errors drop out from the first group of error equations as well.

This concludes our discussion of astrocorrection in a system with a normally free gyrostabilized platform. We can now proceed to a system using a controlled gyroplatform. We will see that the correction techniques considered for a free gyroplatform are equally applicable to the controlled case.

The first two techniques — correction of the system output and formation of $-\Delta m_x$, $-\Delta m_y$, $-\Delta m_z$ from the measured values of x^1, y^1, x^2, y^2 and their derivatives — are as self-evident for the controlled platform as before.

We have seen in the preceding subsection that Eqs. (6.47)—(6.50) remain valid for a controlled gyroplatform, with the only difference that the orientation of the vectors q_1, q_2, q_3 relative to the axes x, y, z is a function of the coordinates measured by the inertial system and time. The orientation of the vectors q_1, q_2, q_3 relative to the axes ξ, η, ζ, on the other hand, remains invariable, as in the case of a free stable platform.

Eqs. (6.53) are now replaced by Eqs. (1.96):

$$\left.\begin{aligned} \dot\theta_x + \omega_y\theta_z - \omega_z\theta_y &= \Delta m_x, \\ \dot\theta_y + \omega_z\theta_x - \omega_x\theta_z &= \Delta m_y, \\ \dot\theta_z + \omega_x\theta_y - \omega_y\theta_x &= \Delta m_z, \end{aligned}\right\} \tag{6.79}$$

or by the equivalent vector equation

$$\dot\theta + \omega \times \theta = \Delta m, \tag{6.80}$$

where dot denotes local differentiation with respect to time in a moving system of coordinates $Oxyz$, rigidly fixed to a platform which rotates with angular velocity ω.

Consider a linear correction routine. The torques M_y^2, M_x^1, M_x^3 are taken in the form

$$\left.\begin{aligned} M_y^2 &= -H_2\boldsymbol{x}\cdot q_s\left(k\theta^s + k_{-1}\int_0^t \theta^s\, dt\right), \\ M_x^1 &= H_1\boldsymbol{y}\cdot q_s\left(k\theta^s + k_{-1}\int_0^t \theta^s\, dt\right), \\ M_x^3 &= H_3\boldsymbol{z}\cdot q_s\left(k\theta^s + k_{-1}\int_0^t \theta^s\, dt\right), \end{aligned}\right\} \tag{6.81}$$

where the components θ^s are obtained using the known x^1, y^1, x^2, y^2 from Eqs. (6.48), and the sum over s goes from 1 to 3.

Application of these correcting torques to the gyroscopes together with the control torque transforms the vector equation of errors of the second group (6.80) to the form

$$\dot{\theta} + \omega \times \theta + k\theta + k_{-1}\int_0^t \theta\, dt = \Delta m. \tag{6.82}$$

or, alternatively,

$$\frac{d\theta}{dt} + k\theta + k_{-1}\int_0^t \theta\, dt = \Delta m \tag{6.83}$$

The integral in the third term on the left, as we see from (6.81), is taken over the directions q_s. These are fixed directions in the axes $O\xi_*\eta_*\zeta_*$ and the integration is thus equivalent to integration along the unit vectors ξ_*, η_*, ζ_*. Eq. (6.83) is thus expressed entirely in the fixed-orientation axes $O\xi_*\eta_*\zeta_*$. Naturally, if the vector Δm is physically determined by its projections on the axes $Oxyz$, Eq. (6.83) contains the corresponding expressions projected onto the axes $O\xi_*\eta_*\zeta_*$.

Eq. (6.83) is formally and essentially identical with Eq. (6.70) previously considered in connection with a stable platform. Its solution is given by (6.75). If the vector Δm is constant in the axes $O\xi_*\eta_*\zeta_*$, Eq. (6.76) and all that follows from it remain valid.

The correcting torques (6.81) thus solve the problem of linear astrocorrection of a controlled gyroscopically stabilized platform. This, however, is only one of several possible solutions.

Another solution can be obtained by taking the correcting torques in the form

$$\left.\begin{aligned} M_z^* &= -H_2\Big(k\boldsymbol{x}\cdot\boldsymbol{q}_s\theta^s + k_{-1}\int_0^t \boldsymbol{x}\cdot\boldsymbol{q}_s\theta^s\, dt - \omega_y\theta_z + \omega_z\theta_y\Big),\\ M_x^* &= H_1\Big(k\boldsymbol{y}\cdot\boldsymbol{q}_s\theta^s + k_{-1}\int_0^t \boldsymbol{y}\cdot\boldsymbol{q}_s\theta^s\, dt - \omega_z\theta_x + \omega_x\theta_z\Big),\\ M_y^* &= H_3\Big(k\boldsymbol{z}\cdot\boldsymbol{q}_s\theta^s + k_{-1}\int_0^t \boldsymbol{z}\cdot\boldsymbol{q}_s\theta^s\, dt - \omega_x\theta_y + \omega_y\theta_x\Big), \end{aligned}\right\} \tag{6.84}$$

where

$$\theta_x = \boldsymbol{x}\cdot\boldsymbol{q}_s\theta^s, \quad \theta_y = \boldsymbol{y}\cdot\boldsymbol{q}_s\theta^s, \quad \theta_z = \boldsymbol{z}\cdot\boldsymbol{q}_s\theta^s. \tag{6.85}$$

Eq. (6.82) is then replaced by

$$\dot{\theta} + k\theta + k_{-1}\int_0^t \theta\, dt = \Delta m. \tag{6.86}$$

In distinction from Eq. (6.83), the differentiation and integration here are taken in the axes x, y, z fixed to the controlled gyroplatform. Eq. (6.76) in this case remains valid if the vector Δm is constant in the axes $Oxyz$ fixed to the inertial platform.

6.3.3. Simultaneous "relay-type" correction using two stars

In the preceding subsection we considered astrocorrection of a free or controlled stable platform with the correcting torques applied to the platform gyros formed as linear functions of θ_x, θ_y, θ_z and their integrals. Linear correction of this kind presupposes knowledge of the coordinates x^1, y^1, x^2, y^2 of the star images in the focal plane of the telescopes.

However, the commonly used phase systems only give the signs of the* coordinates x^1, y^1, x^2, y^2 and their ratios x^1/y^1, x^2/y^2. Let us see how the correction is accomplished in this case.

First let us consider the case of a free gyroscopically stabilized platform. Assuming the signs of the coordinates x^1, y^1, x^2, y^2 known, we will form correcting torques which cancel the platform drift, i.e., keep the coordinates x^1, y^1, x^2, y^2 zero.

From Eqs. (6.48) we see that x^2, x^1, $y^1\,(y^2)$ are proportional to the contravariant components θ^1, θ^2, θ^3 of the vector θ along the vectors q_1, q_2, q_3. Therefore, if $\sin B > 0$, we have

$$\left.\begin{aligned} &\operatorname{sign} x^2 = \operatorname{sign}\theta^1, \quad \operatorname{sign} x^1 = -\operatorname{sign}\theta^2,\\ &\operatorname{sign} y^1 = \operatorname{sign} y^2 = -\operatorname{sign}\theta^3. \end{aligned}\right\} \tag{6.87}$$

On the other hand, Eqs. (6.53) are clearly equivalent to the system

$$\dot\theta^1 = \Delta m^1, \quad \dot\theta^2 = \Delta m^2, \quad \dot\theta^3 = \Delta m^3, \tag{6.88}$$

where Δm^1, Δm^2, Δm^3 are the contravariant components of the gyro errors Δm along the same vectors q_1, q_2, q_3.

Comparison of Eqs. (6.87), (6.88) shows that for correction purposes we require only three triads of correcting torques M^2_{y1}, M^1_{x1}, M^3_{x1}; M^2_{y2}, M^1_{x2}, M^3_{x2}; M^2_{y3}, M^1_{x3}, M^3_{x3}, such that each triad is reduced to precession about the directions q_1, q_2, q_3. These triads of torques should be appropriately commuted in accordance with the signs sign x^1, sign x^2, sign y^1(sign y^2) of the coordinates x^1, x^2, $y^1\,(y^2)$.

Let $H_1 = H_2 = H_3 = H$. Then clearly, for the relevant torques

$$\left.\begin{aligned} \boldsymbol{M}_1 &= -M^2_{y1}\boldsymbol{x} + M^1_{x1}\boldsymbol{y} + M^3_{x1}\boldsymbol{z} = Hkq_1,\\ \boldsymbol{M}_2 &= -M^2_{y2}\boldsymbol{x} + M^1_{x2}\boldsymbol{y} + M^3_{x2}\boldsymbol{z} = Hkq_2,\\ \boldsymbol{M}_3 &= -M^2_{y3}\boldsymbol{x} + M^1_{x3}\boldsymbol{y} + M^3_{x3}\boldsymbol{z} = Hkq_3. \end{aligned}\right\} \tag{6.89}$$

Hence

$$\left.\begin{aligned} &M^2_{y1} = -Hkq_1\cdot\boldsymbol{x}, \quad M^1_{x1} = Hkq_1\cdot\boldsymbol{y}, \quad M^3_{x1} = Hkq_1\cdot\boldsymbol{z},\\ &M^2_{y2} = -Hkq_2\cdot\boldsymbol{x}, \quad M^1_{x2} = Hkq_2\cdot\boldsymbol{y}, \quad M^3_{x2} = Hkq_2\cdot\boldsymbol{z},\\ &M^2_{y3} = -Hkq_3\cdot\boldsymbol{x}, \quad M^1_{x3} = Hkq_3\cdot\boldsymbol{y}, \quad M^3_{x3} = Hkq_3\cdot\boldsymbol{z}. \end{aligned}\right\} \tag{6.90}$$

We form the torques

$$\left.\begin{aligned} &\boldsymbol{M}_{1*} = \boldsymbol{M}_1 \operatorname{sign} x^2 = -Hkq_1 \operatorname{sign} x^2,\\ &\boldsymbol{M}_{2*} = \boldsymbol{M}_2 \operatorname{sign} x^1 = Hkq_2 \operatorname{sign} x^1,\\ &\boldsymbol{M}_{3*} = \boldsymbol{M}_3 \operatorname{sign} y^1 = Hkq_3 \operatorname{sign} y^1 \,(= \boldsymbol{M}_3 \operatorname{sign} y^2 = Hkq_3 \operatorname{sign} y^2). \end{aligned}\right\} \tag{6.91}$$

* The phase systems are discussed in more detail in the next section.

The contravariant components of the resultant correction vector

$$\boldsymbol{M} = \boldsymbol{M}_{1*} + \boldsymbol{M}_{2*} + \boldsymbol{M}_{3*} \tag{6.92}$$

are now given, in virtue of (6.87), by

$$\left.\begin{array}{c} M^1 = -Hk \operatorname{sign} \theta^1, \quad M^2 = Hk \operatorname{sign} \theta^2, \\ M^3 = Hk \operatorname{sign} \theta^3. \end{array}\right\} \tag{6.93}$$

The application of the torque $\boldsymbol{M}$ to the stable platform now leads instead of (6.88) to the following equations of motion:

$$\left.\begin{array}{l} \dot{\theta}^1 + k \operatorname{sign} \theta^1 = \Delta m^1, \\ \dot{\theta}^2 + k \operatorname{sign} \theta^2 = \Delta m^2, \\ \dot{\theta}^3 + k \operatorname{sign} \theta^3 = \Delta m^3. \end{array}\right\} \tag{6.94}$$

From these equations we see that for

$$k > \max \{ |\Delta m^1|, |\Delta m^2|, |\Delta m^3| \} \tag{6.95}$$

the deviations θ^1, θ^2, θ^3, and thus also x^1, x^2, y^1, y^2 are eventually reduced to zero.

Eqs. (6.94) predict undamped oscillations (sliding motion) near the zero values of θ^1, θ^2, θ^3.

To avoid these oscillations, we can introduce additional commutating signals y_1, y_2, y_3 using this time not the equalities

$$y_1 = \operatorname{sign} \theta^1, \quad y_2 = \operatorname{sign} \theta^2, \quad y_3 = \operatorname{sign} \theta^3, \tag{6.96}$$

but a different system:

$$\left.\begin{array}{ll} y_i = 0 & \text{for } |\theta^i| \leqslant c, \\ y_i = \operatorname{sign} \theta^i & \text{for } |\theta^i| > c \\ (i = 1, 2, 3). & \end{array}\right\} \tag{6.97}$$

In this case, there is a dead (insensitive) zone of dimensions $c \times c \times c$ around the point $x^1 = x^2 = y^1 = 0$ as in a normal mechanical relay, and the star image remains inside this zone.

Apart from the (generally small) errors associated with the insensitive zone, the projections Δm_x, Δm_y, Δm_z can be taken equal to zero in the first group of error equations, as in the case of linear correction.

From (6.48), (6.94), (6.95) we see that the performance of the relay-type correction system depends on the angle B between the directions to the two reference stars. The smaller the angle B, the larger the value of $|\Delta m^i|$ for a given $|\Delta m_{..}|$, and therefore k should be increased. However, the performance of linear systems also deteriorates at small angles B. We see from (6.50) that for small values of $\sin B$, small values of the coordinates x^1, x^2 may correspond to large values of θ_x, θ_y, θ_z, i.e., the effective sensitivity of the correction system is lowered as the angle B between the observed stars is made smaller.

Consider relay-type correction for a controlled stable platform. By analogy with Eqs. (6.89)—(6.91), we form the correcting torques using the equalities

$$\left.\begin{aligned} M_y^2 &= -Hkx \cdot q_3 \operatorname{sign} \theta^3, \\ M_x^1 &= Hky \cdot q_3 \operatorname{sign} \theta^3, \\ M_x^3 &= Hkz \cdot q_3 \operatorname{sign} \theta^1, \end{aligned}\right\} \tag{6.98}$$

where the products $x \cdot q_3$, $y \cdot q_3$, $z \cdot q_3$ are no longer constant, as in Eqs. (6.90): they are some functions of time and position.

For the contravariant components θ^1, θ^2, θ^3 of the vector θ along the vectors q_1, q_2, q_3 we again obtain Eqs. (6.94) and all the conclusions that follow.

6.3.4. Alternate two-star correction

The previous linear and relay-type correction systems made use of two telescopes for simultaneous observation of two stars. A different approach to the problem is also possible: one telescope is alternately used to observe two stars. In this arrangement the telescope can be periodically switched from one star to another, i.e., its optical axis is moved from position q_1 pointing to the first star to position q_2 pointing to the second star. Another possibility is to keep the telescope axis fixed, while alternately projecting the images of the two stars onto the optics (with special mirrors or prismatic systems, say).

In either case, a fundamental principle in alternate two-star systems is that the correction torques reducing to zero the deviations x^1, y^1 of the first star image from the center of the focal plane must not increase the coordinates x^2, y^2 of the second star image.

The relay-type correction routine clearly satisfies this requirement and therefore can be readily extended to the case of alternate correction. Indeed, the torque k sign θ^1 entering the first equation (6.94) reduces to zero the component θ^1, and hence the coordinate x^2 as we see from the first equation in (6.48). The platform in this case precesses about the axis q_1, as we see from the first equality in (6.89), and the coordinates x^1, y^1 do not change. The torque k sign θ^2 in the second equation in (6.94) reduces to zero the coordinate x^1 (see second equality in (6.48)). The platform precesses about the axis q_2, and the coordinates x^2, y^2 therefore do not change. Finally, the torque k sign θ^3 simultaneously reduces to zero the coordinates y^1, y^2, irrespective of which particular star is used for correction purposes. The above considerations are clearly valid both for the free and the controlled stable platforms.

The linear correction routine described in the preceding is readily seen to apply to alternate two-star correction of a stable platform. Indeed, when only the first star sighted, x^2 drops out from Eqs. (6.50) and the correcting torques (6.69) thus take the form

$$\left.\begin{aligned} M_y^2 &= -H_2\left(kJ_x + k_{-1}\int_0^t J_x\,dt\right), \\ M_x^1 &= H_1\left(kJ_y + k_{-1}\int_0^t J_y\,dt\right), \end{aligned}\right\} \tag{6.99}$$

$$M_x^3 = H\left(kJ_z + k_{-1}\int_0^t J_z\,dt\right). \qquad (6.99)$$

where, by (6.50), (6.48)

$$J_x = \boldsymbol{x}\cdot(\boldsymbol{q}_2\theta^2 + \boldsymbol{q}_3\theta^3), \quad J_y = \boldsymbol{y}\cdot(\boldsymbol{q}_2\theta^2 + \boldsymbol{q}_3\theta^3), \quad J_z = \boldsymbol{z}\cdot(\boldsymbol{q}_2\theta^2 + \boldsymbol{q}_3\theta^3). \qquad (6.100)$$

The expressions for the torques M_y^2, M_x^1, M_z^1 when the second star is sighted differ from Eqs. (6.99), (6.100) only in that $\boldsymbol{q}_2\theta^2$ is replaced by $\boldsymbol{q}_1\theta^1$.

During the first-star correction interval Eq. (6.70) is thus replaced by the equation

$$\frac{d\theta}{dt} + k(\boldsymbol{q}_2\theta^2 + \boldsymbol{q}_3\theta^3) + k_{-1}\int_0^t (\boldsymbol{q}_2\theta^2 + \boldsymbol{q}_3\theta^3)\,dt = \Delta\boldsymbol{m}. \qquad (6.101)$$

This vector equation is clearly equivalent to a system of scalar equations:

$$\dot\theta = \Delta m^1, \quad \dot\theta^2 + k\theta^2 + k_{-1}\int_0^t \theta^2\,dt = \Delta m^2, \quad \dot\theta^3 + k\theta^3 + k_{-1}\int_0^t \theta^3\,dt = \Delta m^3. \qquad (6.102)$$

The solution of each of the last two equations in (6.102) is analogous to the solution (6.75) of Eq. (6.70). The first equation in (6.102) coincides with the first equation in (6.88). Using Eqs. (6.48), we conclude that tracking the first star reduces the coordinates x^1, y^1 (and thus also y^2) to zero. The coordinates x^2 is not affected, i.e., it varies as if there were no correction of the other coordinates. A similar result is obtained by tracking the second star: the coordinates x^2, y^2 are zeroed and x^1 is not affected. The position clearly does not change if the coefficient k_{-1} in Eqs. (6.99) is zero.

Correcting torques formed in accordance with Eqs. (6.81) also solve the linear correction problem with alternate star tracking in systems with a controlled (and not free) stable platform. Correcting torques corresponding to Eqs. (6.84), on the other hand, are inapplicable to alternate astrocorrection, since these equations contain on the right the projections θ_x, θ_y, θ_z which, in accordance with (6.50), are functions of the coordinates of the two images simultaneously.

It follows from the preceding that simultaneous and alternate observation of two stars through telescopes permits correcting the gyro errors in inertial system.

Tracking a single star, s_1 say, corrects only the error component responsible for θ^2, θ^3. The precession of the inertial platform about the direction to the star, i.e., the error θ^1, is not affected. Total correction of gyro errors is therefore impossible by tracking a single star, and some additional guidance information is required.

6.3.5. The correcting torques

Eqs. (6.50), (6.58), (6.69), (6.81), (6.84), (6.90), (6.98), and (6.99) from which the correcting torques are formed contain the unit vectors q_1, q_2, q_3, or more precisely the projections $q_i \cdot x$, $q_i \cdot y$, $q_i \cdot z$ of these vectors onto the inertial platform axes. The vectors q_1 and q_2 are the unit vectors in the directions of the two stars used in astrocorrection, and the vector q_3 is expressed in terms of q_1 and q_2 by Eq. (6.39).

We will now give a method for calculating the projections of q_1 and q_2 onto the platform axes in two particular cases: when the unperturbed axes maintain a fixed orientation or when they coincide with the co-moving geographical, geocentric, and orthodromic axes.

Having chosen the stars s_1 and s_2 for astrocorrection, we fix the directions q_1 and q_2 in the Earth's coordinate system $O_1\xi\eta\zeta$ at the initial time. The defining relations of the vectors q_1 and q_2 in this system are given in §6.2. Let the initial direction of vectors q_1, q_2 (at the time t^*) be described by the following direction cosines:

$$\begin{array}{cccc} & \xi & \eta & \zeta \\ q_1 & \cos\varphi'^0_{1*}\cos\lambda^0_{1*} & \cos\varphi'^0_{1*}\sin\lambda^0_{1*} & \sin\varphi'^0_{1*} \\ q_2 & \cos\varphi'^0_{2*}\cos\lambda^0_{2*} & \cos\varphi'^0_{2*}\sin\lambda^0_{2*} & \sin\varphi'^0_{2*}, \end{array} \tag{6.103}$$

where φ'^0_{1*}, φ'^0_{2*}, λ^0_{1*}, λ^0_{2*} are the geographical latitudes and longitudes of q_1 and q_2.

If the inertial system uses a free gyroscopically stabilized platform, its initial position relative to the axes $O_1\xi\eta\zeta$ is clearly given. It is described by the direction cosines

$$\begin{array}{cccc} & \xi & \eta & \zeta \\ x & \cos\varphi'^0_x\cos\lambda^0_x & \cos\varphi'^0_x\sin\lambda^0_x & \sin\varphi'^0_x \\ y & \cos\varphi'^0_y\cos\lambda^0_y & \cos\varphi'^0_y\sin\lambda^0_y & \sin\varphi'^0_y \\ z & \cos\varphi'^0_z\cos\lambda^0_z & \cos\varphi'^0_z\sin\lambda^0_z & \sin\varphi'^0_z. \end{array} \tag{6.104}$$

Since the axes x, y, z retain a fixed orientation in space, we obtain the following matrix of direction cosines for the time t^* (assuming the Earth's spin u to be constant, invariably pointing along the axis $O_1\zeta$):

$$\begin{array}{cccc} & \xi & \eta & \zeta \\ x & \cos\varphi'^0_x\cos(\lambda^0_x - ut^*) & \cos\varphi'^0_x\sin(\lambda^0_x - ut^*) & \sin\varphi'^0_x \\ y & \cos\varphi'^0_y\cos(\lambda^0_y - ut^*) & \cos\varphi'^0_y\sin(\lambda^0_y - ut^*) & \sin\varphi'^0_y \\ z & \cos\varphi'^0_z\cos(\lambda^0_z - ut^*) & \cos\varphi'^0_z\sin(\lambda^0_z - ut^*) & \sin\varphi'^0_z. \end{array} \tag{6.105}$$

Matrices (6.105) and (6.103) completely define the projections of the vectors q_1, q_2 (and hence of q_3) on the platform axes x, y, z at the initial time t^*. Since the axes x, y, z retain a fixed orientation, these projections remain constant at any time.

If the unperturbed orientation of the axes x, y, z of the inertial platform coincides with the co-moving geographical axes, their orientation relative to the $O_1\xi\eta\zeta$ axes is described by direction cosines (1.10). Matrix (1.10) together with matrix (6.103) define the projections of the vectors q_1, q_2 on the

axes x, y, z at the initial time t^*. In what follows these projections are assumed constant (contrary to the previous case). To find these projections we extract from matrix (6.103) the following submatrix of direction cosines:

$$\begin{array}{c} \\ q_1 \\ q_2 \end{array} \begin{array}{c} \xi \\ \cos\varphi_{1*}^{\prime 0}\cos\left[\lambda_{1*}^{0}-u(t-t^*)\right] \\ \cos\varphi_{2*}^{\prime 0}\cos\left[\lambda_{2*}^{0}-u(t-t^*)\right] \end{array} \begin{array}{c} \eta \\ \cos\varphi_{1*}^{\prime 0}\sin\left[\lambda_{1*}-u(t-t^*)\right] \\ \cos\varphi_{2*}^{\prime 0}\sin\left[\lambda_{2*}-u(t-t^*)\right] \end{array} \begin{array}{c} \zeta \\ \sin\varphi_{1*}^{\prime 0} \\ \cos\varphi_{2*}^{\prime 0} \end{array} \qquad (6.106)$$

The projections of q_1 and q_2 on the co-moving geocentric and orthodromic axes are similarly obtained. In these cases, the direction cosines (1.14) and (1.16) should be considered jointly with (6.106) and (1.10).

§6.4. ASTROCORRECTION WITH PHASE INDICATION OF STARS

6.4.1. The principle of phase indication

In the previous sections we considered astrocorrection involving telescopic observations of two stars. These observations measured the coordinates x^1, y^1, x^2, y^2 of the star images in the focal plane of the telescopes or at least determined their signs.

Star indication, i.e., determination of the coordinates x^1, y^1, x^2, y^2, can be carried out with the aid of photoelectric cells mounted in the focal plane of the telescopes. Vidicon or Superorthicon tubes can be used. By orienting the two perpendicular sweeps along the axes $x^1(x^2)$ and $y^1(y^2)$, we obtain the coordinates x^1, y^1, x^2, y^2 in terms of proportional time intervals.

Another example of circuits measuring the coordinates x^1, y^1, x^2, y^2 are the so-called phase circuits. Here a photocell or some other photosensitive device is mounted in the focal plane and the light from the star is modulated by a mechanical chopper. The simplest chopper is clearly a semidisk (Figure 6.6), which comprises a transparent disk with one half blackened. When this chopper rotates with angular speed ω, the photocell output contains a component of frequency $f=\frac{2\pi}{\omega}$. By isolating this variable component with a narrow-band filter, we clearly obtain a harmonic modulation signal of the star light. Comparing the phase of this signal with the phase of some master signal, we can determine the angular position of the direction to the star in the telescopic field of view.

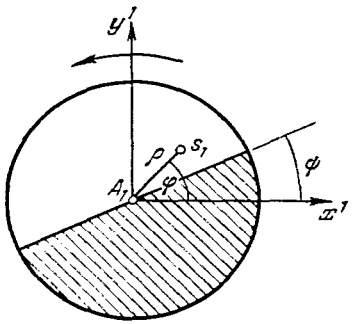

FIGURE 6.6.

This system generally does not measure the deviation of the direction to the star from the telescope axis for an arbitrary position of the star image in the telescopic field, although in principle this is also possible. A semicircular chopping disk whose transparency is a function of the distance from the center is clearly sufficient for this purpose.

If the star is located near the telescope axis, the modulation depth is a function of the distance of the image from the center of the focal plane (because of the finite size of the image). The photocell output signal thus contains in this case information on the deviation of the direction to the star from the optical axis of the telescope.

Television circuits collecting information on the position of the star image in the focal plane are not considered in the following, since their analysis is a pure radio-engineering problem. The analysis of circuits with mechanical choppers, on the other hand, raises a number of problems whose solution requires the application of the methods of theoretical mechanics. We will thus concentrate on phase systems with a mechanical semidisk chopper.

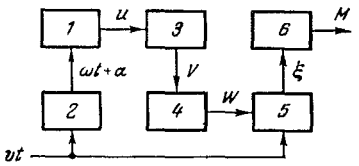

FIGURE 6.7.

The star indication circuit (which isolates the signal received from the star) includes (Figure 6.7) a telescope 1 with a chopper and a photoelectric cell, the chopper drive 2, amplifier 3, resonance filter 4, synchronous phase detector 5, and rectifying filter 6. A reference frequency ω is delivered to units 2 and 5.

If the star is regarded as a point source, the telescope optics are assumed to be ideal, and the error-free case is considered, a point image s_1 will be formed in the focal plane. Let (Figure 6.6) the direction from point A_1 to point s_1 make an angle φ with the axis x^1. Then

$$x^1 = \rho\cos\varphi, \quad y^1 = \rho\sin\varphi, \tag{6.107}$$

where ρ is the distance from the star image s_1 to the optical axis of the telescope. Initially the dividing line between the transparent and the opaque halves of the chopper makes an angle ψ with the axis x^1. If the chopper rotates with angular velocity ω (the sense of rotation is indicated by the arrow in Figure 6.6), the output signal E of the photocell is given by

$$E = \frac{E_0}{2}[1 - \operatorname{sign}\sin(\omega t - \varphi + \psi)]. \tag{6.108}$$

Here the coefficient E_0 characterizes the brightness of the star and the photocell response.

All the information on the position of the star is contained in the phase φ of the signal (6.108), and systems of this kind are therefore known as phase sensitive (or simply phase) systems.

The photocell output signal is amplified and then passed through a resonance filter tuned to the frequency ω. Assuming a linear amplifier and writing μ for its gain, we have for the filter output

$$p_* = \frac{p}{2}[1 - \operatorname{sign}\sin(\omega t - \varphi + \psi)], \tag{6.109}$$

where $p = \mu E_0$.

Consider an ideal filter, which only transmits the frequency ω. $p_s(t)$ is a periodic function of period $2\pi/\omega$. Its Fourier expansion has the form

$$p = \frac{a_0^0}{2} + \sum_{k=1}^{\infty} (a_k^0 \cos k\omega t + b_k^0 \sin k\omega t), \tag{6.110}$$

where

$$\left.\begin{aligned} a_0^0 &= \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} p_s \, dt, \\ a_k^0 &= \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} p_s \cos k\omega t \, dt, \\ b_k^0 &= \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} p_s \sin k\omega t \, dt. \end{aligned}\right\} \tag{6.111}$$

Inserting (6.109) in (6.111) we obtain for the coefficients of the first harmonics

$$a_1^0 = -\frac{2p}{\pi} \sin(\varphi - \psi), \quad b_1^0 = \frac{2p}{\pi} \cos(\varphi - \psi). \tag{6.112}$$

The filtered signal V' thus has the form

$$V' = \frac{2p}{\pi} \sin(\omega t + \psi - \varphi). \tag{6.113}$$

This signal is delivered to a synchronous detector (Figure 6.7). A common multiplicative detector multiplies the input signal by $\sin(\omega t + \alpha)$, and a commutative detector multiplies by sign $\sin(\omega t + \alpha)$. Writing ϑ_1, ϑ_2 for the output signals of a multiplicative and a commutative synchronous detector, we find

$$\left.\begin{aligned} \vartheta_1 &= \frac{2p}{\pi} \sin(\omega t + \psi - \varphi) \sin(\omega t + \alpha), \\ \vartheta_2 &= \frac{2p}{\pi} \sin(\omega t + \psi - \varphi) \operatorname{sign} \sin(\omega t + \alpha). \end{aligned}\right\} \tag{6.114}$$

Here α is the phase difference between the detector master signal of frequency ω and the chopper master signal.

The next element in Figure 6.7 is a rectifier. It transmits only the constant component of the signals (6.114). Inserting the right-hand sides of Eqs. (6.114) for $p_s(t)$ in the first equation in (6.111), we obtain for the constant components

$$\left.\begin{aligned} \bar{\vartheta}_1 &= \frac{p}{\pi} \cos(-\varphi + \psi - \alpha), \\ \bar{\vartheta}_2 &= \frac{4p}{\pi} \cos(-\varphi + \psi - \alpha). \end{aligned}\right\} \tag{6.115}$$

In either case $\bar{\vartheta}$ is proportional to $\cos(-\varphi + \psi - \alpha)$. If the proportionality coefficient is μ', we have

$$\bar{\vartheta} = \mu' \cos(-\varphi + \psi - \alpha). \tag{6.116}$$

Putting

$$\psi - \alpha_1 = \frac{\pi}{2}, \quad \psi - \alpha_2 = \pi, \tag{6.117}$$

we obtain from (6.116)

$$\bar{\vartheta}' = \mu' \sin\varphi, \quad \bar{\vartheta}'' = \mu' \cos\varphi. \tag{6.118}$$

Comparison of (6.118) and (6.107) shows that

$$\bar{\vartheta}'' = \frac{\mu'}{\rho} x^1, \quad \bar{\vartheta}' = \frac{\mu'}{\rho} y^1, \tag{6.119}$$

i.e., $\bar{\vartheta}''$, $\bar{\vartheta}'$ are proportional to the coordinates x^1, y^1 of the star image in the telescope focal plane. Eqs. (6.119) determine only the sign of the coordinates and their ratio. However, we have seen that this is quite sufficient for correcting the gyro errors in inertial systems.

We have so far assumed a constant φ. All the relations remain in force, however, if φ is variable, but the rate of change $\dot{\varphi}$ is small compared to ω (this will be proved in what follows).

6.4.2. The dynamics of a closed phase astrocorrection system near the state of equilibrium. Stability analysis

We will now consider the characteristic features of the different constituent elements of a phase system and how they affect the astrocorrection routine on the whole.*

We no longer assume an ideal telescope and a point image. The image brightness is treated as a function of the distance of the points of the disk from the center. Let, as before (Figure 6.6), ψ be the angle that the dividing line between the two halves of the chopper makes with the axis x^1, $x^1 = \rho\cos\varphi$ and $y^1 = \rho\sin\varphi$ are the coordinates of the image center, and r_0 is the radius of the telescopic field of view. The illuminance E of the photocell (equal to the integral over the brightness of the transparent half of the copper) is then clearly a function of the distances $r_0 - \rho$ and $l = \rho\sin(\varphi - \psi)$ of the image center from the boundaries of the integration domain:

$$E = E[r_0 - \rho, \ \rho\sin(\varphi - \psi)]. \tag{6.120}$$

For a chopper rotating with frequency ω, Eq. (6.120) takes the form

$$E = E[r_0 - \rho, \ \rho\sin(\omega t + \varphi - \psi)]. \tag{6.121}$$

If a linear photocell is used, the output signal is

$$u = \mu E + f(t), \tag{6.122}$$

where the coefficient μ is a response characteristic of the receiver, and $f(t)$

* Devyanin, E.A. On the equations of trackers.—Izvestiya AN SSSR, technical cybernetics, No. 1. 1965.

represents the noise, both determinate (sky background noise, say) and random (detector noise, random background component).*

We further assume that the amplifier (Figure 6.7) is described by the equation

$$P(D)V = R(D)u, \tag{6.123}$$

where $D = \frac{d}{dt}$ is the differentiation operator, $P(D)$ and $R(D)$ are polynomials with constant coefficients, V is the amplifier output signal.

The synchronous detector, as we know,** can be described by the equations

$$\left.\begin{aligned} W_1 &= h(V)\,\Omega(t), \\ W_2 &= h(V)\,\Omega\left(t + \frac{\pi}{2\omega}\right), \end{aligned}\right\} \tag{6.124}$$

where W_1, W_2 are the detector output signal, $h(V)$ is the nonlinear response of the detector, $\Omega(t)$ is a periodic function of frequency ω. For a commutative detector $\Omega(t) = \operatorname{sign}\sin(\omega t + \alpha)$, and for a multiplicative detector $\Omega(t) = \sin(\omega t + \alpha)$, where α is the phase error in the reference signal.

The resonance filters at the amplifier output are also linear with constant parameters. Their equations are written in the form

$$\left.\begin{aligned} Q_1(D)\,\xi &= S_1(D)\,W_1, \\ Q_2(D)\,\eta &= S_2(D)\,W_2, \end{aligned}\right\} \tag{6.125}$$

where ξ and η are the output signals.

Consider the case of linear correction by tracking the first star. Using Eqs. (6.58), (6.50) and seeing that the "images" of the coordinates x^1 and y^1 in Eqs. (6.125) are respectively the variables η and ξ, we form the correcting torques according to the equations

$$\left.\begin{aligned} M_y^2 &= -H_2 k\left(-x\cdot q_2\frac{\eta}{a\sin B} - x\cdot q_3\frac{\xi}{a}\right), \\ M_x^1 &= H_1 k\left(-y\cdot q_2\frac{\eta}{a\sin B} - y\cdot q_3\frac{\xi}{a}\right), \\ M_z^3 &= H_1 k\left(-z\cdot q_2\frac{\eta}{a\sin B} - z\cdot q_3\frac{\xi}{a}\right). \end{aligned}\right\} \tag{6.126}$$

The motion of a stable platform about its unperturbed state is described, according to (6.59), by the equations

$$\left.\begin{aligned} \dot\theta_x - \frac{M_y^2}{H_2} = \Delta m_x, \quad \dot\theta_y + \frac{M_x^1}{H_1} = \Delta m_y, \\ \dot\theta_z + \frac{M_x^3}{H_1} = \Delta m_z. \end{aligned}\right\} \tag{6.127}$$

* The effect of background noise on the performance of astrocorrection systems with radial slit choppers (in particular, semidisk choppers) is considered in Appendix I. The effect of random noise is considered in the last chapter.

** See, e.g., Krasovskii, A.A. and G.S. Pospelov. Osnovy avtomatiki i tekhnicheskoi kibernetiki (Elements of Automation and Engineering Cybernetics). — Gosenergoizdat. 1962; Bol'shakov, I.A. Transmission of regular and random signals through a commutation-type phase detector. — Vestnik MGU, No.6. 1958.

Eqs. (6.120)—(6.127) should be supplemented by expressions relating θ_x, θ_y, θ_z to ρ and φ. By Eqs. (6.50), (6.107), we have

$$\left.\begin{aligned} \theta_x &= -\boldsymbol{x}\cdot\boldsymbol{q}_2\frac{x'}{a\sin B}-\boldsymbol{x}\cdot\boldsymbol{q}_2\frac{y'}{a},\\ \theta_y &= -\boldsymbol{y}\cdot\boldsymbol{q}_2\frac{x'}{a\sin B}-\boldsymbol{y}\cdot\boldsymbol{q}_3\frac{y'}{a},\\ \theta_z &= -\boldsymbol{z}\cdot\boldsymbol{q}_2\frac{x'}{a\sin B}-\boldsymbol{z}\cdot\boldsymbol{q}_3\frac{y'}{a},\\ x' &= \rho\cos\varphi,\quad y' = \rho\sin\varphi. \end{aligned}\right\} \tag{6.128}$$

Let us try to simplify these equations. We start with Eq. (6.120). The function on the right is a periodic function of $\varphi-\psi$ with period of 2π. It can be expanded in a Fourier series with coefficients which are functions of ρ. Thus, we obtain the representation

$$E=\frac{a_0(\rho)}{2}+\sum_{k=1}^{\infty}[a_k(\rho)\cos k(\varphi-\psi)+b_k(\rho)\sin k(\varphi-\psi)]. \tag{6.129}$$

To first approximation, for an ordinary "bell-shaped" dependence of image brightness on the distance from the center, the right-hand side of Eq. (6.120) can be written in the form

$$E = E_0 + C_1(\rho)\sin(\varphi-\psi), \tag{6.130}$$

where the series expansion of $C_1(\rho)$ in powers of ρ starts with the term linear in ρ. For small ρ, Eq. (6.121) therefore takes the form

$$E = E_0 + E_1\rho\sin(\varphi-\psi), \tag{6.131}$$

where E_1 is a constant coefficient.

Consider the following example. Let the star image be a disk of radius ρ_0 and constant brightness B. If $2\rho_0 < r_0$ (r_0 is the radius of the telescopic field of view), the illuminance E is equal to the brightness B multiplied by the image area corresponding to the transparent half of the chopper disk.

Thus

$$E = B\left\{\pi\rho_0^2\left[1-\frac{1}{\pi}\arccos\frac{\rho}{\rho_0}\sin(\varphi-\psi)\right]+\rho\rho_0\sin(\varphi-\psi)\sqrt{1-\frac{\rho^2}{\rho_0^2}\sin^2(\varphi-\psi)}\right\}. \tag{6.132}$$

Suppose the star image lies near the center of the field of view, so that $\rho/\rho_0 \ll 1$. Expanding the right-hand side of Eq. (6.132) in powers of ρ/ρ_0, we obtain an approximate expression

$$E = B\left[\frac{\pi\rho_0^2}{2}+2\rho_0\rho\sin(\varphi-\psi)\right]. \tag{6.133}$$

This representation, as is readily seen, coincides with Eq. (6.131) if we put $E_0=\frac{B\pi\rho_0^2}{2}$, $E_1 = 2B\rho_0$.

The qualitative variation of $C_1(\rho)$ with increasing ρ is on the whole shown by the graph in Figure 6.8. This variation can be fitted with the function

$$\left.\begin{aligned} C_1(\rho) &= E_1\rho && \text{for } \rho\leqslant\rho_1,\\ C_1(\rho) &= E_1\rho_1 = \text{const} && \text{for } \rho>\rho_1. \end{aligned}\right\} \tag{6.134}$$

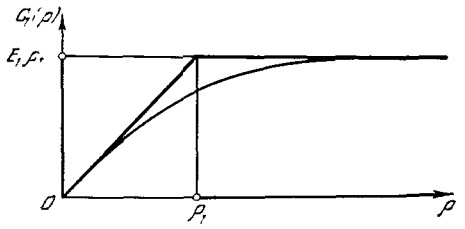

FIGURE 6.8.

Adopting Eq. (6.130) for E and taking $\psi = 0$, we obtain without loss of generality the following equality instead of (6.122):

$$u = \mu C_1(\rho)\sin(\omega t + \varphi) + f_1, \quad f_1 = f(t) + \mu E_0. \tag{6.135}$$

For a multiplicative synchronous detector, taking $h(V) = V$, we write Eqs. (6.124) in the form

$$\left.\begin{aligned} W_1 &= V\sin(\omega t + \alpha), \\ W_2 &= V\cos(\omega t + \alpha). \end{aligned}\right\} \tag{6.136}$$

Eqs. (6.126), (6.128) can also be simplified. Taking $B = \frac{\pi}{2}$, $q_2 = y$, $q_3 = x$, $a = 1$, we obtain

$$\left.\begin{aligned} M_y^2 &= H_2 k\xi, \quad M_x^1 = -H_1 k\eta, \\ \theta_x &= -y^1, \quad \theta_y = -x^1, \\ x^1 &= \rho\cos\varphi, \quad y^1 = \rho\sin\varphi. \end{aligned}\right\} \tag{6.137}$$

The third equation in (6.126) and in (6.128) drops out. Of the three equations in (6.127), we are also left with two only, which using (6.137) take the form

$$\dot\theta_x - k\xi = \Delta m_x, \quad \dot\theta_y - k\eta = \Delta m_y. \tag{6.138}$$

Combining Eqs. (6.135), (6.123), (6.136), (6.125), and the four last equalities in (6.137), we obtain the following set of astrocorrection equations:

$$\left.\begin{aligned} &u = \mu C_1(\rho)\sin(\omega t + \varphi) + f_1, \\ &R(D)u = P(D)V, \\ &W_1 = V\sin(\omega t + \alpha), \quad W_2 = V\cos(\omega t + \alpha), \\ &Q_1(D)\xi = S_1(D)W_1, \quad Q_2(D)\eta = S_2(D)W_2, \\ &\dot\theta_x - k\xi = \Delta m_x, \quad \dot\theta_y - k\eta = \Delta m_y, \\ &\theta_y = \rho\cos\varphi, \quad \theta_x = -\rho\sin\varphi, \\ &\rho = \sqrt{\theta_x^2 + \theta_y^2}. \end{aligned}\right\} \tag{6.139}$$

If the θ_x and θ_y correction channels are symmetric, we have

$$Q_1(D) = Q_2(D), \quad S_1(D) = S_2(D). \tag{6.140}$$

If we now take $C_1(\rho) = E_1\rho$, i.e., use (6.131), the substitution of variables

$$\left.\begin{aligned} a &= \theta_x \sin\omega t + \theta_y \cos\omega t, \\ b &= \theta_x \cos\omega t - \theta_y \sin\omega t \end{aligned}\right\} \tag{6.141}$$

reduces Eqs. (6.139) to a set of linear equations with constant coefficients.

Consider for simplicity the case of an unfiltered commutator output, i.e.,

$$Q_1(D) = Q_2(D) = S_1(D) = S_2(D) = 1; \tag{6.142}$$

the resonance amplifier is tuned to the frequency ν which is close to the chopper frequency ω and it is described by the equation

$$\left(\frac{D^2}{\nu^2} + 2\zeta\frac{D}{\nu} + 1\right)V = u. \tag{6.143}$$

Using (6.131) we get

$$\left.\begin{aligned} &u = \mu E_1[\rho\sin(\omega t + \varphi) + f_2], \quad f_2 = \frac{f_1}{\mu E_1}, \\ &\left(\frac{D^2}{\nu^2} + 2\zeta\frac{D}{\nu} + 1\right)V = u, \\ &\xi = V\sin(\omega t + \alpha), \quad \eta = V\cos(\omega t + \alpha), \\ &\dot\theta_x - k\xi = \Delta m_x, \quad \dot\theta_y - k\eta = \Delta m_y, \\ &\theta_y = \rho\cos\varphi, \quad \theta_x = -\rho\sin\varphi, \\ &\rho = \sqrt{\theta_x^2 + \theta_y^2}. \end{aligned}\right\} \tag{6.144}$$

Substituting the variables from (6.141), we obtain

$$\left.\begin{aligned} &u = \mu E_1(a + f_2), \\ &\left(\frac{D^2}{\nu^2} + 2\zeta\frac{D}{\nu} + 1\right)V = u, \\ &\dot a - \omega b - kV\cos\alpha = \Delta m_x \sin\omega t + \Delta m_y \cos\omega t, \\ &\dot b + \omega a - kV\sin\alpha = \Delta m_x \cos\omega t - \Delta m_y \sin\omega t. \end{aligned}\right\} \tag{6.145}$$

We can now proceed with stability analysis. The characteristic equation of the homogeneous system (6.145) has the form

$$\begin{aligned} \Delta(p) = {} & p^4 + 2\zeta\nu p^3 + (\omega^2 + \nu^2)p^2 + \\ & + 2\nu(\zeta\omega^2 - n\nu\cos\alpha)p + \omega^2\nu(\omega - 2n\sin\alpha), \end{aligned} \tag{6.146}$$

where $n = \frac{\mu E_1 k}{2}$.

Hurwitz's criterion gives the following stability conditions:

$$\left.\begin{aligned} 2\zeta\nu &> 0, \\ \zeta\nu + n\cos\alpha &> 0, \\ n[-\zeta\cos\alpha(\nu^2 - \omega^2) + 2\zeta^2\nu\omega\sin\alpha - n\cos^2\alpha] &> 0, \\ \omega - 2n\sin\alpha &> 0. \end{aligned}\right\} \tag{6.147}$$

The first inequality is always satisfied. The second inequality follows from the third and the fourth. The stability region of the system for $n > 0$ is thus determined by the two inequalities

$$\left.\begin{aligned} -\zeta\cos\alpha(\nu^2 - \omega^2) + 2\zeta^2\nu\omega\sin\alpha - n\cos^2\alpha &> 0, \\ \omega - 2n\sin\alpha &> 0. \end{aligned}\right\} \tag{6.148}$$

The stability region for $n > 0$ in the n, α plane is shown in Figure 6.9. The stability limit crosses the horizontal axis $n = 0$ at the point where

$$\operatorname{tg}\alpha = \frac{\nu^2 - \omega^2}{2\zeta\omega\nu}. \tag{6.149}$$

For $\alpha = \pi$ we get

$$n = \zeta(\nu^2 - \omega^2). \tag{6.150}$$

The stability region in Figure 6.9 is constructed assuming $\omega^2 - \nu^2 > 0$.

For negative n, an analogous stability region is evidently obtained for $\pi < \alpha \leq 2\pi$.

Figure 6.9 leads to the following conclusion. To ensure stable operation of the system, the reference signal of frequency ω should be phase shifted by α. The optimum phase shift for small $\nu^2 - \omega^2$ is close to $\alpha = \frac{\pi}{2}$. This shift compensates the phase difference between the resonance filter output signal V and the input signal u. The smaller ζ, i.e., the higher the Q-factor of the amplifier resonance circuit, the closer are the side branches of the stability limit to the horizontal axis $n = 0$, and the narrower are the tolerance limits of the phase shift α. The latter is associated with the phase—frequency response of the resonance filter: the lower the damping ζ, the greater is the phase difference between V and u near the frequency ω. For $\zeta = 0$, the system is inherently unstable.

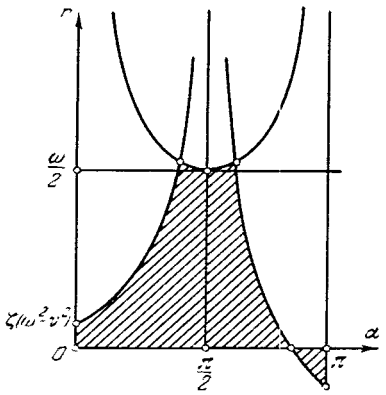

FIGURE 6.9.

Physically this is associated with the finite passband of the filter around the frequency ω in a closed system with the star image moving in the focal plane. As ζ is reduced, the filter passband also decreases.

As we see from Figure 6.9, the stability region rapidly tapers for large n, and n cannot exceed $\omega/2$ (for $\alpha = \pi/2$). Qualitatively this is quite understandable. Large n correspond to rapid displacements of the star image over the chopper during the correction procedure. The chopping rate should clearly be greater than the migration velocity of the star image. Otherwise, the correction system cannot function. Therefore, for a given finite correction speed, the chopping frequency ω cannot be made arbitrarily small.

§6.5. COMBINED ASTRO-DOPPLER CORRECTION

6.5.1. System with simultaneous delivery of Doppler and astrocorrection signals to gyro torquers

The error equations of an autonomous inertial system fall into two groups of differential equations which can be solved independently. Astrocorrection of the gyro elements in inertial systems, as we have seen in the preceding, affect only the second group of error equations. The first group does not change. More precisely, the left-hand sides of the error equations in the first group do not change. The gyro errors Δm_x, Δm_y, Δm_z on the right either drop out altogether (after the termination of the transients in the astrocorrection circuits) or take the form (6.66), (6.77).

In the preceding chapters we saw that altimeter correction and the fundamental Doppler correction techniques conversely change only the first group of error equations, and do not affect the second group. In this case, the effect of joint astro-Doppler correction (together with altimeter correction) is simply the sum of the individual effects of each correction technique separately.

However, in the particular case of an inertial system using a controlled stable platform (or a controlled gyroframe), the Doppler correction can be devised so that it does affect the second group of error equations. This case was studied in 5.6.1. Its characteristic feature is that the Doppler correction signals are delivered to the control torquers of the gyroplatform. Joint astro-Doppler correction therefore requires a special analysis in this case, as the two groups of error equations are no longer separated.

Consider, as in 5.6.1, a two-accelerometer inertial system which measures the orthodromic coordinates. This system uses a controlled gyroplatform with the unperturbed z axis directed along the radius-vector r, i.e., along the geocentric vertical, and the accelerometers point along the axes x and y of the platform lying in the horizontal plane. The ideal operation equations of this system without Doppler and astronomical correction are Eqs. (2.23).

The astro-Doppler correction signals are introduced into the system in the following way. The first two equations in (2.23) are replaced by Eqs. (5.49). The other equations in (2.23), except those for the control torques M_y^2, M_x^1, M_x^3, remain as before. The control torques are defined by the equations

$$\left.\begin{aligned} M_y^2 &= -H_2\left(\omega_x - k_2\frac{\Delta V_y}{r}\right) + M_{y*}^2, \\ M_x^1 &= H_1\left(\omega_y + k_2\frac{\Delta V_x}{r}\right) + M_{x*}^1, \\ M_x^3 &= H_3\left[\left(\omega_y + k_2\frac{\Delta V_x}{r}\right)\operatorname{tg} z + \right. \\ &\quad \left. + \frac{u}{\cos z}\left(\beta_{31}\cos S + \beta_{32}\sin S\right)\right] + M_{x*}^3. \end{aligned}\right\} \qquad (6.151)$$

Here M_{y*}^2, M_{x*}^1, M_{x*}^3 are the astrocorrection torques. If they are zero, Eqs. (6.151) for M_y^2, M_x^1, M_x^3 reduce to Eqs. (5.321).

The astrocorrection torques are formed using Eqs. (6.81) where we may take for simplicity $k_{-1} = 0$. Then

$$\left.\begin{aligned} M^{\cdot}_{y_0} &= -H_2 k\theta_x, \qquad M^{1}_{x_0} = H_1 k\theta_y, \\ M^{1}_{z_0} &= H_3 k\theta_z. \end{aligned}\right\} \tag{6.152}$$

We can now write the error equations of this system.

Varying Eqs. (5.49) together with the third and the fourth equations in (2.23), we obtain Eqs. (5.322),

$$\left.\begin{aligned} &2\omega_y \Delta\dot{r} + \dot{r}\,\delta\omega_y + (\dot{\omega}_y + \omega_x\omega_z)\Delta r + \\ &+ \frac{d}{dt}(r\,\delta\omega_y) = \delta n_x - r\omega_z\delta\omega_x - r\omega_x\delta\omega_z - k_1\delta\Delta V_x, \\ &-2\omega_x\Delta\dot{r} - \dot{r}\,\delta\omega_x + (-\dot{\omega}_x + \omega_y\omega_z)\Delta r - \\ &- \frac{d}{dt}(r\,\delta\omega_x) = \delta n_y - r\omega_y\delta\omega_z - r\omega_z\delta\omega_y - k_1\delta\Delta V_y. \end{aligned}\right\} \tag{6.153}$$

If the unperturbed platform axes xyz are designated $x_0y_0z_0$, the deviation of the platform from its unperturbed attitude is described by small angles α, β, γ in accordance with the matrix of direction cosines (4.22). The variations $\delta n_{x_0}, \delta n_{y_0}$ are then given by Eqs. (4.26), (4.27), (4.7), (4.8), (4.12).

From (6.152), (6.153), (4.23), we obtain

$$\left.\begin{aligned} \delta\omega_{x_0} &= \dot{\alpha} + \omega_{y_0}\gamma - \omega_{z_0}\beta - \Delta\omega_{x_0} + k_2\frac{\delta\Delta V_{y_0}}{r} - k\theta_{x_0}, \\ \delta\omega_{y_0} &= \dot{\beta} - \omega_{x_0}\gamma + \omega_{z_0}\alpha - \Delta\omega_{y_0} - k_2\frac{\delta\Delta V_{x_0}}{r} - k\theta_{y_0}, \\ \delta\omega_{z_0} &= \dot{\gamma} + \omega_{x_0}\beta - \omega_{y_0}\alpha - \Delta\omega_{z_0} - \\ &\qquad - k_2\frac{\delta\Delta V_{x_0}}{r}\operatorname{tg} z - k\theta_{z_0}. \end{aligned}\right\} \tag{6.154}$$

For $k = 0$ these equations reduce to Eqs. (5.323). $\delta\Delta V_{x_0}$ and $\delta\Delta V_{y_0}$ on the right in Eqs. (6.154), as in (5.323), are given by (5.58).

Eqs. (5.58) contain the variations δv_{x_0} and δv_{y_0}. By the third and the fourth equations in (2.23), we have

$$\left.\begin{aligned} \delta v_{x_0} &= -\omega_{x_0}\Delta r - r\,\delta\omega_{x_0}, \\ \delta v_{y_0} &= \omega_{y_0}\Delta r + r\,\delta\omega_{y_0}. \end{aligned}\right\} \tag{6.155}$$

Inserting these expressions for δv_{x_0}, δv_{y_0} in (5.58), we get

$$\left.\begin{aligned} \delta\Delta V_{x_0} &= r\,\delta\omega_{y_0} + \omega_{y_0}\Delta r + \rho(-u_{z_0}\theta_{x_0} + u_{x_0}\theta_{z_0}) - \\ &\qquad - \rho u_{z_0}\alpha + r\omega_x\gamma + \dot{r}\beta - \Delta V_{Dx_0}, \\ \delta\Delta V_{y_0} &= -r\,\delta\omega_{x_0} - \omega_{x_0}\Delta r - \rho(u_{z_0}\theta_{y_0} - u_{y_0}\theta_{z_0}) - \\ &\qquad - \rho u_{z_0}\beta + r\omega_{y_0}\gamma - \dot{r}\alpha - \Delta V_{Dy_0}. \end{aligned}\right\} \tag{6.156}$$

From (6.154), (6.156) we can now find $\delta\omega_{x_0}$, $\delta\omega_{y_0}$, $\delta\omega_{z_0}$:

$$\left.\begin{aligned} \delta\omega_{x_0} &= \frac{1}{1+k_2}\left\{\dot{\alpha} + \omega_{y_0}\gamma - \omega_{z_0}\beta - \Delta\omega_{x_0} - k\theta_{x_0} + \right. \\ &+ \frac{k_2}{r}[-\omega_{x_0}\Delta r - \rho(u_{z_0}\theta_{y_0} - u_{y_0}\theta_{z_0}) - \rho u_{z_0}\beta + \\ &\left. + r\omega_{y_0}\gamma - \dot{r}\alpha - \Delta V_{Dy_0}]\right\}, \end{aligned}\right\} \tag{6.157}$$

$$\left.\begin{aligned}
\delta\omega_{y_0} &= \frac{1}{1+k_2}\Big\{\dot\beta - \omega_{x_0}\gamma + \omega_{z_0}\alpha - \Delta\omega_{y_0} - \\
&\quad - k\theta_{y_0} - \frac{k_2}{r}\left[\omega_{y_0}\Delta r + \rho(-u_{z_0}\theta_{x_0} + u_{x_0}\theta_{z_0}) - \right. \\
&\quad \left. - \rho u_{z_0}\alpha + r\omega_{x_0}\gamma + \dot r\beta - \Delta V_{Dx_0}\right]\Big\}, \\
\delta\omega_{z_0} &= \gamma + \omega_{z_0}\beta - \omega_{y_0}\alpha - \Delta\omega_{z_0} - k\theta_{z_0} - \\
&\quad - \frac{k_2\,\mathrm{tg}\,z}{1+k_2}\Big[\dot\beta - \omega_{x_0}\gamma + \omega_{z_0}\alpha - \Delta\omega_{y_0} - k\theta_{y_0} + \\
&\quad + \frac{\Delta r}{r}\,\omega_{y_0} + \frac{\rho}{r}(-u_{z_0}\theta_{x_0} + u_{x_0}\theta_{z_0}) - \\
&\quad - \frac{\rho}{r}\,u_{z_0}\alpha + \omega_{x_0}\gamma + \frac{\dot r}{r}\beta - \frac{\Delta V_{Dx_0}}{r}\Big].
\end{aligned}\right\} \tag{6.157}$$

Inserting these expressions in Eqs. (6.156), we obtain

$$\left.\begin{aligned}
\delta\Delta V_{x_0} &= \frac{r}{1+k_2}\Big[\dot\beta + \Big(\omega_{z_0} - \frac{\rho}{r}\,u_{z_0}\Big)\alpha - \Delta\omega_{y_0} - \\
&\quad - k\theta_{y_0} + \frac{\Delta r}{r}\,\omega_y + \frac{\rho}{r}(-u_{z_0}\theta_{x_0} + u_{x_0}\theta_{z_0}) + \frac{\dot r}{r}\beta - \frac{\Delta V_{Dx_0}}{r}\Big], \\
\delta\Delta V_{y_0} &= -\frac{r}{1+k_2}\Big[\dot\alpha - \Big(\omega_{z_0} - \frac{\rho}{r}\,u_{z_0}\Big)\beta - \\
&\quad - \Delta\omega_{x_0} - k\theta_{x_0} + \frac{\Delta r}{r}\,\omega_{x_0} + \frac{\rho}{r}(u_{z_0}\theta_{y_0} - u_{y_0}\theta_{z_0}) + \frac{\dot r}{r}\alpha + \frac{\Delta V_{Dy_0}}{r}\Big].
\end{aligned}\right\} \tag{6.158}$$

To complete the derivation of the first group of error equations, it only remains to insert (6.157), (6.158) in (6.153). For δn_{x_0}, δn_{y_0} in these equations we should insert their expressions from (4.26), (4.27), (4.7), (4.8), (4.12).

Let us now proceed with a derivation of the second group of error equations. Varying the fifth and the sixth equations in (2.23), we obtain Eqs. (5.327) and then Eqs. (5.329). Inserting for α_2, β_2, γ_2 their expressions in terms of α, β, γ, θ_{x_0}, θ_{y_0}, θ_{z_0} from (4.39), we write for (5.329)

$$\left.\begin{aligned}
\dot\theta_{x_0} + \dot\alpha + \omega_{y_0}(\theta_{z_0} + \gamma) - \omega_{z_0}(\theta_{y_0} + \beta) &= \delta\omega_{x_0}, \\
\dot\theta_{y_0} + \dot\beta + \omega_{z_0}(\theta_{x_0} + \alpha) - \omega_{x_0}(\theta_{z_0} + \gamma) &= \delta\omega_{y_0}, \\
\dot\theta_{z_0} + \dot\gamma + \omega_{x_0}(\theta_{y_0} + \beta) - \omega_{y_0}(\theta_{x_0} + \alpha) &= \delta\omega_{z_0}.
\end{aligned}\right\} \tag{6.159}$$

These equations, together with (6.157), comprise the second group of error equations.

Further investigation of the error equations of this system is carried out assuming small angular velocity components ω_{x_0}, ω_{y_0}, ω_{z_0} of the axes $x_0y_0z_0$. For simplicity we further take $\rho = r = \text{const}$ and $z = 0$. Then dropping the terms with ω_{x_0}, ω_{y_0}, ω_{z_0}, u_{x_0}, u_{y_0}, u_{z_0} and their derivatives, we obtain from (6.153)

$$\left.\begin{aligned}
\frac{d}{dt}(r\,\delta\omega_{y_0}) &= \delta n_{x_0} - k_1\delta\Delta V_{x_0}, \\
-\frac{d}{dt}(r\,\delta\omega_{x_0}) &= \delta n_{y_0} - k_1\delta\Delta V_{y_0},
\end{aligned}\right\} \tag{6.160}$$

where, by (4.26),

$$\delta n_{x_0} = -\omega_0^2 r\beta + \Delta n_{x_0}, \quad \delta n_{y_0} = \omega_0^2 r\alpha + \Delta n_{y_0}. \tag{6.161}$$

The same simplifications reduce Eqs. (6.157) for $\delta\omega_{x_2}$, $\delta\omega_{y_2}$, $\delta\omega_{z_2}$ to the form

$$\left.\begin{aligned}
\delta\omega_{x_2} &= \frac{1}{1+k_2}\left(\dot{\alpha} - \Delta\omega_{x_0} - k\theta_{x_2} - k_2\frac{\Delta V_{Dy_2}}{r}\right),\\
\delta\omega_{y_2} &= \frac{1}{1+k_2}\left(\dot{\beta} - \Delta\omega_{y_0} - k\theta_{y_2} + k_2\frac{\Delta V_{Dx_2}}{r}\right),\\
\delta\omega_{z_2} &= \dot{\gamma} - \Delta\omega_{z_0} - k\theta_{z_2}.
\end{aligned}\right\} \tag{6.162}$$

Similarly, Eqs. (6.156) are replaced by the simpler expressions

$$\left.\begin{aligned}
\delta\Delta V_{x_2} &= \frac{r}{1+k_2}\left(\dot{\beta} - \Delta\omega_{y_0} - k\theta_{y_2} - \frac{\Delta V_{Dx_2}}{r}\right),\\
\delta\Delta V_{y_2} &= -\frac{r}{1+k_2}\left(\dot{\alpha} - \Delta\omega_{x_0} - k\theta_{x_2} + \frac{\Delta V_{Dy_2}}{r}\right).
\end{aligned}\right\} \tag{6.163}$$

Finally, instead of Eqs. (6.159) we obtain

$$\left.\begin{aligned}
\dot{\theta}_{x_2} + \dot{\alpha} &= \delta\omega_{x_2}, \quad \dot{\theta}_{y_2} + \dot{\beta} = \delta\omega_{y_2},\\
\dot{\theta}_{z_2} + \dot{\gamma} &= \delta\omega_{z_2}.
\end{aligned}\right\} \tag{6.164}$$

The error equations can now be written in a fairly simple form. Thus, substituting in Eqs. (6.160)—(6.163)

$$\delta x = r\beta, \quad \delta y = -r\alpha \tag{6.165}$$

and seeing that $r = \text{const}$, we obtain

$$\left.\begin{aligned}
\delta\ddot{x} + k_1\delta\dot{x}_1 + (1+k_2)\omega_0^2\delta x - rk\dot{\theta}_{y_2} - rk_1k\theta_{y_2} &=\\
= (1+k_2)\Delta n_{x_0} + r\Delta\dot{\omega}_{y_2} &+\\
+ k_1(\Delta V_{Dx_0} + r\Delta\omega_{y_0}) - k_2\Delta\dot{V}_{Dx_2},&\\
\delta\ddot{y} + k_1\delta\dot{y} + (1+k_2)\omega_0^2\delta y + rk\dot{\theta}_{x_2} + rk_1k\theta_{x_2} &=\\
= (1+k_2)\Delta n_{y_0} - r\Delta\dot{\omega}_{x_2} &+\\
+ k_1(\Delta V_{Dy_0} - r\Delta\omega_{x_2}) - k_2\Delta\dot{V}_{Dy_2}.&
\end{aligned}\right\} \tag{6.166}$$

Eqs. (6.166) are the first group of error equations of our system. The second group is obtained from (6.164), (6.162), (6.165):

$$\left.\begin{aligned}
(1+k_2)\dot{\theta}_{x_2} + k\theta_{x_2} - k_2\frac{\delta\dot{y}}{r} &= -\Delta\omega_{x_0} - \frac{k_2}{r}\Delta V_{Dy_0},\\
(1+k_2)\dot{\theta}_{y_2} + k\theta_{y_2} + k_2\frac{\delta\dot{x}}{r} &= -\Delta\omega_{y_0} + \frac{k_2}{r}\Delta V_{Dx_0},\\
\dot{\theta}_{z_2} + k\theta_{z_2} &= -\Delta\omega_{z_0}.
\end{aligned}\right\} \tag{6.167}$$

The total position errors are written, as for an autonomous system, in the form

$$\left.\begin{aligned}
\delta x_2 &= \delta x + r\theta_{y_2},\\
\delta y_2 &= \delta y - r\theta_{x_2}.
\end{aligned}\right\} \tag{6.168}$$

For $k = 0$ Eqs. (6.166)—(6.168) are readily seen to be equivalent to Eqs. (5.335), (5.337), (5.340). Indeed, for $k = 0$ Eqs. (6.166) go into Eqs. (5.335). To establish the equivalence of Eqs. (5.337), (5.340) and Eqs. (6.167), (6.168),

we make the substitution of variables

$$\left.\begin{aligned} \vartheta_{x_0} &= (1+k_2)\,\theta_{x_0} - k_2\frac{\delta y}{r}, \\ \vartheta_{y_0} &= (1+k_2)\,\theta_{y_0} + k_2\frac{\delta x}{r}, \\ \vartheta_{z_0} &= \theta_{z_0}. \end{aligned}\right\} \tag{6.169}$$

Consider the differential equations (6.166), (6.167). The first equation in (6.166) and the second in (6.167) constitute a third-order system of differential equations:

$$\left.\begin{aligned} \delta\ddot{x} + k_1\delta\dot{x} + \omega_0^2(1+k_2)\,\delta x - kr\,(\dot{\theta}_{y_0} + k_1\theta_{y_0}) = \\ = \Delta n_{x_0}(1+k_2) + r\,\Delta\dot{\omega}_{y_0} + \\ + k_1\,(\Delta V_{Dx_0} + r\,\Delta\omega_{y_0}) - k_2\,\Delta\dot{V}_{Dx_0}, \\ (1+k_2)\,\dot{\theta}_{y_0} + k\theta_{y_0} + k_2\frac{\delta\dot{x}}{r} = -\Delta\omega_{y_0} + \frac{k_2}{r}\,\Delta V_{Dx_0}. \end{aligned}\right\} \tag{6.170}$$

A similar system is formed by the second equation in (6.166) and the first in (6.167).

The characteristic equation of each of these two systems is

$$p^3 + (k_1+k)\,p^2 + \left[\omega_0^2(1+k_2) + k_1k\right]p + k\omega_0^2 = 0. \tag{6.171}$$

The coefficients in this equation are such that an appropriate choice of k, k_1, k_2 ensures any required value of the roots. The stability of the astro-Doppler correction routine thus can always be guaranteed. The free oscillations in the system are damped.

Note, however, that for $k \to \infty$ Eq. (6.171) can be factored:

$$\left.\begin{aligned} p + k &= 0, \\ p^2 + k_1p + \omega_0^2 &= 0. \end{aligned}\right\} \tag{6.172}$$

In other words, for sufficiently effective astrocorrection, i.e., for large k, the natural frequencies of the inertial system do not change, while their change is clearly one of the main aims of Doppler correction.

If the stability of the system is ensured by an appropriate choice of the coefficients k, k_1, k_2, the steady-state errors in a system with constant instrumental errors are given by

$$\left.\begin{aligned} \theta_{y_0} &= \frac{1}{k}\left(-\Delta\omega_{y_0} + \frac{k_2}{r}\,\Delta V_{Dx_0}\right), \\ \theta_{x_0} &= \frac{1}{k}\left(-\Delta\omega_{x_0} - \frac{k_2}{r}\,\Delta V_{Dy_0}\right), \\ \theta_{z_0} &= -\frac{1}{k}\,\Delta\omega_{z_0}, \\ \delta x &= \frac{\Delta n_{x_0}}{\omega_0^2} + \frac{k_1}{\omega_0^2}\,\Delta V_{Dx_0}, \\ \delta y &= \frac{\Delta n_{y_0}}{\omega_0^2} + \frac{k_1}{\omega_0^2}\,\Delta V_{Dy_0}. \end{aligned}\right\} \tag{6.173}$$

Therefore, in accordance with Eqs. (6.168), the steady-state total position errors δx_2, δy_2 are given by

$$\left.\delta x_2 = \frac{1}{\omega_0^2}\,(\Delta n_{x_0} + k_1\,\Delta V_{Dx_0}) + \frac{1}{k}\,(k_2\,\Delta V_{Dx_0} - r\,\Delta\omega_{y_0}),\right\} \tag{6.174}$$

$$\delta y_2 = \frac{1}{\omega_0^2}(\Delta n_{y_2} + k_1 \Delta V_{Dy_2}) + \frac{1}{k}(k_2 \Delta V_{Dy_2} + r \Delta\omega_{x_2}). \quad \Big\} \tag{6.174}$$

It is interesting to compare the steady-state errors in combined astro-Doppler correction with those in systems with two separable groups of error equations.

Let the Doppler correction be introduced in accordance with Eqs. (5.196) or, equivalently, Eqs. (5.217).

Without astrocorrections, the simplifications used in the derivation of (6.166) give Eqs. (5.311), (5.312) for the first group of error equations. Replacing Δm_{x_2}, Δm_{y_2} with $-\Delta\omega_{x_2}$, $-\Delta\omega_{y_2}$, we obtain

$$\begin{aligned} &\delta\ddot{x} + k_1\delta\dot{x} + (1+k_2)\omega_0^2\delta x = (1+k_2)\Delta n_{x_2} + \\ &\quad + r\Delta\dot{\omega}_{y_2} - k_2\Delta\dot{V}_{Dx_2} + k_1(\Delta V_{Dx_2} - r\Delta\omega_{y_2}), \\ &\delta\ddot{y} + k_1\delta\dot{y} + (1+k_2)\omega_0^2\delta y = (1+k_2)\Delta n_{y_2} - \\ &\quad - r\Delta\dot{\omega}_{x_2} - k_2\Delta\dot{V}_{Dy_2} + k_1(\Delta V_{Dy_2} - r\Delta\omega_{x_2}). \end{aligned} \quad \Big\} \tag{6.175}$$

The second group of error equations includes

$$\dot{\theta}_{x_2} = -\Delta\omega_{x_2}, \quad \dot{\theta}_{y_2} = -\Delta\omega_{y_2}, \quad \dot{\theta}_{z_2} = -\Delta\omega_{z_2}. \tag{6.176}$$

The total position errors are calculated from (6.168).

If now astrocorrection is introduced, forming the correcting torques by Eqs. (6.152), the second group of error equations take the form

$$\begin{aligned} \dot{\theta}_{x_2} + k\theta_{x_2} = -\Delta\omega_{x_2}, \quad \dot{\theta}_{y_2} + k\theta_{y_2} = -\Delta\omega_{y_2}, \\ \dot{\theta}_{z_2} + k\theta_{z_2} = -\Delta\omega_{z_2}. \end{aligned} \quad \Big\} \tag{6.177}$$

Eqs. (6.175) also change. As we have seen in 6.3.2, $\Delta\omega_{x_2}$, $\Delta\omega_{y_2}$ are replaced with

$$\Delta\omega'_{x_2} = \Delta\omega_{x_2} + k\theta_{x_2}, \quad \Delta\omega'_{y_2} = \Delta\omega_{y_2} + k\theta_{y_2}, \tag{6.178}$$

which in our particular case follows directly from a comparison of Eqs. (6.176) and (6.177).

For a system with astro-Doppler correction based on Eqs. (5.196), (6.152), the simplified system of error equations is

$$\begin{aligned} &\delta\ddot{x} + k_1\delta\dot{x} + (1+k_2)\omega_0^2\delta x = \\ &\quad = (1+k_2)\Delta n_{x_2} + r(\Delta\dot{\omega}_{y_2} + k\dot{\theta}_{y_2}) - k_2\Delta\dot{V}_{Dx_2} + \\ &\qquad + k_1(\Delta V_{Dx_2} + r\Delta\omega_{y_2} + rk\theta_{y_2}), \\ &\delta\ddot{y} + k_1\delta y + (1+k_2)\omega_0^2\delta y = \\ &\quad = (1+k_2)\Delta n_{y_2} - r(\Delta\dot{\omega}_{x_2} + k\dot{\theta}_{x_2}) - k_2\Delta\dot{V}_{Dy_2} + \\ &\qquad + k_1(\Delta V_{Dy_2} - r\Delta\omega_{x_2} - rk\theta_{x_2}), \\ &\dot{\theta}_{x_2} + k\theta_{x_2} = -\Delta\omega_{x_2}, \quad \dot{\theta}_{y_2} + k\theta_{y_2} = -\Delta\omega_{y_2}, \\ &\dot{\theta}_{z_2} + k\theta_{z_2} = -\Delta\omega_{z_2}, \\ &\delta x_2 = \delta x + r\theta_{y_2}, \quad \delta y_2 = \delta y - r\theta_{x_2}. \end{aligned} \quad \Big\} \tag{6.179}$$

For constant instrumental errors, the steady-state errors from (6.179) are

$$\left.\begin{aligned}
&\theta_{x_0} = -\frac{\Delta\omega_{x_0}}{k}, \quad \theta_{y_0} = -\frac{\Delta\omega_{y_0}}{k}, \quad \theta_{z_0} = -\frac{\Delta\omega_{z_0}}{k},\\
&\delta x = \frac{\Delta n_{x_0}}{\omega_0^2} + \frac{k_1 \Delta V_{Dx_0}}{\omega_0^2(1+k_2)},\\
&\delta y = \frac{\Delta n_{y_0}}{\omega_0^2} + \frac{k_1 \Delta V_{Dy_0}}{\omega_0^2(1+k_2)},\\
&\delta x_2 = \frac{\Delta n_{x_0}}{\omega_0^2} + \frac{k_1 \Delta V_{Dx_0}}{\omega_0^2(1+k_2)} - \frac{r\,\Delta\omega_{y_0}}{k},\\
&\delta y_2 = \frac{\Delta n_{y_0}}{\omega_0^2} + \frac{k_1 \Delta V_{Dy_0}}{\omega_0^2(1+k_2)} + \frac{r\,\Delta\omega_{x_0}}{k}.
\end{aligned}\right\} \qquad (6.180)$$

Similarly for a system with astro-Doppler correction based on Eqs. (5.198), (6.152), the error equations are obtained from (5.313), (6.177), (6.178), (6.168):

$$\left.\begin{aligned}
&\delta\ddot{x} + k_1\delta\dot{x} + \omega_0^2(1+k_2)\delta x =\\
&\quad = \Delta n_{x_0}(1+k_2) + \frac{k_1}{\omega_0^2}\Delta\dot{n}_{x_0} + r(\Delta\dot{\omega}_{y_0} + k\dot{\theta}_{y_0}) -\\
&\qquad - k_2\Delta\dot{V}_{Dx_0} - \frac{k_1}{\omega_0^2}\Delta\ddot{V}_{Dx_0},\\
&\delta\ddot{y} + k_1\delta\dot{y} + \omega_0^2(1+k_2)\delta y =\\
&\quad = \Delta n_{y_0}(1+k_2) + \frac{k_1}{\omega_0^2}\Delta\dot{n}_{y_0} - r(\Delta\dot{\omega}_{x_0} + k\dot{\theta}_{x_0}) -\\
&\qquad - k_2\Delta\dot{V}_{Dy_0} - \frac{k_1}{\omega_0^2}\Delta\ddot{V}_{Dy_0},\\
&\dot{\theta}_{x_0} + k\theta_{x_0} = -\Delta\omega_{x_0}, \quad \dot{\theta}_{y_0} + k\theta_{y_0} = -\Delta\omega_{y_0},\\
&\dot{\theta}_{z_0} + k\theta_{z_0} = -\Delta\omega_{z_0},\\
&\delta x_2 = \delta x + r\theta_{y_0}, \quad \delta y_2 = \delta y - r\theta_{x_0}.
\end{aligned}\right\} \qquad (6.181)$$

The steady-state errors for constant instrumental errors are thus

$$\left.\begin{aligned}
&\theta_{x_0} = -\frac{\Delta\omega_{x_0}}{k}, \quad \theta_{y_0} = -\frac{\Delta\omega_{y_0}}{k},\\
&\theta_{z_0} = -\frac{\Delta\omega_{z_0}}{k},\\
&\delta x = \frac{\Delta n_{x_0}}{\omega_0^2}, \quad \delta y = \frac{\Delta n_{y_0}}{\omega_0^2},\\
&\delta x_2 = \frac{\Delta n_{x_0}}{\omega_0^2} - \frac{r\,\Delta\omega_{y_0}}{k},\\
&\delta y_2 = \frac{\Delta n_{y_0}}{\omega_0^2} + \frac{r\,\Delta\omega_{x_0}}{k}.
\end{aligned}\right\} \qquad (6.182)$$

Comparison of Eqs. (6.173), (6.174) with Eqs. (6.180), (6.182) shows that (6.180) and (6.182) give smaller errors. This applies both to orientation and position errors.

In conclusion note that the solutions of these error equations show that for combined astro-Doppler correction the initial errors vanish with time, and the initial setting of the inertial system can be fairly approximate. The permissible deviations of the initial conditions from their exact values may be quite large.

6.5.2. One-star correction

In the previous sections we established that the time-proportional errors associated with the gyro drift can be eliminated with the aid of simultaneous or alternate two-star astrocorrection. Astro-Doppler correction also requires two stars, if the effect of the gyro errors is to be completely compensated. However, in some cases it suffices to reduce the effect of instrumental errors to a certain preset level, which is determined by the overall operating precision of the inertial system. In this case, astro-correction routines using a single star may prove to be quite adequate (at least for constant instrumental errors).

Before proceeding with a discussion of these possibilities, we should consider one point which has been raised in the preceding. In a two-accelerometer autonomous inertial system (with the accelerometers lying in the horizontal plane along the axes x and y of the platform, whose unperturbed z axis points in the direction of r) and in systems with Doppler correction, the (constant) gyro errors do not produce a growing deviation of the platform z axis from the vertical (the direction of the radius-vector r). These deviations remain bounded. Only the azimuthal orientation error grows with time, i.e., the deviation of the axes x and y from the unperturbed orientation in the horizontal plane. The platform thus only drifts about the z axis. Indeed, the platform orientation errors θ_{1x_2}, θ_{1y_2}, θ_{1z_2} are given by

$$\left.\begin{aligned} \theta_{1x_2} = -\frac{\delta y}{r}, \quad \theta_{1y} = \frac{\delta x}{r}, \\ \theta_{1z_2} = -\theta_{z_2}. \end{aligned}\right\} \tag{6.183}$$

Without astrocorrection, the angle θ_{1z_2} thus always increases with time, since

$$\theta_{z_2} = -\Delta\omega_{z_2} t, \tag{6.184}$$

whereas the solutions δx, δy of the first group of error equations are bounded for constant instrumental errors, so that θ_{1x_2}, θ_{1y_2} are also bounded.

In systems with Doppler correction, the steady-state values of the angles θ_{1x_2}, θ_{1y_2} associated with constant errors $\Delta\omega_{x_2}$, $\Delta\omega_{y_2}$ contain a factor $\frac{1}{1+k_2}$. This is clear from Eqs. (5.311)—(5.314) and (5.335). If a sufficiently large k_2 is chosen, the steady-state angles θ_{1x_2}, θ_{1y_2} can be made as small as desired. The axis z of the inertial platform will then remain caged to the vertical with high precision.

The total position errors in systems without astrocorrection contain growing components which depend on $\Delta\omega_{x_2}$, $\Delta\omega_{y_2}$.

However, if the Doppler signals are incorporated in the platform control torques in accordance with Eqs. (5.321), the drift components for a given duration of system operation can be made acceptably small by an appropriate choice of k_2. This follows from Eqs. (5.350). Thus, if the only instrumental errors are gyro errors, the Doppler correction is unable to deal with one serious error only, the azimuthal error θ_{z_2} of the platform orientation. This error is described by Eq. (6.184) and it obviously can be eliminated by using one-star astrocorrection.

Strictly speaking, the preceding argument is applicable only in the absence of any other instrumental errors apart from gyro errors, or when the other errors are small compared to the latter. Thus, we see from (5.350), (5.352) that the effect of Doppler errors grows in proportion to time. Two-star astronomical correction eliminates these errors, as is evident from Eqs. (6.174) for the total position errors.

Further note that the fairly exact vertical in systems with Doppler correction (the angles θ_{1x_0} and θ_{1y_0} being small) in practice suggests the following autocorrection routine. It suffices to measure the angles between the z axis (the vertical) and the axes of the two tracking telescopes and then calculate the current position from equations of the form (6.38). This correction, however, again requires two stars.

Chapter 7

DYNAMICS OF INERTIAL SYSTEMS WITH RANDOM INSTRUMENTAL ERRORS

§7.1. FUNDAMENTAL RELATIONS OF THE CORRELATION THEORY OF RANDOM PROCESSES

The instrumental errors in inertial systems and correction devices can be random, as well as determinate, functions of time. In this case the errors in the navigation parameters also contain random components, and the error equations of inertial systems can be analyzed by the methods of the theory of random processes and statistical dynamics. In what follows we give (without proof) the fundamental relations of this theory.* We will only consider the correlation theory of real stationary random processes (and processes with stationary coupling).

A real random function or random process $X(t)$ is described by a set of real functions $x(t)$ with certain statistical properties. A random process $X(t)$ is completely defined by the corresponding probability density distribution functions $f_1, f_2 \ldots$. The k-th order distribution function $f_k(x_1, t_1, x_2, t_2, \ldots, x_k, t_k)$ of a random process $X(t)$ is defined as the joint probability that the function $X(t)$ at the time t_1 takes a value $x_1 < X(t_1) < x_1 + dx_1$, at the time t_2 a value $x_2 < X(t_2) < x_2 + dx_2$, ... and at the time t_k a value $x_k < X(t_k) < x_k + dx_k$. If the k-th order probability density is known, we in fact have the entire set of probability densities $f_1, f_2, \ldots, f_k$, since**

$$f_{k-1} = \int_{-\infty}^{\infty} f_k \, dx_k. \tag{7.1}$$

Two fundamental concepts of the correlation theory are the expectation value and the correlation function of a random process. The expectation value and the correlation function, as we shall see, are only functions of the first and second order probability densities, i.e., they contain much less information on the random process than the complete system of probability density functions. In most applied problems, however, these two characteristics are quite adequate.

* A detailed and comprehensive treatment of all the relevant topics can be found in Pugachev, V.S. Teoriya sluchainykh funktsii i ee primenenie k zadacham avtomaticheskogo upravleniya (Theory of Random Functions and Its Application to Automatic Control). — Gostekhizdat. 1957.

** Here and in what follows the improper integrals are taken in the sense of the Cauchy principal value.

The expectation value $M[X(t)] = m_x(t)$ of a random function $X(t)$ is defined as

$$m_x(t) = \int_{-\infty}^{\infty} x f_1(x, t)\, dx, \tag{7.2}$$

and the correlation function K_x is defined by the relation

$$K_x(t, t') = M\{[X(t) - m_x(t)][X(t') - m_x(t')]\} = \\ = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [x - m_x(t)][x' - m_x(t')] f_2(x, x', t, t')\, dx\, dx'. \tag{7.3}$$

The dispersion D_x of a random function $X(t)$ is equal to the value of the correlation function at $t' = t$, i.e.,

$$D_x = K_x(t, t) = M[X(t) - m_x(t)]^2. \tag{7.4}$$

The dispersion D_x of a random function $X(t)$ is thus the mathematical expectation of the square of the deviation of the random function from its expectation value. The root mean square deviation $\bar{x}(t)$ of the function $X(t)$ from its expectation value is therefore equal to $\sqrt{D_x(t)}$.

Apart from the correlation functions $K_x(t, t')$, $K_y(t, t')$ of the random processes $X(t)$ and $Y(t)$, we also define the cross-correlation function

$$K_{xy}(t, t') = M\{[X(t) - m_x(t)][Y(t') - m_y(t')]\}. \tag{7.5}$$

If $K_{xy}(t, t') \equiv 0$, the random functions $X(t)$ and $Y(t)$ are said to be uncorrelated. For real random functions

$$K_{xy}(t, t') = K_{yx}(t', t). \tag{7.6}$$

If

$$Z(t) = X(t) + Y(t), \tag{7.7}$$

we have from (7.3), (7.5), (7.6)

$$K_z(t, t') = K_x(t, t') + K_y(t, t') + K_{xy}(t, t') + K_{xy}(t', t). \tag{7.8}$$

Hence it follows that the dispersion of the sum of uncorrelated random functions $X(t)$ and $Y(t)$ is equal to the sum of the dispersions of these functions.

If

$$Y(t) = \frac{d^n X(t)}{dt^n}, \tag{7.9}$$

the correlation function of the random process $Y(t)$ is

$$K_y(t, t') = \frac{\partial^{2n} K_x(t, t')}{\partial t^n\, \partial t'^n}. \tag{7.10}$$

Similarly, if

$$Y(t) = \frac{d^p X(t)}{dt^p}, \quad Z(t) = \frac{d^q X(t)}{dt^q}, \tag{7.11}$$

the cross-correlation function of the random processes $Y(t)$ and $Z(t)$ is

$$K_{yz}(t, t') = \frac{\partial^{n+m} K_y(t, t')}{\partial t^n \partial t'^m}. \tag{7.12}$$

If a random process is given by

$$X(t) = Af(t), \tag{7.13}$$

where A is a random variable which is not a function of time and $f(t)$ is a determinate function of time, we have

$$K_x(t, t') = M[A]^2 f(t) f(t') = D_A f(t) f(t'). \tag{7.14}$$

Similarly for two random processes

$$X = Af(t) \quad \text{and} \quad Y = B\varphi(t) \tag{7.15}$$

the cross-correlation function is

$$K_{xy}(t, t') = M[AB] f(t) \varphi(t'), \tag{7.16}$$

where $M[AB]$ is the cross-correlation moment of the random variables A and B.

In what follows we will mainly deal with stationary and stationarily coupled random processes. Their probability density functions are independent of the choice of the zero point on the time scale. Correspondingly, the correlation and cross-correlation functions depend only on the difference $t - t'$ of the arguments t and t':

$$k_x = k_x(\tau), \quad k_{xy} = k_{xy}(\tau), \quad \tau = t - t'. \tag{7.17}$$

In this case evidently

$$D_x = k_x(0). \tag{7.18}$$

From the definition of the correlation function $k_x(\tau)$ and the cross-correlation functions $k_{xy}(\tau)$, $k_{yx}(\tau)$ we have the following properties:

$$\left.\begin{aligned} &k_x(\tau) = k_x(-\tau), \quad k_{xy}(\tau) = k_{yx}(-\tau), \\ &|k_x(\tau)| \leqslant k_x(0) = D_x = \text{const}, \\ &|k_{xy}(\tau)| \leqslant \sqrt{D_x D_y} = \text{const} \end{aligned}\right\} \tag{7.19}$$

Note that if

$$Z(t) = \frac{d^n X(t)}{dt^n}, \tag{7.20}$$

we have from (7.10)

$$k_z(\tau) = (-1)^n \frac{d^{2n} k_x(\tau)}{d\tau^{2n}}, \tag{7.21}$$

whence it follows that all the derivatives of a stationary random function (if they exist) are also stationary random functions. The inverse proposition,

however, is not true. Integration of a stationary random function may give a *nonstationary function.*

Another characteristic of a stationary random process, besides its correlation function, is the spectral density $S_x(\omega)$, which is defined as the Fourier transform of the correlation function $k_x(\tau)$:

$$S_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} k_x(\tau)\, e^{-j\omega\tau}\, d\tau. \tag{7.22}$$

The correlation function in its turn is expressed in terms of the spectral density by the integral

$$k_x(\tau) = \int_{-\infty}^{\infty} S_x(\omega)\, e^{j\omega\tau}\, d\omega, \tag{7.23}$$

i.e., the correlation function is the Fourier transform of the spectral density.

By (7.23)

$$D_x = k_x(0) = \int_{-\infty}^{\infty} S_x(\omega)\, d\omega. \tag{7.24}$$

Since $k(\tau)$ is an even function, Eq. (7.22) suggests that $S_x(\omega)$ is also an even function.

From (7.8) and (7.22) it follows that the spectral density of two uncorrelated stationary random functions is equal to the sum of their spectral densities.

From the theory of stationary random functions, any such function $X(t)$ is representable over the interval $-T < t < T$ by the expansion

$$X(t) = m_x(t) + \sum_{l=-\infty}^{\infty} a_l e^{j\omega_l t}. \tag{7.25}$$

Here a_l are uncorrelated random variables of zero mathematical expectation, and

$$\omega_l = \frac{\pi l}{2T}. \tag{7.26}$$

The dispersions D_l of the random variables a_l are equal to the Fourier expansion coefficients of the correlation function $k_x(\tau)$ over the interval $-2T < \tau < 2T$:

$$k_x(\tau) = \sum_{l=-\infty}^{\infty} D_l e^{j\omega_l \tau}, \quad \omega_l = \frac{\pi l}{2T}. \tag{7.27}$$

Thus,

$$D_l = M[a_l]^2 = \frac{1}{4T} \int_{-2T}^{2T} k_x(\tau)\, e^{j\omega_l \tau}\, d\tau. \tag{7.28}$$

Taking the limit $T \to \infty$ in (7.27) we obtain (7.23). Taking the limit $T \to \infty$ in (7.25) we obtain an integral representation of the stationary random function $X(t)$:

$$X(t) = m_x(t) + \int_{-\infty}^{\infty} Q(\omega)\, e^{j\omega t}\, d\omega, \tag{7.29}$$

where

$$Q(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(t)\, e^{j\omega t}\, dt, \tag{7.30}$$

so that its correlation function is expressed by

$$K_Q(\omega, \omega') = S_x(\omega)\,\delta(\omega - \omega'), \tag{7.31}$$

where $\delta(\omega - \omega')$ is Dirac's delta function.*

Consider the following problem. Let the differential equation between the random functions $X(t)$ and $Y(t)$ be

$$P(D)\,Y = e(D)\,X, \tag{7.32}$$

where $P(D)$ and $e(D)$ are polynomials with constant coefficients in the differentiation operator $D = \frac{d}{dt}$. Let the function $X(t)$ be stationary and let the expectation value of $X(t)$ and its correlation function $k_x(\tau)$ or spectral density $S_x(\omega)$ be known. Let further the initial conditions of $Y(t)$ be some given random variables. We seek the expectation value and the dispersion of the random function $Y(t)$ which solves Eq. (7.32).

The solution of the homogeneous ($X(t) \equiv 0$) equation (7.32) for random initial conditions is clearly obtained by the same methods as the solution for determinate initial conditions. It is given by the sum of some functions of time with random coefficients, which depend on the random initial conditions. If the expectation values, the dispersions, and the cross-correlation moments of the initial conditions are given, the expectation value and the dispersion of the solution of the homogeneous equation (7.32) can be found without difficulty. If the stability conditions are satisfied, i.e., if the roots α_i of the characteristic equation $P(\alpha) = 0$ have negative real parts, the expectation value and the dispersion of the homogeneous equation (7.32) clearly go to zero with time.

Let us consider the solution of the inhomogeneous equation (7.32) with zero initial conditions. Applying the expectation operation to the two sides of (7.32) and seeing that it commutates with the differentiation operation, we obtain the following equation between the expectation values $m_x(t)$ and $m_y(t)$:

$$P(D)\,m_y(t) = e(D)\,m_x(t). \tag{7.33}$$

This equation is clearly solved by the usual techniques.

It now remains to find the dispersion D_y of the solution of the inhomogeneous equation (7.32). On the right we clearly can substitute for the function

* Generally it is assumed that the correlation function may have discontinuities of the first kind and may incorporate delta functions.

$X(t)$ the centered random process

$$X^0(t) = X(t) - m_x(t) \tag{7.34}$$

with zero expectation value. The corresponding value of $Y(t)$ is designated $Y^0(t)$. This is evidently also a centered random process (of zero expectation value).

Eq. (7.32) is thus replaced by the equation

$$P(D)\,Y^0(t) = e(D)\,X^0(t). \tag{7.35}$$

Let $g(t)$ be the solution of Eq. (7.35) with the unit step function $\delta(t)$ substituted for $X^0(t)$ on the right. $g(t)$ is generally called a weight function or a unit step response.

The solution of Eq. (7.35) with zero initial conditions can thus formally be written in the form

$$Y^0(t) = \int_{t_0}^{t} g(t-\tau)\,X^0(\tau)\,d\tau. \tag{7.36}$$

From this equality and the definitions of correlation function (7.3) and dispersion (7.4) we get

$$\left.\begin{aligned} K_y(t, t') &= \int_{t_0}^{t}\int_{t_0}^{t'} g(t-\tau)\,g(t'-\tau')\,k_x(\tau, \tau')\,d\tau\,d\tau', \\ D_y(t) &= K_y(t, t). \end{aligned}\right\} \tag{7.37}$$

The dispersion $D_y(t)$ of a random function $Y^0(t)$ when the function $X^0(t)$ is stationary can also be found by a different method. From (7.25), the stationary function $X^0(t)$ can be represented by the expansion

$$X^0(t) = \sum_{l=-\infty}^{\infty} a_l e^{j\omega_l t}. \tag{7.38}$$

For each elementary term on the right in Eq. (7.38) we can find a solution $y_l(t, \omega_l)$ of Eq. (7.35). From (7.36) we have

$$y_l(t, \omega_l) = \int_{t_0}^{t} g(t-\tau)\,e^{j\omega_l \tau}\,d\tau. \tag{7.39}$$

Now

$$Y^0(t) = \sum_{l=-\infty}^{\infty} a_l y_l(t, \omega_l), \tag{7.40}$$

and since the random variables a_l are uncorrelated we get

$$D_y(t) = \sum_{l=-\infty}^{\infty} M[a_l]^2\,|\,y(t, \omega_l)\,|^2. \tag{7.41}$$

Now, by (7.28)

$$D_y(t) = \sum_{l=-\infty}^{\infty} D_l\,|\,y_l(t, \omega_l)\,|^2. \tag{7.42}$$

Taking the limit $T \to \infty$ in Eqs. (7.28), (7.39), (7.42) and using Eqs. (7.27), (7.22), (7.23), we obtain for the dispersion

$$D_y(t) = \int_{-\infty}^{\infty} S_x(\omega) |y(t, \omega)|^2 d\omega, \tag{7.43}$$

where

$$y(t, \omega) = \int_{t_0}^{t} g(t-\tau) e^{j\omega\tau} d\tau. \tag{7.44}$$

Eqs. (7.37), (7.42), (7.43) give the dispersion of the random process $Y^0(t)$, and hence of the process $Y(t)$ as a function of time. If the stability conditions are satisfied, we can find the dispersion D_y from the damping of the transients (when the random process $Y^0(t)$ becomes stationary, assuming a stationary $X^0(t)$ by taking $D_y(t=\infty)$. Then clearly

$$|y_i(t, \omega_i)| = |\Phi(j\omega_i)|, \tag{7.45}$$

where $\Phi(j\omega_i)$ stands for the transfer function

$$\Phi(j\omega_i) = \frac{Q(j\omega_i)}{P(j\omega_i)}. \tag{7.46}$$

Eq. (7.42) is now replaced by the series

$$D_y(\infty) = \sum_{i=-\infty}^{\infty} D_i |\Phi(j\omega_i)|^2, \tag{7.47}$$

and (7.43) by the integral

$$D_y(\infty) = \int_{-\infty}^{\infty} S_x(\omega) |\Phi(j\omega)|^2 d\omega. \tag{7.48}$$

We have considered the case of one random function on the right in Eq. (7.35).

If the right-hand side of Eq. (7.35) contains several random functions $X_i(t)$, i.e., if the right-hand side is representable as a sum

$$F(t) = \sum_{i=1}^{n} e_i(D) X_i(t), \tag{7.49}$$

the above methods can be applied to determine the dispersions $D_{y_i}(t)$ corresponding to each random function $X_i(t)$. If no two random functions $X_i(t)$ are correlated, the dispersion $D_y(t)$ corresponding to the sum on the right in (7.49) is simply the sum of the dispersions $D_{y_i}(t)$. If, however, the random functions $X_i(t)$ are correlated, the dispersion can be calculated from relations which follow from Eqs. (7.4), (7.8). In this case we require the cross-correlation functions $K_{y_i y_j}(t, t')$. Clearly, if the solution $y_i(t)$ of Eq. (7.35) with the sum (7.49) on the right corresponds to unit step response $g_i(t)$, and the solution $y_j(t)$ to the unit step response $g_j(t)$, we have

$$K_{y_i y_j}(t, t') = \int_{t_0}^{t} \int_{t_0}^{t'} g_i(t-\tau) g_j(t'-\tau') K_{x_i x_j}(\tau, \tau') d\tau d\tau', \tag{7.50}$$

where $K_{x_i x_j}(\tau, \tau')$ is the cross-correlation function of the random processes $X_i(t)$ and $X_j(t)$ which enter (7.49).

Now, if the right-hand side of Eq. (7.35) is not correlated with the random initial conditions, the dispersion of the solution of Eq. (7.35) with nonzero initial conditions is obtained by taking the sum of the dispersion $D_y(t)$ above with the dispersion of the solution of the corresponding homogeneous equation. If, however, there is correlation between the random initial conditions and the random functions on the right, this correlation clearly must be allowed for in the calculation of the total dispersion.

Note that if several differential equations of the type (7.35) are given,

$$P_i(D)\,Y_i(t) = e_i(D)\,X_i(t), \tag{7.51}$$

and their right-hand sides are correlated, the dispersion of the random function

$$Z(t) = \sum_i Y_i(t) \tag{7.52}$$

should be calculated allowing for the fact that the functions $Y_i(t)$ are also correlated. The calculation is carried out using expressions of the form (7.8), and the cross-correlation functions $K_{y_i y_j}(t, t')$ are found from relations analogous to (7.50).

Any problem of random motions, in particular the analysis of any linear dynamic system reduces to the solution of equations of the form (7.35). The above relations give the expectation value and the dispersion if the statistical characteristics of the random disturbances and the initial conditions are known.

We have so far assumed that the expectation values, the dispersions, and the correlation moments of the random initial conditions are known. The expectation values and the correlation functions (or spectral densities) of the random functions on the right in Eq. (7.35) were also assumed to be given, as well as the cross-correlation functions of these random processes between themselves and with the initial conditions. All these starting statistical characteristics in every particular problem should be either evident from a priori considerations or determined in special tests.

Experimental determination of the expectation values and correlation and cross-correlation functions is a fairly complex undertaking. It cannot be solved unless the relevant random processes are treated as ergodic. In the ergodicity hypothesis is satisfied, one realization $x(t)$ of the random process $X(t)$ in a sufficiently large time interval $0 < t < T$ is all that is required for the determination of the expectation value and the correlation function. Then

$$\left.\begin{aligned} m_x &\approx \frac{1}{T}\int_0^T x(t)\,dt, \\ K_x(\tau) &\approx \frac{1}{T}\int_0^t [x(t+\tau) - m_x][x(t) - m_x]\,dt. \end{aligned}\right\} \tag{7.53}$$

Similarly, from two simultaneous realizations $x(t)$, $y(t)$ of the two random processes $X(t)$, $Y(t)$ we find their cross-correlation function.

§7.2. THE EFFECT OF RANDOM INSTRUMENTAL ERRORS ON AUTONOMOUS INERTIAL SYSTEMS

In connection with autonomous inertial systems we will consider two cases only: the case of a stationary object in the axes $O_1\xi_*\eta_*\zeta_*$ and the case of a satellite in a circular orbit. These are fairly typical cases: the first provides an adequate approximation to slow motion, and the second is representative of satellite motion with small perturbations. These two cases, as we shall see, provide ample information concerning the dependence of system errors on random instrumental errors when the motion parameters change.

7.2.1. The case of a stationary object

Consider an object which is stationary in the axes $O_1\xi_*\eta_*\zeta_*$. The error equations of an autonomous inertial system in this case reduce to the form (see §1.2)

$$\left.\begin{aligned} &\delta\ddot{x} + \omega_0^2\delta x = \Delta n_x - r\,\Delta\dot{m}_y,\\ &\delta\ddot{y} + \omega_0^2\delta y = \Delta n_y + r\,\Delta\dot{m}_x,\\ &\delta\ddot{z} - 2\omega_0^2\delta z = \Delta n_z,\\ &\dot{\theta}_x = \Delta m_x,\ \dot{\theta}_y = \Delta m_y,\ \dot{\theta}_z = \Delta m_z,\\ &\delta x_2 = \delta x + r\theta_y,\ \delta y_2 = \delta y - r\theta_x,\ \delta z_2 = \delta z\\ &\theta_{1x} = -\frac{\delta y}{r},\ \theta_{1y} = \frac{\delta x}{r},\ \theta_{1z} = -\theta_z. \end{aligned}\right\} \tag{7.54}$$

Here δx_2, δy_2, δz_2 are the total position errors, and θ_{1x}, θ_{1y}, θ_{1z} are the platform orientation errors.

As Eqs. (7.54) are symmetric with respect to the variables δx and δy, we need only consider the truncated system

$$\left.\begin{aligned} &\delta\ddot{x} + \omega_0^2\delta x = \Delta n_x - r\,\Delta\dot{m}_y,\\ &\dot{\theta}_y = \Delta m_y,\\ &\delta x_2 = \delta x + r\theta_y,\\ &\theta_{1y} = \frac{\delta x}{r} \end{aligned}\right\} \tag{7.55}$$

plus the additional equation

$$\delta\ddot{z}_2 - 2\omega_0^2\delta z_2 = \Delta n_z. \tag{7.56}$$

Let us consider the accelerometer errors Δn_x, Δn_z and the gyro errors Δm_y as centered random stationary processes. Suppose these processes are uncorrelated and their spectral densities are given by

$$\left.\begin{aligned} S_{\Delta n_x} &= \frac{D_{\Delta n_x}}{\pi}\,\frac{\alpha_x}{\alpha_x^2+\omega^2},\\ S_{\Delta n_z} &= \frac{D_{\Delta n_z}}{\pi}\,\frac{\alpha_z}{\alpha_z^2+\omega^2}, \end{aligned}\right\} \tag{7.57}$$

$$S_{\Delta m_y} = \frac{D_{\Delta m_y}}{\pi} \frac{\beta_y}{\beta_y^2 + \omega^2}. \quad \Big\} \tag{7.57}$$

Finally let the initial conditions of Eqs. (7.55)—(7.56) be independent centered random variables δx^0, $\delta \dot{x}^0$, θ_y^0, δz^0, $\delta \dot{z}^0$ with known dispersions $D_{\delta x^0}$, $D_{\delta \dot{x}^0}$, $D_{\theta_y^0}$, $D_{\delta z^0}$, $D_{\delta \dot{z}^0}$. We assume that the initial conditions are uncorrelated with instrumental errors.

Let us find the dispersions of the random functions δz_2, δx_2 and θ_{1y}.

We start with Eqs. (7.56). Since δz^0, $\delta \dot{z}^0$, Δn_z are independent (uncorrelated), the dispersion $D_{\delta z_2}$ can be written as the sum

$$D_{\delta z_2} = D_{\delta z_2}^{\delta z^0} + D_{\delta z_2}^{\delta \dot{z}^0} + D_{\delta z_2}^{\Delta n_z}. \tag{7.58}$$

The solution of Eq. (7.56) is

$$\delta z_2 = \delta z^0 \operatorname{ch} \omega_0 \sqrt{2}\, t + \frac{\delta \dot{z}^0}{\omega_0 \sqrt{2}} \operatorname{sh} \omega_0 \sqrt{2}\, t, \tag{7.59}$$

and by (7.14) we have

$$D_{\delta z_2}^{\delta z^0} = D_{\delta z^0} \operatorname{ch}^2 \omega_0 \sqrt{2}\, t, \quad D_{\delta z_2}^{\delta \dot{z}^0} = \frac{D_{\delta \dot{z}^0}}{2\omega_0^2} \operatorname{sh}^2 \omega_0 \sqrt{2}\, t. \tag{7.60}$$

It now remains to calculate the dispersion $D_{\delta z_2}^{\Delta n_z}$ of the solution of Eq. (7.56) for zero initial conditions. To this end, note that the unit step response of Eq. (7.56) is

$$g(t) = \frac{1}{\omega_0 \sqrt{2}} \operatorname{sh} \omega_0 \sqrt{2}\, t, \tag{7.61}$$

and the correlation function $K_{\Delta n_z}(t, t')$ of the random process $\Delta n_z(t)$ in virtue of Eqs. (7.23), (7.57) is

$$K_{\Delta n_z}(t, t') = D_{\Delta n_z} e^{-\alpha_z |t - t'|}. \tag{7.62}$$

Therefore, by (7.37), we have for $t_0 = 0$

$$D_{\delta z_2}^{\Delta n_z}(t) = \frac{D_{\Delta n_z}}{2\omega_0^2} \int_0^t \int_0^t \operatorname{sh} \omega_0 \sqrt{2}\,(t - \tau) \operatorname{sh} \omega_0 \sqrt{2}\,(t - \tau') e^{-\alpha_z |\tau - \tau'|}\, d\tau\, d\tau'. \tag{7.62a}$$

Integrating* in the right-hand side of Eq. (7.62a), we find

$$D_{\delta z_2}^{\Delta n_z}(t) = \frac{D_{\Delta n_z}}{2\omega_0^2 (2\omega_0^2 - \alpha_z^2)} \left\{ -\alpha_z \left(\frac{1}{2\omega_0 \sqrt{2}} \operatorname{sh} \omega_0 \sqrt{2}\, t - t \right) + \right.$$
$$\left. + \frac{4\omega_0^2}{2\omega_0^2 - \alpha_z^2} \left[1 - e^{-\alpha_z t} \left(\operatorname{ch} \omega_0 \sqrt{2}\, t + \frac{\alpha_z}{\omega_0 \sqrt{2}} \operatorname{sh} \omega_0 \sqrt{2}\, t \right) \right] + \operatorname{sh}^2 \omega_0 \sqrt{2}\, t \right\}. \tag{7.63}$$

It is readily seen that $D_{\delta z_2}^{\Delta n_z}(0) = 0$, as it should. For small t, series-expanding the right-hand side of (7.63), we obtain the following approximation:

$$D_{\delta z_2}^{\Delta n_z}(t) = D_{\Delta n_z} \left(\frac{t^4}{4} - \alpha_z \frac{t^5}{15} \right). \tag{7.64}$$

* When evaluating the integral, the domain of integration should be divided into two subintervals: $\tau - \tau' > 0$ and $\tau - \tau' < 0$.

For sufficiently large t, when $\operatorname{sh}\omega_0\sqrt{2}t \approx \operatorname{ch}\omega_0\sqrt{2}t \approx \frac{1}{2}e^{\omega_0\sqrt{2}t}$, we obtain from (7.63) the following asymptotic expression for the dispersion:

$$D_{\delta x_2}^{\Delta\eta_z} = \frac{D_{\Delta\eta_z} e^{2\omega_0\sqrt{2}t}}{8\omega_0^3\sqrt{2}(\omega_0\sqrt{2}+\alpha_z)}. \tag{7.65}$$

If $\alpha_z \gg \omega_0$, we have

$$D_{\delta x_2}^{\Delta\eta_z} \approx \frac{D_{\Delta\eta_z} e^{2\omega_0\sqrt{2}t}}{8\omega_0^3\alpha_z\sqrt{2}}. \tag{7.66}$$

Now consider Eqs. (7.55). To find the dispersions $D_{\delta x_2}(t)$ and $D_{\theta_y}(t)$ we clearly require the correlation functions of the random processes δx and θ_y, as well as their cross-correlation function.

By analogy with (7.58), we have

$$K_{\theta_y}(t, t') = K_{\theta_y}^{\theta_y^0}(t, t') + K_{\theta_y}^{\Delta m_y}(t, t'). \tag{7.67}$$

Clearly,

$$K_{\theta_y}^{\theta_y^0} = D_{\theta_y^0}. \tag{7.68}$$

The unit step response of the second equation in (7.55) is unity, and the correlation function $K_{\Delta m_y}(t, t')$ of the random process $\Delta m_y(t)$ is

$$K_{\Delta m_y}(t, t') = D_{\Delta m_y} e^{-\beta_y|t-t'|}. \tag{7.69}$$

Therefore ($t_0 = 0$) we have from the first equation in (7.37)

$$K_{\theta_y}^{\Delta m_y}(t, t') = D_{\Delta m_y} \int_0^t \int_0^t e^{-\beta_y|\tau-\tau'|}\, d\tau\, d\tau', \tag{7.70}$$

whence

$$K_{\theta_y}^{\Delta m_y} = D_{\Delta m_y}\left[\frac{2}{\beta_y}\left(\frac{t+t'}{2} - \frac{|t-t'|}{2}\right) + \right.$$
$$\left. + \frac{1}{\beta_y^2}\left(-e^{-\beta_y|t-t'|} + e^{-\beta_y t'} + e^{-\beta_y t} - 1\right)\right]. \tag{7.71}$$

Now,

$$\left.\begin{aligned} D_{\theta_y}^{\Delta m_y}(t) &= 2D_{\Delta m_y}\left[\frac{t}{\beta_y} + \frac{1}{\beta_y^2}\left(e^{-\beta_y t} - 1\right)\right], \\ D_{\theta_y}(t) &= D_{\theta_y^0} + D_{\theta_y}^{\Delta m_y}. \end{aligned}\right\} \tag{7.72}$$

With time the second term in brackets becomes ignorable and for sufficiently large t we have the asymptotic approximation

$$D_{\theta_y} = \frac{2D_{\Delta m_y} t}{\beta_y}. \tag{7.73}$$

For small t we have the approximation

$$D_{\theta_y} = D_{\Delta m_y} t^2 + D_{\theta_y^0}. \tag{7.74}$$

Consider the first equation in (7.55). Since by assumption Δn_x and Δm_y are uncorrelated random functions, we may write

$$D_{\delta x}(t) = D_{\delta x}^{\delta x^0} + D_{\delta x}^{\delta \dot{x}^0} + D_{\delta x}^{\Delta n_x} + D_{\delta x}^{\Delta m_y}. \tag{7.75}$$

Here, evidently,

$$D_{\delta x}^{\delta x^0} = D_{\delta x^0} \cos^2 \omega_0 t, \quad D_{\delta x}^{\delta \dot{x}^0} = \frac{D_{\delta \dot{x}^0}}{\omega_0^2} \sin^2 \omega_0 t. \tag{7.76}$$

and the problem reduces to the calculation of $D_{\delta x}^{\Delta n_x}$ and $D_{\delta x}^{\Delta m_y}$.

Leaving only Δn_x on the right in the first equation in (7.55), we note that the unit step response in this case is

$$g(t) = \frac{1}{\omega_0} \sin \omega_0 t. \tag{7.77}$$

Hence and using (7.37), (7.57), (7.62) we get

$$D_{\delta x}^{\Delta n_x}(t) = \frac{D_{\Delta n_x}}{\omega_0^2} \int_0^t \int_0^t \sin \omega_0 (t - \tau) \sin \omega_0 (t - \tau') e^{-\alpha_x |\tau - \tau'|} d\tau \, d\tau'. \tag{7.78}$$

Evaluating the integral on the right, we find

$$D_{\delta x}^{\Delta n_x}(t) = \frac{D_{\Delta n_x}}{\alpha_x^2 + \omega_0^2} \left\{ \frac{\alpha_x}{\omega_0^2} \left(t - \frac{1}{2\omega_0} \sin 2\omega_0 t \right) + \right.$$
$$\left. + \frac{2}{\alpha_x^2 + \omega_0^2} \left[1 - e^{-\alpha_x t} \left(\frac{\alpha_x}{\omega_0} \sin \omega_0 t + \cos \omega_0 t \right) \right] - \frac{1}{\omega_0^2} \sin^2 \omega_0 t \right\}. \tag{7.79}$$

For small t we have the approximation

$$D_{\delta x}^{\Delta n_x}(t) = D_{\Delta n_x} \frac{t^4}{4}. \tag{7.80}$$

Conversely, for large t, Eq. (7.79) gives the asymptotic expression

$$D_{\delta x}^{\Delta n_x}(t) = \frac{D_{\Delta n_x}}{\alpha_x^2 + \omega_0^2} \left[\frac{\alpha_x}{\omega_0^2} \left(t - \frac{1}{2\omega_0} \sin 2\omega_0 t \right) - \frac{1}{\omega_0^2} \sin^2 \omega_0 t \right] \tag{7.81}$$

or approximately

$$D_{\delta x}^{\Delta n_x}(t) = \frac{D_{\Delta n_x}}{\alpha_x \omega_0^2} t, \tag{7.82}$$

if we may take

$$\alpha_x^2 \gg \omega_0^2. \tag{7.83}$$

We now leave only $-r \Delta \dot{m}_y$ on the right in the first equation in (7.55). The unit step response to this disturbance is given by Eq. (7.77). At the same time, by (7.69) and (7.21),

$$K_{\Delta \dot{m}_y}(t, t') = D_{\Delta m_y} \left[2\beta_y \delta(t - t') - \beta_y^2 e^{-\beta_y |t - t'|} \right]. \tag{7.84}$$

Therefore,

$$D_{\delta x}^{\Delta m_y}(t) = \frac{\beta_y r^2 D_{\Delta m_y}}{\omega_0^2} \int_0^t \int_0^t \sin\omega_0(t-\tau)\sin\omega_0(t-\tau') \times \\ \times \left[2\delta(\tau-\tau') - \beta_y e^{-\beta_y|\tau-\tau'|}\right] d\tau\, d\tau'. \tag{7.85}$$

But

$$\left.\begin{aligned} &\int_0^t \int_0^t \sin\omega_0(t-\tau)\sin\omega_0(t-\tau')\,\delta(\tau-\tau')\,d\tau\,d\tau' = \frac{1}{2}\left(t - \frac{1}{2\omega_0}\sin 2\omega_0 t\right), \\ &\int_0^t \int_0^t \sin\omega_0(t-\tau)\sin\omega_0(t-\tau')\,e^{-\beta_y|\tau-\tau'|}\,d\tau\,d\tau' = \\ &= \int_0^t \left[\sin\omega_0(t-\tau')\int_0^{\tau'} \sin\omega_0(t-\tau)\,e^{-\beta_y(\tau'-\tau)}\,d\tau + \right. \\ &\left. + \sin\omega_0(t-\tau')\int_{\tau'}^t \sin\omega_0(t-\tau)\,e^{-\beta_y(\tau-\tau')}\,d\tau\right] d\tau' = \\ &= \frac{1}{\beta_y^2+\omega_0^2}\left\{\beta_y\left(t - \frac{1}{2\omega_0}\sin 2\omega_0 t\right) + \right. \\ &\left. + \frac{2\omega_0^2}{\beta_y^2+\omega_0^2}\left[1 - e^{-\beta_y t}\left(\frac{\beta_y}{\omega_0}\sin\omega_0 t + \cos\omega_0 t\right)\right] - \sin^2\omega_0 t\right\}. \end{aligned}\right\} \tag{7.86}$$

Therefore

$$D_{\delta x}^{\Delta m_y}(t) = \frac{r^2\beta_y D_{\Delta m_y}}{\beta_y^2+\omega_0^2}\left\{t - \frac{1}{2\omega_0}\sin 2\omega_0 t - \right. \\ \left. - \frac{2\beta_y}{\beta_y^2+\omega_0^2}\left[1 - e^{-\beta_y t}\left(\frac{\beta_y}{\omega_0}\sin\omega_0 t + \cos\omega_0 t\right)\right] + \frac{\beta_y}{\omega_0^2}\sin^2\omega_0 t\right\}. \tag{7.87}$$

Series expanding the right-hand side in powers of t, we obtain the approximate expression

$$D_{\delta x}^{\Delta m_y}(t) = \frac{2r^2\beta_y(\omega_0^2+2\beta_y^2)D_{\Delta m_y} t^3}{3(\beta_y^2+\omega_0^2)}. \tag{7.88}$$

If $\beta_y^2 \gg \omega_0^2$, Eq. (7.88) can be simplified to the form

$$D_{\delta x}^{\Delta m_y}(t) = \frac{4}{3} r^2\beta_y D_{\Delta m_y} t^3. \tag{7.89}$$

For large t, we obtain from (7.87) the following asymptotic expression for the dispersion:

$$D_{\delta x}^{\Delta m_y}(t) = \frac{r^2\beta_y D_{\Delta m_y}}{\beta_y^2+\omega_0^2}\left(t - \frac{1}{2\omega_0}\sin 2\omega_0 t - \frac{2\beta_y}{\beta_y^2+\omega_0^2} + \frac{\beta_y}{\omega_0^2}\sin^2\omega_0 t\right). \tag{7.90}$$

Approximately,

$$D_{\delta x}^{\Delta m_y}(t) = \frac{r^2 D_{\Delta m_y} t}{\beta_y}. \tag{7.91}$$

If now Eqs. (7.76), (7.79), (7.87) are inserted in the right-hand side of Eq. (7.75), we obtain an expression for the dispersion $D_{\delta x}(t)$ of the random function δx:

$$D_{\delta x} = D_{\delta x^0}\cos^2\omega_0 t + \frac{1}{\omega_0^2} D_{\delta \dot{x}^0}\sin^2\omega_0 t + \\ + \frac{D_{\Delta n_x}}{\alpha_x^2+\omega_0^2}\left\{\frac{\alpha_x}{\omega_0^2}\left(t - \frac{1}{2\omega_0}\sin 2\omega_0 t\right) + \right. \\ + \frac{2}{\alpha_x^2+\omega_0^2}\left[1 - e^{-\alpha_x t}\left(\frac{\alpha_x}{\omega_0}\sin\omega_0 t + \cos\omega_0 t\right)\right] - \\ \left. - \frac{1}{\omega_0^2}\sin^2\omega_0 t\right\} + \frac{r^2\beta_y D_{\Delta m_y}}{\beta_y^2+\omega_0^2}\left\{t - \frac{1}{2\omega_0}\sin 2\omega_0 t - \right. \\ \left. - \frac{2\beta_y}{\beta_y^2+\omega_0^2}\left[1 - e^{-\beta_y t}\left(\frac{\beta_y}{\omega_0}\sin\omega_0 t + \cos\omega_0 t\right)\right] + \frac{\beta_y}{\omega_0^2}\sin^2\omega_0 t\right\}. \qquad (7.92)$$

From the last equation in (7.55) we thus find for the dispersion of θ_{1y}

$$D_{\theta_{1y}}(t) = \frac{1}{r^2} D_{\delta x}(t). \qquad (7.93)$$

For small t, for example, we obtain the following approximation:

$$D_{\theta_{1y}}(t) = \frac{D_{\delta x^0}}{r^2} + \frac{D_{\delta \dot{x}^0}}{r^2} t^2 + \frac{4}{3}\beta_y D_{\Delta m_y} t^3 + \frac{1}{4r^2} D_{\Delta n_x} t^4. \qquad (7.94)$$

For large t we have asymptotically

$$D_{\theta_{1y}} = \frac{D_{\delta x^0}}{r^2}\cos^2\omega_0 t + \frac{D_{\delta \dot{x}^0}}{r^2\omega_0^2}\sin^2\omega_0 t + \\ + \frac{D_{\Delta n_x}}{r^2(\alpha_x^2+\omega_0^2)}\left[\frac{\alpha_x}{\omega_0^2}\left(t - \frac{1}{2\omega_0}\sin 2\omega_0 t\right) + \frac{2}{\alpha_x^2+\omega_0^2} - \frac{1}{\omega_0^2}\sin^2\omega_0 t\right] + \\ + \frac{\beta_y D_{\Delta m_y}}{\beta_y^2+\omega_0^2}\left[t - \frac{1}{2\omega_0}\sin 2\omega_0 t - \frac{2\beta_y}{\beta_y^2+\omega_0^2} + \frac{\beta_y}{\omega_0^2}\sin^2\omega_0 t\right]. \qquad (7.95)$$

Hence it is clear that the contribution of the time-proportional terms to the dispersion $D_{\theta_{1y}}$ becomes progressively more pronounced with time. For sufficiently large t (assuming ω_0^2 to be small compared to α_x^2 and β_y^2) we thus have the approximate equality

$$D_{\theta_{1y}} = \left(\frac{D_{\Delta n_x}}{r^2\alpha_x\omega_0^2} + \frac{D_{\Delta m_y}}{\beta_y}\right)t. \qquad (7.96)$$

To conclude the solution of our problem, we still require the dispersion $D_{\delta x_2}$ of the total error δx_2 in the coordinate x. From the third equality in (7.55) and (7.8) we have

$$K_{\delta x_2}(t, t') = K_{\delta x}(t, t') + r^2 K_{\theta_y}(t, t') + r\left[K_{\delta x, \theta_y}(t, t') + K_{\delta x, \theta_y}(t', t)\right]. \qquad (7.97)$$

Therefore

$$D_{\delta x_2}(t) = D_{\delta x}(t) + r^2 D_{\theta_y}(t) + 2r K_{\delta x, \theta_y}(t, t). \qquad (7.98)$$

The expressions $D_{\delta x}(t)$ and $D_{\theta_y}(t)$ have been determined. They are given by Eq. (7.92) and the second equality in (7.72), respectively. We only need $K_{\delta x, \theta_y}$.

Eqs. (7.55) for δx and θ_y contain the random functions $\Delta \dot{m}_y$ and Δm_y on the right. They are clearly correlated, and by Eqs. (7.12) and (7.69) their cross-correlation function is

$$K_{\Delta \dot{m}_y, \Delta m_y}(t, t') = -D_{\Delta m_y} \beta_y e^{-\beta_y |t-t'|} \operatorname{sign}(t-t'). \tag{7.99}$$

Therefore

$$K_{\delta x, \theta_y}(t, t) = \frac{r \beta_y D_{\Delta m_y}}{\omega_0} \int_0^t \int_0^t \sin \omega_0 (t-\tau)\, e^{-\beta_y |\tau - \tau'|} \operatorname{sign}(\tau - \tau')\, d\tau\, d\tau'. \tag{7.100}$$

Integrating in the right-hand side in (7.100), we find

$$K_{\delta x, \theta_y}(t, t) = \frac{r D_{\Delta m_y}}{\beta_y^2 + \omega_0^2} \Big[1 - e^{-\beta_y t} - \\ - e^{-\beta_y t} \Big(\frac{\beta_y}{\omega_0} \sin \omega_0 t + \cos \omega_0 t \Big) - \frac{\beta_y}{\omega_0} \sin \omega_0 t + \cos \omega_0 t \Big]. \tag{7.101}$$

For small t we have approximately

$$K_{\delta x, \theta_y}(t, t) = -\frac{r \beta_y D_{\Delta m_y}}{6} t^3. \tag{7.102}$$

For large t asymptotically

$$K_{\delta x, \theta_y}(t, t) = \frac{r D_{\Delta m_y}}{\beta_y^2 + \omega_0^2} \Big(1 - \frac{\beta_y}{\omega_0} \sin \omega_0 t + \cos \omega_0 t \Big). \tag{7.103}$$

From Eqs. (7.98), (7.101), (7.92), and (7.72) we now find the following expression for the dispersion in the error δx_2:

$$D_{\delta x_2} = D_{\delta x^0} \cos^2 \omega_0 t + \frac{1}{\omega_0^2} D_{\dot{\delta x}^0} \sin^2 \omega_0 t + r^2 D_{\theta_y^0} + \\ + \frac{D_{\Delta n_x}}{\alpha_x^2 + \omega_0^2} \Big\{ \frac{\alpha_x}{\omega_0^2} \Big(t - \frac{1}{2\omega_0} \sin 2\omega_0 t \Big) + \\ + \frac{2}{\alpha_x^2 + \omega_0^2} \Big[1 - e^{-\alpha_x t} \Big(\frac{\alpha_x}{\omega_0} \sin \omega_0 t + \cos \omega_0 t \Big) \Big] - \frac{1}{\omega_0^2} \sin^2 \omega_0 t \Big\} + \\ + \frac{r^2 \omega_0^2 D_{\Delta m_y}}{\beta_y^2 (\beta_y^2 + \omega_0^2)} \Big[\frac{\beta_y (3\beta_y^2 + 2\omega_0^2)}{\omega_0^2} t - \frac{2(\beta_y^4 + \omega_0^2 \beta_y^2 + \omega_0^4)}{\omega_0^3 (\beta_y^2 + \omega_0^2)} + \\ + \frac{\omega_0^2}{\beta_y^2 + \omega_0^2} e^{-\beta_y t} + \frac{2\beta_y^2}{\omega_0^2} \cos \omega_0 t - \frac{2\beta_y^3}{\omega_0^3} \sin \omega_0 t - \\ - \frac{\beta_y^3}{2\omega_0^3} \sin 2\omega_0 t + \frac{\beta_y^4}{\omega_0^4} \sin^2 \omega_0 t - \frac{2\beta_y^3}{\omega_0 (\beta_y^2 + \omega_0^2)} e^{-\beta_y t} \sin \omega_0 t \Big]. \tag{7.104}$$

For small t we have from Eqs. (7.102), (7.94), (7.74)

$$D_{\delta x_2} = D_{\delta x^0} + r^2 D_{\theta_y^0} + D_{\dot{\delta x}^0} t^2 + r^2 \beta_y D_{\Delta m_y} t^3 + \frac{1}{4} D_{\Delta n_x} t^4. \tag{7.105}$$

For large t, we have as in (7.96)

$$D_{\delta x_2} = \left(\frac{D_{\Delta n_x}}{\alpha_x \omega_0^2} + \frac{3r^2 D_{\Delta m_y}}{\beta_y} \right) t. \tag{7.106}$$

Note that the dispersion $D_{\delta x_2}$ can be found by a somewhat different technique. Consider the equation

$$\delta \ddot{x}_2 + \omega_0^2 \delta x_2 = \Delta n_x + r\omega_0^2 \theta_y, \tag{7.107}$$

which is obtained from the first three equations in (7.55). Remember that in virtue of the second and third of these equations the initial conditions of Eq. (7.107) have the form

$$\delta x_2^0 = \delta x^0 + r\theta_y^0, \quad \delta \dot{x}_2^0 = \delta \dot{x}^0 + r\,\Delta m_y^0. \tag{7.108}$$

The initial value of the derivative $\delta \dot{x}_2^0$ is therefore correlated with the random function $\theta_y(t)$.

The dispersion $D_{\delta x_2}$ clearly can be written as the sum

$$D_{\delta x_2} = D_{\delta x_2}^{\delta x^0} + D_{\delta x_2}^{\delta \dot{x}^0} + D_{\delta x_2}^{\Delta n_x} + \tilde{D}_{\delta x_2}. \tag{7.109}$$

All the three terms are given by Eqs. (7.76) and (7.79). The term $\tilde{D}_{\delta x_2}$ is obtained from the equality

$$\begin{aligned} \tilde{D}_{\delta x_2} = {} & r^2 D_{\theta_y^0} \cos^2 \omega_0 t + \frac{r^2}{\omega_0^2} D_{\Delta m_y} \sin^2 \omega_0 t + \\ & + r^2 \omega_0^2 \int_0^t \int_0^t K_{\theta_y}(\tau, \tau') \sin \omega_0 (t - \tau) \sin \omega_0 (t - \tau')\, d\tau\, d\tau' + \\ & + 2r^2 \omega_0 \cos \omega_0 t \int_0^t K_{\theta_y}(\tau, 0) \sin \omega_0 (t - \tau)\, d\tau + \\ & + 2r^2 \sin \omega_0 t \int_0^t K_{\theta_y, \Delta m_y}(\tau, 0) \sin \omega_0 (t - \tau)\, d\tau. \end{aligned} \tag{7.110}$$

The correlation function $K_{\theta_y}(\tau, \tau')$ have been calculated above. By (7.67), (7.68), (7.71) we have

$$\begin{aligned} K_{\theta_y}(\tau, \tau') = {} & D_{\theta_y^0} + D_{\Delta m_y} \left[\frac{2}{\beta_y} \left(\frac{\tau + \tau'}{2} - \frac{|\tau - \tau'|}{2} \right) + \right. \\ & \left. + \frac{1}{\beta_y^2} \left(- e^{-\beta_y |\tau - \tau'|} + e^{-\beta_y \tau'} + e^{-\beta_y \tau} - 1 \right) \right] \end{aligned} \tag{7.111}$$

whence

$$K_{\theta_y}(\tau, 0) = D_{\theta_y^0} + D_{\Delta m_y} \left[\frac{1}{\beta_y} (\tau - |\tau|) + \frac{1}{\beta_y^2} \left(- e^{-\beta_y |\tau|} + e^{-\beta_y \tau} \right) \right]. \tag{7.112}$$

The integrand in the last term on the right in Eq. (7.110) contains the cross-correlation function $K_{\theta_y, \Delta m_y}$. Since $\Delta m_y = \dot{\theta}_y$, we have

$$K_{\theta_y, \Delta m_y} = K_{\theta_y, \dot{\theta}_y}. \tag{7.113}$$

From (7.111), (7.12) we therefore have

$$K_{\theta_y, \Delta m_y}(\tau, \tau') = \frac{D_{\Delta m_y}}{\beta_y}\left[1 - e^{-\beta_y \tau} + \operatorname{sign}(\tau - \tau')(1 - e^{-\beta_y |\tau - \tau'|})\right], \tag{7.114}$$

whence

$$K_{\theta_y, \Delta m_y}(\tau, 0) = \frac{D_{\Delta m_y}}{\beta_y} \operatorname{sign} \tau (1 - e^{-\beta_y |\tau|}). \tag{7.115}$$

We can now evaluate the integrals in Eq. (7.110). Using (7.111) we find

$$\begin{aligned}
&\int_0^t \int_0^t K_{\theta_y}(\tau, \tau') \sin \omega_0 (t - \tau') \sin \omega_0 (t - \tau)\, d\tau\, d\tau' = \\
&= \frac{1}{\omega_0^2} D_{\theta_y^0}(1 - \cos \omega_0 t)^2 + \\
&+ \frac{D_{\Delta m_y}}{\beta_y^2(\beta_y^2 + \omega_0^2)}\left[-\frac{2(\beta_y^4 + \omega_0^2\beta_y + \omega_0^4)}{\omega_0^2(\beta_y^2 + \omega_0^2)} + \frac{\beta_y(3\beta_y^2 + 2\omega_0^2)}{\omega_0^2} t + \right. \\
&+ 2e^{-\beta_y t} - \frac{2\beta_y(2\beta_y^2 + \omega_0^2)}{\omega_0^3} \sin \omega_0 t + \frac{2\beta_y^2}{\omega_0^2} \cos \omega_0 t + \\
&+ \frac{\beta_y^2}{2\omega_0^2} \sin 2\omega_0 t + \frac{\beta_y^2}{\omega_0^2} \sin^2 \omega_0 t - \frac{2\beta_y^2}{\beta^2 + \omega_0^2} e^{-\beta_y t} \cos \omega_0 t + \\
&\left. + \frac{2\omega_0\beta_y}{\beta_y^2 + \omega_0^2} e^{-\beta_y t} \sin \omega_0 t\right].
\end{aligned} \tag{7.116}$$

By (7.115)

$$\begin{aligned}
&\int_0^t K_{\theta_y, \Delta m_y}(\tau, 0) \sin \omega_0 (t - \tau)\, d\tau = \\
&= \frac{D_{\Delta m_y}}{\beta_y}\left\{\frac{1}{\omega_0}(1 - \cos \omega_0 t) - \frac{\omega_0}{\beta_y^2 + \omega_0^2} e^{-\beta_y t} + \right. \\
&\left. + \frac{1}{\beta_y^2 + \omega_0^2}[-\beta_y \sin \omega_0 t + \omega_0 \cos \omega_0 t]\right\}.
\end{aligned} \tag{7.117}$$

Finally using expression (7.112) for the function $K_{\theta_y}(\tau, 0)$, we find

$$\int_0^t K_{\theta_y}(\tau, 0) \sin \omega_0 (t - \tau)\, d\tau = \frac{D_{\theta_y^0}}{\omega_0^2}(1 - \cos \omega_0 t). \tag{7.118}$$

Inserting the integrals (7.116)—(7.118) on the right in Eq. (7.110), we obtain an expression for $\tilde{D}_{\delta x_2}$. Substitution of this expression together with expressions (7.76), (7.79) for the functions $D_{\delta x_2}^{\delta x^0}$, $D_{\delta x_2}^{\delta \dot{x}^0}$, $D_{\delta x_2}^{\Delta m_x}$ in Eq. (7.109) brings us back to Eq. (7.104).

In conclusion note that we have used Eqs. (7.37) in calculating the dispersions of the random functions δx, δx_2, θ_y, θ_{1y}. These equations express the dispersions in terms of the correlation function of the right-hand side of the differential equation. Similar manipulations naturally can be carried out using Eq. (7.43). These calculations, however, involve improper integrals and are fairly complicated.

7.2.2. The case of a satellite in a circular orbit

For a satellite in a circular orbit, the error equations projected on the orbital axes have the form

$$\left.\begin{aligned}
&\delta\ddot{x} + 2\omega_0\delta\dot{z} = \Delta n_x - r\,\Delta\dot{m}_y,\\
&\delta\ddot{y} + \omega_0^2\delta y = \Delta n_y + r\,\Delta\dot{m}_x - r\omega_0\,\Delta m_z,\\
&\delta\ddot{z} - 3\omega_0^2\delta z - 2\omega_0\delta\dot{x} = \Delta n_z + 2r\omega_0\,\Delta m_y;\\
&\dot{\theta}_x + \omega_0\theta_z = \Delta m_x,\\
&\dot{\theta}_y = \Delta m_y,\\
&\dot{\theta}_z - \omega_0\theta_x = \Delta m_z;\\
&\theta_{1x} = -\frac{\delta y}{r} \quad \theta_{1y} = \frac{\delta x}{r}, \quad \theta_{1z} = -\theta_z;\\
&\delta x_2 = \delta x + r\theta_y, \quad \delta y_2 = \delta y - r\theta_x,\\
&\delta z_2 = \delta z.
\end{aligned}\right\} \tag{7.119}$$

They are obtained from the general equations (1.95)—(1.97) if we take

$$x = y = 0, \quad r = z, \quad \omega_x = \omega_z = 0, \quad \omega_y = \omega_0. \tag{7.120}$$

The solution of the first three equations in (7.119) is given by (1.152). According to these expressions,

$$\left.\begin{aligned}
\delta x = {}&\frac{1}{\omega_0}\int_0^t \Delta n_x\,[-3\omega_0(t-\tau) + 4\sin\omega_0(t-\tau)]\,d\tau -\\
&- r\int_0^t \Delta m_y\,d\tau - \frac{2}{\omega_0}\int_0^t \Delta n_z\,[1-\cos\omega_0(t-\tau)]\,d\tau +\\
&+ \delta x^0 + \frac{\delta\dot{x}^0 + r\,\Delta m_y^0}{\omega_0}(4\sin\omega_0 t - 3\omega_0 t) +\\
&+ 6\,\delta z^0(\sin\omega_0 t - \omega_0 t) + \frac{2\,\delta\dot{z}^0}{\omega_0}(\cos\omega_0 t - 1),\\
\delta y = {}&\frac{1}{\omega_0}\int_0^t (\Delta n_y - r\omega_0\,\Delta m_z)\sin\omega_0(t-\tau)\,d\tau +\\
&+ r\int_0^t \Delta m_x\cos\omega_0(t-\tau)\,d\tau + \delta y^0\cos\omega_0 t +\\
&+ \frac{\delta\dot{y}^0 - r\,\Delta m_x^0}{\omega_0}\sin\omega_0 t,\\
\delta z = {}&\frac{2}{\omega_0}\int_0^t \Delta n_x\,[1-\cos\omega_0(t-\tau)]\,d\tau +\\
&+ \frac{1}{\omega_0}\int_0^t \Delta n_z\sin\omega_0(t-\tau)\,d\tau + \delta z^0(4-3\cos\omega_0 t) +\\
&+ \frac{\delta\dot{z}^0}{\omega_0}\sin\omega_0 t + \frac{2\left(\delta\dot{x}^0 + r\,\Delta m_y^0\right)}{\omega_0}(1-\cos\omega_0 t).
\end{aligned}\right\} \tag{7.121}$$

The solution of the system comprising the fourth and sixth equations in (7.119) is found in the form

$$\left.\begin{aligned}\theta_x &= \theta_x^0 \cos\omega_0 t - \dot{\theta}_z^0 \sin\omega_0 t + \\ &+ \int_0^t [\Delta m_x \cos\omega_0 (t-\tau) - \Delta m_z \sin\omega_0 (t-\tau)]\, d\tau, \\ \theta_z &= \theta_z^0 \sin\omega_0 t + \dot{\theta}_z^0 \cos\omega_0 t + \\ &+ \int_0^t [\Delta m_x \sin\omega_0 (t-\tau) + \Delta m_z \cos\omega_0 (t-\tau)]\, d\tau.\end{aligned}\right\} \tag{7.122}$$

This follows from Eqs. (1.179) if we put $\omega_z = \omega_x = \text{const}$. The solution of the fifth equation in (7.119) is clearly

$$\theta_y = \theta_y^0 + \int_0^t \Delta m_y\, d\tau. \tag{7.123}$$

In what follows we assume that the expectation values of the random initial conditions and the random instrumental errors are all zero. The random functions Δn_x, Δn_y, Δn_z, Δm_x, Δm_y, Δm_z are assumed to be uncorrelated with one another and with the initial conditions δx^0, δy^0, δz^0, $\delta\dot{x}^0$, $\delta\dot{y}^0$, $\delta\dot{z}^0$, θ_x^0, θ_y^0, θ_z^0. The initial conditions are uncorrelated either.

Let us find the dispersions $D_{\theta_{1x}}$, $D_{\theta_{1y}}$, $D_{\theta_{1z}}$ of the platform orientation errors and the dispersions $D_{\delta x_2}$, $D_{\delta y_2}$, $D_{\delta z_2}$ of the total position errors using the known dispersions of the initial conditions and the given correlation functions $K_{\Delta n_x}$, $K_{\Delta n_y}$, $K_{\Delta n_z}$, $K_{\Delta m_x}$, $K_{\Delta m_y}$, $K_{\Delta m_z}$ of the instrumental errors. The correlation functions, by analogy with Eqs. (7.62), (7.69), are taken in the form

$$\left.\begin{aligned}K_{\Delta n_{x,y,z}} &= D_{\Delta n_{x,y,z}} e^{-\alpha_{x,y,z}|t-t'|}, \\ K_{\Delta m_{x,y,z}} &= D_{\Delta m_{x,y,z}} e^{-\beta_{x,y,z}|t-t'|}.\end{aligned}\right\} \tag{7.124}$$

By the seventh, eighth, and ninth equations in (7.119), we have

$$D_{\theta_{1x}} = \frac{1}{r^2} D_{\delta y}, \quad D_{\theta_{1y}} = \frac{1}{r^2} D_{\delta x}, \quad D_{\theta_{1z}} = D_{\theta_z}. \tag{7.125}$$

From Eq. (7.121) for δx we see that the dispersion $D_{\delta x}$ may be written as the sum

$$D_{\delta x} = D_{\delta x}^{\delta x^0} + D_{\delta x}^{\delta \dot{x}^0} + D_{\delta x}^{\delta z^0} + D_{\delta x}^{\delta \dot{z}^0} + D_{\delta x}^{\Delta n_x} + D_{\delta x}^{\Delta n_z} + D_{\delta x}^{\Delta m_y}, \tag{7.126}$$

where

$$\left.\begin{aligned}D_{\delta x}^{\delta x^0} &= D_{\delta x^0}, \\ D_{\delta x}^{\delta \dot{x}^0} &= \frac{1}{\omega_0^2} D_{\delta \dot{x}^0} (4\sin\omega_0 t - 3\omega_0 t)^2, \\ D_{\delta x}^{\delta z^0} &= 36 D_{\delta z^0} (\sin\omega_0 t - \omega_0 t)^2, \\ D_{\delta x}^{\delta \dot{z}^0} &= \frac{4}{\omega_0^2} D_{\delta \dot{z}^0} (\cos\omega_0 t - 1)^2,\end{aligned}\right\} \tag{7.127}$$

and the other terms are given by

$$\left.\begin{aligned}
D_{\delta x}^{\Delta n_y} &= \frac{1}{\omega_0^2}\int_0^t\int_0^t K_{\Delta n_y}(\tau,\tau')[-3\omega_0(t-\tau)+ \\
&\quad +4\sin\omega_0(t-\tau)][-3\omega_0(t-\tau')+ \\
&\quad +4\sin\omega_0(t-\tau')]\,d\tau\,d\tau', \\
D_{\delta x}^{\Delta n_z} &= \frac{4}{\omega_0^2}\int_0^t\int_0^t K_{\Delta n_z}(\tau,\tau')[1-\cos\omega_0(t-\tau)]\times \\
&\quad \times[1-\cos\omega_0(t-\tau')]\,d\tau\,d\tau', \\
D_{\delta x}^{\Delta m_y} &= r^2\int_0^t\int_0^t K_{\Delta m_y}(\tau,\tau')\,d\tau\,d\tau'+ \\
&\quad +\frac{r^2}{\omega_0^2}D_{\Delta m_y}(4\sin\omega_0 t-3\omega_0 t)^2- \\
&\quad -\frac{2r^2}{\omega_0^2}(4\sin\omega_0 t-3\omega_0 t)\int_0^t K_{\Delta m_y}(\tau,0)\,d\tau.
\end{aligned}\right\} \tag{7.128}$$

Inserting the correlation function $K_{\Delta n_x}(\tau,\tau')$ in the first equation, we get

$$\begin{aligned}
D_{\delta x}^{\Delta n_x} &= 9D_{\Delta n_x}\int_0^t\int_0^t (t-\tau)(t-\tau')\,e^{-\alpha_x|\tau-\tau'|}\,d\tau\,d\tau'- \\
&\quad -\frac{24D_{\Delta n_x}}{\omega_0}\int_0^t\int_0^t (t-\tau)\,e^{-\alpha_x|\tau-\tau'|}\sin\omega_0(t-\tau')\,d\tau\,d\tau'+ \\
&\quad +\frac{16D_{\Delta n_x}}{\omega_0^2}\int_0^t\int_0^t e^{-\alpha_x|\tau-\tau'|}\sin\omega_0(t-\tau)\sin\omega_0(t-\tau')\,d\tau\,d\tau'.
\end{aligned} \tag{7.129}$$

The last integral in (7.129) has been evaluated earlier; it is given by (7.79). For the first two integrals we have

$$\left.\begin{aligned}
&\int_0^t\int_0^t (t-\tau)(t-\tau')\,e^{-\alpha_x|\tau-\tau'|}\,d\tau\,d\tau' = \\
&\qquad = \frac{2}{3\alpha_x}t^3-\frac{1}{\alpha_x^2}t^2+\frac{2}{\alpha_x^4}[1-(\alpha_x t+1)e^{-\alpha_x t}], \\
&\int_0^t\int_0^t (t-\tau)\,e^{-\alpha_x|\tau-\tau'|}\sin\omega_0(t-\tau')\,d\tau\,d\tau' = \\
&\qquad = -\frac{2(\alpha_x^2+\omega_0^2)}{\alpha_x\omega_0(\alpha_x^2+\omega_0^2)}\left(\frac{\sin\omega_0 t}{\omega_0}-t\cos\omega_0 t\right)+ \\
&\qquad + \frac{1}{\alpha_x^2(\alpha_x^2+\omega_0^2)}\{\omega_0[1-(\alpha_x t+1)e^{-\alpha_x t}]- \\
&\qquad - e^{-\alpha_x t}(\alpha_x\sin\omega_0 t+\omega_0\cos\omega_0 t)- \\
&\qquad\qquad -\alpha_x^2 t\sin\omega_0 t+\omega_0\cos\omega_0 t\}.
\end{aligned}\right\} \tag{7.130}$$

Inserting these expressions for the integrals in (7.129) and collecting like terms, we obtain

$$D_{\delta x}^{\Delta n_x} = D_{\Delta n_x}\left\{\frac{6t^3}{\alpha_x} - \frac{9t^2}{\alpha_x^2} + \frac{16\alpha_x t}{\omega_0^2(\alpha_x^2+\omega_0^2)} + \right.$$
$$+ \frac{1}{\alpha_x^2+\omega_0^2}\left[\frac{26\alpha_x^4 + 12\alpha_x^2\omega_0^2 + 18\omega_0^4}{\alpha_x^4(\alpha_x^2+\omega_0^2)} + \right.$$
$$+ \frac{6(\alpha_x^2-3\omega_0^2)}{\alpha_x^4}(\alpha_x t+1)e^{-\alpha_x t} +$$
$$+ \frac{24(2\alpha_x^2-\omega_0^2)}{\alpha_x\omega_0^3}(\omega_0 t\cos\omega_0 t - \sin\omega_0 t) +$$
$$+ \frac{24}{\omega_0} t\sin\omega_0 t - \frac{24}{\alpha_x^2}\cos\omega_0 t - \frac{8\alpha_x}{\omega_0^3}\sin 2\omega_0 t -$$
$$\left.\left. - \frac{16}{\omega_0^2}\sin^2\omega_0 t - \frac{8(\alpha_x^2-3\omega_0^2)}{\alpha_x^2(\alpha_x^2+\omega_0^2)}\left(\frac{\alpha_x}{\omega_0}\sin\omega_0 t + \cos\omega_0 t\right)e^{-\alpha_x t}\right]\right\}. \tag{7.131}$$

As in (7.129), we further obtain

$$\int_0^t\int_0^t K_{\Delta n_z}(\tau,\tau')[1-\cos\omega_0(t-\tau)][1-\cos\omega_0(t-\tau')]\,d\tau\,d\tau' =$$
$$= \int_0^t\int_0^t K_{\Delta n_z}(\tau,\tau')\,d\tau\,d\tau' - 2\int_0^t\int_0^t K_{\Delta n_z}(\tau,\tau')\cos\omega_0(t-\tau')\,d\tau\,d\tau' +$$
$$+ \int_0^t\int_0^t K_{\Delta n_z}(\tau,\tau')\cos\omega_0(t-\tau)\cos\omega_0(t-\tau')\,d\tau\,d\tau'. \tag{7.132}$$

But

$$\left.\begin{aligned}
&\int_0^t\int_0^t K_{\Delta n_z}(\tau,\tau')\cos\omega_0(t-\tau')\,d\tau\,d\tau' = \\
&= \frac{D_{\Delta n_z}}{\alpha_z^2+\omega_0^2}\left[(e^{-\alpha_z t}-1)\left(1+\cos\omega_0 t + \frac{\omega_0}{\alpha_z}\sin\omega_0 t\right) + 2\,\frac{\alpha_z}{\omega_0}\sin\omega_0 t\right], \\
&\int_0^t\int_0^t K_{\Delta n_z}(\tau,\tau')\cos\omega_0(t-\tau)\cos\omega_0(t-\tau')\,d\tau\,d\tau' = \\
&= D_{\Delta n_z}\left\{\frac{\alpha_z}{\alpha_z^2+\omega_0^2}\left(t+\frac{1}{2\omega_0}\sin 2\omega_0 t\right) + \right. \\
&\left. + \frac{2\alpha_z}{(\alpha_z^2+\omega_0^2)^2}e^{-\alpha_z t}(\alpha_z\cos\omega_0 t - \omega_0\sin\omega_0 t) + \frac{1}{\alpha_z^2+\omega_0^2}\sin^2\omega_0 t - \frac{2\alpha_z^2}{(\alpha_z^2+\omega_0^2)^2}\right\}.
\end{aligned}\right\} \tag{7.133}$$

Therefore, using the first equation in (7.72), we obtain instead of the second relation in (7.128)

$$D_{\delta x}^{\Delta n_z} = \frac{4D_{\Delta n_z}}{\omega_0^2(\alpha_z^2+\omega_0^2)}\left[\frac{3\alpha_z^2+2\omega_0^2}{\alpha_z} t - \frac{2\alpha_z^4+2\alpha_z^2\omega_0^2+2\omega_0^4}{\alpha_z^2(\alpha_z^2+\omega_0^2)} + \frac{2\omega_0^2}{\alpha_z^2}e^{-\alpha_z t} + \right.$$
$$+ \frac{\alpha_z}{2\omega_0}\sin 2\omega_0 t - \frac{2(2\alpha_z^2+\omega_0^2)}{\alpha_z\omega_0}\sin\omega_0 t + 2\cos\omega_0 t + \sin^2\omega_0 t +$$
$$\left. + \frac{2\omega_0^2}{\alpha_z(\alpha_z^2+\omega_0^2)}\left(\sin\omega_0 t - \frac{\alpha_z}{\omega_0}\cos\omega_0 t\right)e^{-\alpha_z t}\right]. \tag{7.134}$$

Finally, again using the first equation in (7.72), we obtain from the third equality in (7.128)

$$D_{\delta x}^{\Delta m_y}(t) = r^2 D_{\Delta m_y}\left[9t^2 - \frac{4}{\beta_y}t - \frac{2}{\beta_y^2} + \right.$$
$$+ \frac{16}{\omega_0^2}\sin^2\omega_0 t + \frac{8}{\beta_y\omega_0}\left(1 - e^{-\beta_y t}\right)\sin\omega_0 t -$$
$$\left. - \frac{24}{\omega_0}t\sin\omega_0 t + \frac{2}{\beta_y^2}e^{-\beta_y t} + \frac{6}{\beta_y}te^{-\beta_y t}\right]. \qquad (7.135)$$

Eqs. (7.126), (7.127), (7.131), (7.134), (7.135) together with the second equation in (7.125) give the dispersion $D_{\theta_{1y}}$ of the angle θ_{1y} at any time. In particular, for small t, we have the approximation

$$D_{\theta_{1y}} = \frac{1}{r^2}\left(D_{\delta x^0} + D_{\delta\dot{x}^0}t^2 + D_{\delta z^0}\omega_0^2 t^3 + \right.$$
$$\left. + D_{\delta\dot{z}^0}\omega_0^2 t^4 + \frac{1}{4}D_{\Delta n_x}t^4 + \frac{1}{9}D_{\Delta n_z}\omega_0^2 t^6\right) + \frac{2}{3}\beta_y D_{\Delta m_y}t^3. \qquad (7.136)$$

As t increases, the terms with $e^{-\alpha t}$ and $e^{-\beta t}$ in Eqs. (7.131), (7.134), (7.135) for $D_{\delta x}^{\Delta n_x}$, $D_{\delta x}^{\Delta n_z}$, $D_{\delta x}^{\Delta m_y}$ rapidly decrease (at least when $\alpha, \beta \gg \omega_0$) and cease to affect the dispersion $D_{\theta_{1y}}$. For sufficiently large t, $D_{\theta_{1y}}$ is thus determined by the first term on the right in (7.131), since in this case we have approximately

$$D_{\theta_{1y}} = \frac{1}{r^2}\left[\frac{6}{\alpha_x}D_{\Delta n_x}t^3 + 9\left(D_{\delta\dot{x}^0} + 4D_{\delta z^0}\omega_0^2\right)t^2 + \frac{12}{\omega_0^2\alpha_z}D_{\Delta n_z}t\right] + 9D_{\Delta m_y}t^2. \qquad (7.137)$$

We will now find the dispersion $D_{\theta_{1x}}(t)$. As we see from the first equation in (7.125), this requires knowledge of $D_{\delta y}(t)$. By the second equation in (7.121), we may write

$$D_{\delta y} = D_{\delta y}^{\delta y^0} + D_{\delta y}^{\delta\dot{y}^0} + D_{\delta y}^{\Delta n_y} + D_{\delta y}^{\Delta m_x} + D_{\delta y}^{\Delta m_z}, \qquad (7.138)$$

where

$$\left.\begin{aligned} D_{\delta y}^{\delta y^0} &= D_{\delta y^0}\cos^2\omega_0 t, \\ D_{\delta y}^{\delta\dot{y}^0} &= \frac{1}{\omega_0^2}D_{\delta\dot{y}^0}\sin^2\omega_0 t, \end{aligned}\right\} \qquad (7.139)$$

and the other terms are represented in integral form

$$\left.\begin{aligned} D_{\delta y}^{\Delta n_y} &= \frac{1}{\omega_0^2}\int_0^t\int_0^t K_{\Delta n_y}(\tau, \tau')\sin\omega_0(t-\tau)\times \\ &\qquad \times\sin\omega_0(t-\tau')\,d\tau\,d\tau', \\ D_{\delta y}^{\Delta m_z} &= r^2\int_0^t\int_0^t K_{\Delta m_z}(\tau, \tau')\sin\omega_0(t-\tau)\times \\ &\qquad \times\sin\omega_0(t-\tau')\,d\tau\,d\tau', \\ D_{\delta y}^{\Delta m_x} &= r^2\int_0^t\int_0^t K_{\Delta m_x}(\tau, \tau')\cos\omega_0(t-\tau)\times \\ &\quad \times\cos\omega_0(t-\tau')\,d\tau\,d\tau' + \frac{r^2}{\omega_0^2}D_{\Delta m_x}\sin^2\omega_0 t - \\ &\quad - \frac{2r^2}{\omega_0}\sin\omega_0 t\int_0^t K_{\Delta m_x}(\tau, 0)\cos\omega_0(t-\tau)\,d\tau. \end{aligned}\right\} \qquad (7.140)$$

Using (7.78), (7.79), we find

$$\left.\begin{aligned} D_{\delta y}^{\Delta m_y} &= \frac{D_{\Delta m_y}}{\alpha_y^2+\omega_0^2}\left\{\frac{\alpha_y}{\omega_0^2}\left(t-\frac{1}{2\omega_0}\sin 2\omega_0 t\right)+\right. \\ &\left.+\frac{2}{\alpha_y^2+\omega_0^2}\left[1-e^{-\alpha_y t}\left(\frac{\alpha_y}{\omega_0}\sin\omega_0 t+\cos\omega_0 t\right)\right]-\frac{1}{\omega_0^2}\sin^2\omega_0 t\right\}, \\ D_{\delta y}^{\Delta m_z} &= \frac{r^2\omega_0^2 D_{\Delta m_z}}{\beta_z^2+\omega_0^2}\left\{\frac{\beta_z}{\omega_0^2}\left(t-\frac{1}{2\omega_0}\sin 2\omega_0 t\right)+\right. \\ &\left.+\frac{2}{\beta_z^2+\omega_0^2}\left[1-e^{-\beta_z t}\left(\frac{\beta_z}{\omega_0}\sin\omega_0 t+\cos\omega_0 t\right)\right]-\frac{1}{\omega_0^2}\sin^2\omega_0 t\right\}. \end{aligned}\right\} \tag{7.141}$$

Similarly from the third equation in (7.140), using the second equality in (7.133), we find

$$D_{\delta y}^{\Delta m_x} = \frac{r\cdot D_{\Delta m_x}}{\beta_x^2+\omega_0^2}\left[\beta_x\left(t-\frac{1}{2\omega_0}\sin 2\omega_0 t\right)+\right.$$
$$\left.+\frac{\beta_x^2}{\omega_0^2}\sin^2\omega_0 t-\frac{2\beta_x^2}{\beta_x^2+\omega_0^2}+\frac{2\beta_x^2}{\beta_x^2+\omega_0^2}\left(\cos\omega_0 t+\frac{\beta_x}{\omega_0}\sin\omega_0 t\right)e^{-\beta_x t}\right]. \tag{7.142}$$

Eqs. (7.138), (7.139), (7.141), (7.142) together with the second equation in (7.125) give the dispersion $D_{\theta_{1x}}$ of the angle θ_{1x}. For small t we have the approximation

$$D_{\theta_{1x}} = \frac{1}{r^2}\left(D_{\delta y^0}+D_{\delta\dot{y}^0}t^2+\frac{1}{4}D_{\Delta m_y}t^4\right)+\frac{2\beta_x}{3}D_{\Delta m_x}t^3+\frac{\omega_0^2}{4}D_{\Delta m_z}t^4. \tag{7.143}$$

For fairly large t the dispersion is determined by the time-proportional terms on the right in Eqs. (7.141), (7.142), so that approximately ($\alpha \gg \omega_0$, $\beta \gg \omega_0$)

$$D_{\theta_{1x}} = \left(\frac{1}{r^2\alpha_y\omega_0^2}D_{\Delta m_y}+\frac{1}{\beta_z}D_{\Delta m_z}+\frac{1}{\beta_x}D_{\Delta m_x}\right)t. \tag{7.144}$$

It now remains to find the dispersion $D_{\theta_{1z}}$. By the last equation in (7.125) and the second equation in (7.122)

$$D_{\theta_{1z}} = D_{\theta_{1z}}^{\theta_x^0}+D_{\theta_{1z}}^{\theta_z^0}+D_{\theta_{1z}}^{\Delta m_x}+D_{\theta_{1z}}^{\Delta m_z}, \tag{7.145}$$

where the first two terms are

$$D_{\theta_{1z}}^{\theta_x^0} = D_{\theta_x^0}\sin^2\omega_0 t, \quad D_{\theta_{1z}}^{\theta_z^0} = D_{\theta_z^0}\cos^2\omega_0 t, \tag{7.146}$$

and the last two terms are expressed in integral form

$$\left.\begin{aligned} D_{\theta_{1z}}^{\Delta m_x} &= \int_0^t\int_0^t K_{\Delta m_x}(\tau,\tau')\sin\omega_0(t-\tau)\times \\ &\qquad\times\sin\omega_0(t-\tau')\,d\tau\,d\tau', \\ D_{\theta_{1z}}^{\Delta m_z} &= \int_0^t\int_0^t K_{\Delta m_z}(\tau,\tau')\cos\omega_0(t-\tau)\times \\ &\qquad\times\cos\omega_0(t-\tau')\,d\tau\,d\tau'. \end{aligned}\right\} \tag{7.147}$$

Using (7.79) and (7.133) we find

$$\left.\begin{aligned} D_{\theta_{1z}}^{\Delta m_x} &= \frac{\omega_0^2}{\beta_x^2+\omega_0^2} D_{\Delta m_x}\left\{\frac{\beta_x}{\omega_0^2}\left(t-\frac{1}{2\omega_0}\sin 2\omega_0 t\right)+\right. \\ &\left.+\frac{2}{\beta_x^2+\omega_0^2}\left[1-\left(\frac{\beta_x}{\omega_0}\sin\omega_0 t+\cos\omega_0 t\right)e^{-\beta_x t}-\frac{1}{\omega_0^2}\sin^2\omega_0 t\right\}, \right. \\ D_{\theta_{1z}}^{\Delta m_z} &= \frac{1}{\beta_z^2+\omega_0^2} D_{\Delta m_z}\left\{\beta_z\left(t+\frac{1}{2\omega_0}\sin 2\omega_0 t\right)+\right. \\ &\left.+\frac{2\beta_z}{\beta_z^2+\omega_0^2}\left[(\beta_z\cos\omega_0 t-\omega_0\sin\omega_0 t)e^{-\beta_z t}-\beta_z\right]+\sin^2\omega_0 t\right\}. \end{aligned}\right\} \tag{7.148}$$

From (7.145), (7.146), (7.148) we see that for small t

$$D_{\theta_{1z}} = D_{\theta_z^0}+\left(\omega_0^2 D_{\theta_x^0}+\frac{1}{2}D_{\Delta m_z}\right)t^2+\frac{\omega_0^2}{4}D_{\Delta m_x}t^4, \tag{7.149}$$

and for large t we approximately have

$$D_{\theta_{1z}} = \left(\frac{D_{\Delta m_x}}{\beta_x}+\frac{D_{\Delta m_z}}{\beta_z}\right)t. \tag{7.150}$$

We have so far dealt with the dispersions $D_{\theta_{1x}}$, $D_{\theta_{1y}}$, $D_{\theta_{1z}}$ of the random orientation errors of the platform. We can now proceed to calculate the dispersions $D_{\delta x_2}$, $D_{\delta y_2}$, $D_{\delta z_2}$ of the total position errors δx_2, δy_2, δz_2.

We start with $D_{\delta z_2}$. By the last equality in (7.119), $\delta z_2 = \delta z$, and therefore the last relation in (7.121) gives

$$D_{\delta z_2} = D_{\delta z_2}^{\delta\dot{x}^0}+D_{\delta z_2}^{\delta z^0}+D_{\delta z_2}^{\delta\dot{z}^0}+D_{\delta z_2}^{\Delta n_x}+D_{\delta z_2}^{\Delta n_z}+D_{\delta z_2}^{\Delta m_y}. \tag{7.151}$$

Here evidently

$$\left.\begin{aligned} D_{\delta z_2}^{\delta\dot{x}^0} &= \frac{4}{\omega_0^2} D_{\delta\dot{x}^0}(1-\cos\omega_0 t)^2, \\ D_{\delta z_2}^{\delta z^0} &= D_{\delta z^0}(4-3\cos\omega_0 t)^2, \\ D_{\delta z_2}^{\delta\dot{z}^0} &= \frac{1}{\omega_0^2} D_{\delta\dot{z}^0}\sin^2\omega_0 t, \\ D_{\delta z_2}^{\Delta m_y} &= \frac{4r^2}{\omega_0^2} D_{\Delta m_y}(1-\cos\omega_0 t)^2, \end{aligned}\right\} \tag{7.152}$$

and the other two terms are given in integral form

$$\left.\begin{aligned} D_{\delta z_2}^{\Delta n_x} &= \frac{4}{\omega_0^2}\int_0^t\int_0^t K_{\Delta n_x}(\tau,\tau')\left[1-\cos\omega_0(t-\tau)\right]\left[1-\cos\omega_0(t-\tau')\right]d\tau\,d\tau', \\ D_{\delta z_2}^{\Delta n_z} &= \frac{1}{\omega_0^2}\int_0^t\int_0^t K_{\Delta n_z}(\tau,\tau')\sin\omega_0(t-\tau)\ \sin\omega_0(t-\tau')\,d\tau\,d\tau'. \end{aligned}\right\} \tag{7.153}$$

The integrals on the right in (7.153) are not new. The first differs from the integral in the second equality in (7.128) only in that $K_{\Delta n_z}$ has been replaced with $K_{\Delta n_x}$, a point of no consequence for the evaluation of this

Therefore, by analogy with (7.134), we find

$$D_{\delta z_1}^{\Delta n_x} = \frac{4D_{\Delta n_x}}{\omega_0^2(\alpha_x^2+\omega_0^2)}\left[\frac{3\alpha_x^2+2\omega_0^2}{\alpha_x}t - \right.$$
$$- \frac{2(\alpha_x^4+\alpha_x^2\omega_0^2+\omega_0^4)}{\alpha_x^2(\alpha_x^2+\omega_0^2)} + \frac{2\omega_0^2}{\alpha_x^2}e^{-\alpha_x t} + \frac{\alpha_x}{2\omega_0}\sin 2\omega_0 t -$$
$$- \frac{2(2\alpha_x^2+\omega_0^2)}{\alpha_x\omega_0}\sin\omega_0 t + 2\cos\omega_0 t + \sin^2\omega_0 t +$$
$$\left. + \frac{2\omega_0^2}{\alpha_x(\alpha_x^2+\omega_0^2)}\left(\sin\omega_0 t - \frac{\alpha_x}{\omega_0}\cos\omega_0 t\right)e^{-\alpha_x t}\right]. \qquad (7.154)$$

The second integral in (7.153) is evaluated by comparison with Eqs. (7.78), (7.79). We find

$$D_{\delta z_2}^{\Delta n_z} = \frac{D_{\Delta n_z}}{\alpha_z^2+\omega_0^2}\left\{\frac{\alpha_z}{\omega_0^2}\left(t - \frac{1}{2\omega_0}\sin 2\omega_0 t\right) + \right.$$
$$\left. + \frac{2}{\alpha_z^2+\omega_0^2}\left[1 - \left(\frac{\alpha_z}{\omega_0}\sin\omega_0 t + \cos\omega_0 t\right)e^{-\alpha_z t}\right] - \frac{1}{\omega_0^2}\sin^2\omega_0 t\right\}. \qquad (7.155)$$

From Eqs. (7.151), (7.152), (7.154), (7.155), expanding the relevant $D_{\delta z_i}$ functions in Taylor series and retaining only the first nonzero terms, we obtain for small t:

$$D_{\delta z_2} = D_{\delta z^0} + (D_{\delta \dot z^0} + D_{\delta \dot z^0} + r^2 D_{\Delta n_y})t^2 + \frac{1}{4}D_{\Delta n_x}t^4 + \frac{1}{9}D_{\Delta n_x}\omega_0^2 t^6. \qquad (7.156)$$

For large t we approximately have ($\alpha_x^2 \gg \omega_0^2$)

$$D_{\delta z_2} = \left(\frac{12}{\alpha_x}D_{\Delta n_x} + \frac{1}{\alpha_z}D_{\Delta n_z}\right)\frac{t}{\omega_0^2}. \qquad (7.157)$$

Let us now calculate $D_{\delta x_2}$, $D_{\delta v_2}$. Using the solutions (7.121), (7.122), we find the cross-correlation functions $K_{\delta x, \theta_y}(t, t')$, $K_{\delta y, \theta_x}(t, t')$. Then, according to the corresponding equalities at the end of (7.119), we write

$$\left.\begin{aligned} D_{\delta x_2} &= D_{\delta x} + r^2 D_{\theta_y} + 2rK_{\delta x, \theta_y}(t, t), \\ D_{\delta y_2} &= D_{\delta y} + r^2 D_{\theta_x} - 2rK_{\delta y, \theta_x}(t, t). \end{aligned}\right\} \qquad (7.158)$$

The dispersions $D_{\delta x}$ and $D_{\delta y}$ on the right have been calculated before, as well as the dispersion D_{θ_y}. The dispersion D_{θ_x}, however, has to be calculated anew, together with the cross-correlation functions $K_{\delta x, \theta_y}$ and $K_{\delta y, \theta_x}$.

From the first equation in (7.122) we get

$$D_{\theta_x} = D_{\theta_x^0}\cos^2\omega_0 t + D_{\theta_z^0}\sin^2\omega_0 t +$$
$$+ \int_0^t\int_0^t K_{\Delta m_x}(\tau, \tau')\cos\omega_0(t-\tau)\cos\omega_0(t-\tau')\,d\tau\,d\tau' +$$
$$+ \int_0^t\int_0^t K_{\Delta m_z}(\tau, \tau')\sin\omega_0(t-\tau)\sin\omega_0(t-\tau')\,d\tau\,d\tau'. \qquad (7.159)$$

From the first equation in (7.121) and Eq. (7.123) we have

$$K_{\delta x, \theta_y}(t, t) = -r\int_0^t\int_0^t K_{\Delta m_y}(\tau, \tau')\,d\tau\,d\tau' + \frac{r}{\omega_0}(4\sin\omega_0 t - 3\omega_0 t)\int_0^t K_{\Delta m_y}(\tau, 0)\,d\tau. \qquad (7.160)$$

Finally, from the second equation in (7.121) and the first equation in (7.122), we have

$$K_{\delta y,\theta_x}(t,t) = r\int_0^t\int_0^t K_{\Delta m_x}(\tau,\tau')\cos\omega_0(t-\tau)\cos\omega_0(t-\tau')\,d\tau\,d\tau' + \\ + r\int_0^t\int_0^t K_{\Delta m_z}(\tau,\tau')\sin\omega_0(t-\tau)\sin\omega_0(t-\tau')\,d\tau\,d\tau' - \\ - \frac{r}{\omega_0}\sin\omega_0 t\int_0^t K_{\Delta m_x}(\tau,0)\cos\omega_0(t-\tau)\,d\tau. \qquad (7.161)$$

Turning to the first equation in (7.158), we insert for $D_{\delta x}$ its expression from (7.126), for D_{θ_y} from (7.72), and for $K_{\delta x,\theta_y}(t,t)$ from (7.160). In (7.126) the last term is replaced in accordance with the last equation in (7.128). All this gives

$$D_{\delta x_2} = D_{\delta x}^{\delta x^0} + D_{\delta x}^{\delta \dot{x}^0} + D_{\delta x}^{\delta z^0} + D_{\delta x}^{\delta \dot{z}^0} + \\ + D_{\delta x}^{\Delta n_x} + D_{\delta x}^{\Delta n_z} + r^2 D_{\theta_y^0} + \frac{r^2}{\omega_0^2}\Delta m_y(4\sin\omega_0 t - 3\omega_0 t)^2. \qquad (7.162)$$

Here the functions $D_{\delta x}^{\delta x^0}$, $D_{\delta x}^{\delta \dot{x}^0}$, $D_{\delta x}^{\delta z^0}$, $D_{\delta x}^{\delta \dot{z}^0}$, $D_{\delta x}^{\Delta n_x}$, $D_{\delta x}^{\Delta n_z}$ are given by Eqs. (7.127), (7.131), (7.134).

Turning to the second equality in (7.158), we insert for $D_{\delta y}$ its expression from (7.138), for D_{θ_x} from (7.159), and for $K_{\delta y,\theta_x}(t,t)$ from (7.161). As with $D_{\delta x_2}$, the last two terms on the right in Eq. (7.138) are replaced according to the last two expressions in (7.140). This gives

$$D_{\delta y_2} = D_{\delta y}^{\delta y^0} + D_{\delta y}^{\delta \dot{y}^0} + D_{\delta y}^{\Delta n_y} + \frac{r^2}{\omega_0^2} D_{\Delta m_x}\sin^2\omega_0 t + \\ + r^2 D_{\theta_x^0}\cos^2\omega_0 t + r^2 D_{\theta_z^0}\sin^2\omega_0 t. \qquad (7.163)$$

The functions $D_{\delta y}^{\delta y^0}$, $D_{\delta y}^{\delta \dot{y}^0}$ are given by Eqs. (7.139), and $D_{\delta y}^{\Delta n_y}$ by the first relation in (7.141).

For small t, Eqs. (7.162), (7.163) give the approximation

$$\left.\begin{aligned} D_{\delta x_2} &= D_{\delta x^0} + \left(D_{\delta \dot{x}^0} + r^2 D_{\Delta m_y}\right)t^2 + \\ &+ D_{\delta z^0}\omega_0^2 t^3 + \left(\omega_0^2 D_{\delta \dot{z}^0} + \frac{1}{4}D_{\Delta n_x}\right)t^4 + \frac{1}{9}D_{\Delta n_z}\omega_0^2 t^6, \\ D_{\delta y_2} &= D_{\delta y^0} + r^2 D_{\theta_x^0} + \left(D_{\delta \dot{y}^0} + r^2 D_{\Delta m_x} + \right. \\ &\left. + r^2 D_{\theta_z^0}\omega_0^2\right)t^2 + \frac{1}{4}D_{\Delta n_y}t^4. \end{aligned}\right\} \qquad (7.164)$$

For large t, the dispersions $D_{\delta x_2}$, $D_{\delta y_2}$ are expressed by

$$\left.\begin{aligned} D_{\delta x_2} &= \frac{12}{\alpha_z\omega_0^2}D_{\Delta n_z}t + 9\left(4D_{\delta z^0}\omega_0^2 + D_{\delta \dot{x}^0} + r^2 D_{\Delta m_y}\right)t^2 + \frac{6}{\alpha_x}D_{\Delta n_x}t^3, \\ D_{\delta y_2} &= \frac{1}{\omega_0^2\alpha_y}D_{\Delta n_y}t. \end{aligned}\right\} \qquad (7.164a)$$

Note that Eqs. (7.164) for $D_{\delta x_2}$, $D_{\delta y_2}$ contain only the initial values of the random errors θ_x, θ_y, θ_z, Δm_x, Δm_y, and Δm_z is not included at all. When taking the sums on the right in Eqs. (7.158), the integrals over

$K_{\Delta m_x}(\tau, \tau')$, $K_{\Delta m_y}(\tau, \tau')$, $K_{\Delta m_z}(\tau, \tau')$ mutually cancel. The reason for this becomes evident if we approach Eqs. (7.119) somewhat differently.

In the first three equations in (7.119) we substitute the variables

$$\delta x = \delta x_2 - r\theta_y, \quad \delta y = \delta y_2 + r\theta_x, \quad \delta z_2 = \delta z, \tag{7.165}$$

using the last equations in (7.119). At the same time we express $\Delta\dot{m}_x$, $\Delta\dot{m}_y$, $\Delta\dot{m}_z$, Δm_z on the right in terms of θ_x, θ_y, θ_z and their derivatives using the fourth, fifth, and sixth equations in (7.119). For the total position errors δx_2, δy_2, δz_2 we then have the differential equations

$$\left.\begin{aligned} \delta\ddot{x}_2 + 2\omega_0^2\delta\dot{z} &= \Delta n_x, \\ \delta\ddot{y}_2 + \omega_0^2\delta y_2 &= \Delta n_y, \\ \delta\ddot{z}_2 - 3\omega_0^2\delta z_2 - 2\omega_0\delta\dot{x}_2 &= \Delta n_z \end{aligned}\right\} \tag{7.166}$$

with the initial conditions

$$\left.\begin{aligned} \delta x_2^0 &= \delta x^0 + r\theta_y^0, \quad \delta y_2^0 = \delta y^0 - r\theta_x^0, \quad \delta z_2^0 = \delta z^0, \\ \delta\dot{x}_2^0 &= \delta\dot{x}^0 + r\dot{\theta}_y^0, \quad \delta\dot{y}_2^0 = \delta\dot{y}^0 - r\dot{\theta}_x^0, \quad \delta\dot{z}_2^0 = \delta\dot{z}^0. \end{aligned}\right\} \tag{7.167}$$

As we see, Eqs. (7.166) on the right do not contain the instrumental errors Δm_x, Δm_y, Δm_z. Their initial values enter only the initial conditions.*

If we write for δx_2, δy_2, δz_2 a solution analogous to (7.121) for δx, δy, δz and calculate the dispersions $D_{\delta x_2}$, $D_{\delta y_2}$ according to this solution, we naturally end up again with Eqs. (7.162), (7.163). The reader will easily verify it for himself.

In conclusion note that we twice used the differential equations for the total position errors δx_2, δy_2, δz_2. For a stationary object these are Eqs. (7.107) and for a satellite in a circular orbit Eqs. (7.166). Equations of this kind naturally can be obtained in a general case as well, a point which may prove helpful in some problems. Thus, turning to the general error equations (1.95)—(1.97), we can find the corresponding equations for δx_2, δy_2, δz_2 substituting the variables from (1.97) in Eqs. (1.95). Making use of (1.97), we eliminate the instrumental errors Δm_x, Δm_y, Δm_z and their derivatives from the right sides of the equations. Instead terms containing the angles θ_x, θ_y, θ_z from Eqs. (1.97) are introduced.

§7.3. THE EFFECT OF RANDOM INSTRUMENTAL ERRORS ON AIDED SYSTEMS

7.3.1. Altimeter correction

In §2.2 we have derived the error equations of inertial systems with altimeter correction. Two alternative systems were considered: those with

* This is also applicable to satellite motion in an elliptical orbit. The elastic suspension of an ideal linear accelerometer placed at the center of mass of an object moving in a pure gravitational field is not strained (the accelerometer readings are zero) irrespective of the actual orientation of its sensitive axis.

three arbitrarily oriented accelerometers, and with two accelerometers in the plane of the geographical or geocentric horizon.

For three-accelerometer systems the error equations projected on the axes xyz with the z axis directed along r comprise Eqs. (2.16), Eqs. (1.96), and the algebraic relations (1.108), (1.118). The error equations of two-accelerometer systems include Eqs. (2.28), Eqs. (1.96), and again the algebraic relations (1.108), (1.118).

Let us consider these equations when the instrumental errors are random processes with correlation functions of the form (7.124). As in the previous section, we will only consider the case of a stationary object in the $O_1\xi_*\eta_*\zeta_*$ axes and the case of a satellite in a circular orbit.

First we will deal with the error equations of three-accelerometer systems. For a stationary object we have

$$\left.\begin{aligned}
&\delta\ddot{x} + \omega_0^2\delta x = \Delta n_x - r\,\Delta\dot{m}_y,\\
&\delta\ddot{y} + \omega_0^2\delta y = \Delta n_y + r\,\Delta\dot{m}_x,\\
&\delta\ddot{z} + \omega_0^2\delta z = \Delta n_z + 3\omega_0^2\Delta r;\\
&\dot{\theta}_x = \Delta m_x,\quad \dot{\theta}_y = \Delta m_y,\quad \dot{\theta}_z = \Delta m_z;\\
&\theta_{1x} = -\frac{\delta y}{r},\quad \theta_{1y} = \frac{\delta x}{r},\quad \theta_{1z} = -\theta_z;\\
&\delta x_2 = \delta x + r\theta_y,\quad \delta y_2 = \delta y - r\theta_x,\\
&\delta z_2 = \delta z.
\end{aligned}\right\} \tag{7.168}$$

Comparison with Eqs. (7.54) investigated in 7.2.1 shows that only the third equation is different. The dispersions $D_{\delta x}$, $D_{\delta y}$, D_{θ_x}, D_{θ_y}, D_{θ_z}, $D_{\theta_{1x}}$, $D_{\theta_{1y}}$, $D_{\theta_{1z}}$, $D_{\delta x_2}$, $D_{\delta y_2}$ of the solutions of the corresponding equations in (7.168) are therefore expressed by the previous relations, derived in connection with Eqs. (7.54). It thus remains to find the dispersion $D_{\delta z} = D_{\delta z_2}$ of the solution of the third equation in (7.168).

Note that the left-hand side of the third equation in (7.168) has the same form as the left-hand sides of the first two equations in (7.168). They all contain the accelerometer errors Δn_x, Δn_y, Δn_z in equivalent position. Thus, if the dispersion $D_{\delta z}$ of the solution of the third equation in (7.168) is written as the sum

$$D_{\delta z} = D_{\delta z}^{\Delta n_z} + D_{\delta z}^{\delta z^0} + D_{\delta z}^{\delta \dot{z}^0} + D_{\delta z}^{\Delta r}, \tag{7.169}$$

the first three terms can be obtained from (7.79), (7.76) by a suitable change of indices. These terms thus take the form

$$\left.\begin{aligned}
D_{\delta z}^{\Delta n_z} &= \frac{D_{\Delta n_z}}{\alpha_z^2+\omega_0^2}\left\{\frac{\alpha_z}{\omega_0^2}\left(t - \frac{1}{2\omega_0}\sin 2\omega_0 t\right) + \right.\\
&\left. + \frac{2}{\alpha_z^2+\omega_0^2}\left[1 - \left(\frac{\alpha_z}{\omega_0}\sin\omega_0 t + \cos\omega_0 t\right)e^{-\alpha_z t}\right] - \frac{1}{\omega_0^2}\sin^2\omega_0 t\right\},\\
D_{\delta z}^{\delta z^0} &= D_{\delta z^0}\cos^2\omega_0 t, \qquad D_{\delta z}^{\delta \dot{z}^0} = D_{\delta z^0}\sin^2\omega_0 t.
\end{aligned}\right\} \tag{7.170}$$

If the correlation function of the random altimeter error Δr is given by

$$K_{\Delta r}(t, t') = D_{\Delta r}e^{-\alpha_r|t-t'|}, \tag{7.171}$$

the fourth term in (7.169) clearly can be written in a form similar to the first expression in (7.170):

$$D_{\delta z}^{\Delta r} = \frac{2\omega_0^4}{\alpha_r^2+\omega_0^2} D_{\Delta r}\left\{\frac{\alpha_r}{\omega_0^2}\left(t-\frac{1}{2\omega_0}\sin 2\omega_0 t\right)+\right.$$
$$\left.+\frac{2}{\alpha_r^2+\omega_0^2}\left[1-\left(\frac{\alpha_r}{\omega_0}\sin\omega_0 t+\cos\omega_0 t\right)e^{-\alpha_r t}\right]-\frac{1}{\omega_0^2}\sin^2\omega_0 t\right\}. \qquad (7.172)$$

Eqs. (7.169), (7.170), (7.172) give the dispersion $D_{\delta z}$ of the error $\delta z = \delta z_2$ at any time. For small t we have the approximation

$$D_{\delta z} = D_{\delta z_0} + D_{\delta \dot z_0} t^2 + \left(D_{\Delta n_z} + 9\omega_0^2 D_{\Delta r}\right)\frac{t^4}{4}. \qquad (7.173)$$

For large t, $D_{\delta z}$ is asymptotically expressed by

$$D_{\delta z} = \left(\frac{D_{\Delta n_z}}{\alpha_r\omega_0^2} + \frac{\omega_0^2 D_{\Delta r}}{\alpha_r}\right)t. \qquad (7.174)$$

The dispersion $D_{\delta z}$ grows in proportion to time, whereas in autonomous inertial systems (see Eqs. (7.58), (7.60), (7.66)) the dispersion $D_{\delta z}$ is an exponentially increasing function of time (for sufficiently large t).

For a satellite in a circular orbit, the error equations of a three-accelerometer system are obtained from Eqs. (2.16), (1.96), (1.108), (1.118) where we should take

$$\omega_x = \omega_z = 0, \quad r = \text{const}, \quad \omega_y = \omega_0 = \sqrt{\frac{\mu}{r^3}} = \text{const}, \qquad (7.175)$$

and have the form

$$\left.\begin{aligned}
&\delta\ddot x + 2\omega_0\delta\dot z = \Delta n_x - r\,\Delta\dot m_y,\\
&\delta\ddot y + \omega_0^2\delta y = \Delta n_y + r\,\Delta\dot m_x - r\omega_0\,\Delta m_z,\\
&\delta\ddot z - 2\omega_0\delta\dot x = \Delta n_z + 2r\omega_0\,\Delta m_y + 3\omega_0^2\,\Delta r;\\
&\dot\theta_x + \omega_0\theta_z = \Delta m_x,\\
&\dot\theta_y = \Delta m_y,\\
&\dot\theta_z - \omega_0\theta_z = \Delta m_z;\\
&\theta_{1x} = -\frac{\delta y}{r}, \quad \theta_{1y} = \frac{\delta x}{r}, \quad \theta_{1z} = -\theta_z;\\
&\delta x_2 = \delta x + r\theta_y, \quad \delta y_2 = \delta y - r\theta_x,\\
&\delta z_2 = \delta z.
\end{aligned}\right\} \qquad (7.176)$$

Comparison with Eqs. (7.119) again shows that only the third equation is different. However, the generalization of the solution of Eqs. (7.119) to Eqs. (7.176) is a somewhat more complicated problem than in the case of Eqs. (7.54) and (7.163). Nevertheless, comparison of (7.176) and (7.119) shows that the dispersions D_{θ_x}, D_{θ_y}, D_{θ_z}, $D_{\delta y}$, and also the dispersions $D_{\theta_{1x}}$, $D_{\delta y_2}$, $D_{\theta_{1z}}$ have the same form in the case of Eqs. (7.176) as in the case of Eqs. (7.119). It therefore remains to find the dispersions $D_{\delta z} = D_{\delta z_2}$, $D_{\delta x_2}$, $D_{\theta_{1y}}$.

The solution of the system comprising the first and the third equations from (7.176) can be found using (2.28) where we put $r = 0$, $\omega_y = \omega_0$, and

express α_{ij}, β_{ij}, C_i^0 from (2.230) with $e = 0$, $\nu = \omega_0$. This gives

$$\left.\begin{aligned}
\delta x &= \frac{1}{2\omega_0}\int_0^t (\Delta n_x - r\,\Delta\dot{m}_y)\sin 2\omega_0(t-\tau)\,d\tau + \\
&\quad + \frac{1}{2\omega_0}\int_0^t (\Delta n_z + 2r\omega_0\,\Delta m_y + 3\omega_0^2\,\Delta r)\times \\
&\quad \times [\cos 2\omega_0(t-\tau) - 1]\,d\tau + \delta x^0 + \frac{\delta\dot{x}^0}{2\omega_0}\sin 2\omega_0 t + \frac{\delta\dot{z}^0}{2\omega_0}(\cos 2\omega_0 t - 1), \\
\delta z &= \frac{1}{2\omega_0}\int_0^t (\Delta n_z + 2r\omega_0\,\Delta m_y + 3\omega_0^2\,\Delta r)\times \\
&\quad \times \sin 2\omega_0(t-\tau)\,d\tau - \frac{1}{2\omega_0}\int_0^t (\Delta n_x - r\,\Delta\dot{m}_y)\times \\
&\quad \times [\cos 2\omega_0(t-\tau) - 1]\,d\tau + \delta z^0 + \frac{\delta\dot{z}^0}{2\omega_0}\sin 2\omega_0 t - \frac{\delta\dot{x}^0}{2\omega_0}(\cos 2\omega_0 t - 1).
\end{aligned}\right\} \qquad (7.177)$$

These expressions can be somewhat simplified if the terms with Δm_y are integrated by parts. We obtain

$$\left.\begin{aligned}
&-\frac{r}{2\omega_0}\int_0^t \Delta\dot{m}_y \sin 2\omega_0(t-\tau)\,d\tau = \\
&\quad = \frac{r\,\Delta m_y^0}{2\omega_0}\sin 2\omega_0 t - r\int_0^t \Delta m_y \cos 2\omega_0(t-\tau)\,d\tau, \\
&\frac{r}{2\omega_0}\int_0^t \Delta\dot{m}_y\,[\cos 2\omega_0(t-\tau) - 1]\,d\tau = \\
&\quad = -\frac{r\,\Delta m_y^0}{2\omega_0}(\cos 2\omega_0 t - 1) - \\
&\quad - r\int_0^t \Delta m_y \sin 2\omega_0(t-\tau)\,d\tau.
\end{aligned}\right\} \qquad (7.178)$$

Now

$$\left.\begin{aligned}
\delta x &= \frac{1}{2\omega_0}\int_0^t \Delta n_x \sin 2\omega_0(t-\tau)\,d\tau + \\
&\quad + \frac{1}{2\omega_0}\int_0^t (\Delta n_z + 3\omega_0^2\,\Delta r)\,[\cos 2\omega_0(t-\tau) - 1]\,d\tau - \\
&\quad - r\int_0^t \Delta m_y\,d\tau + \frac{r\,\Delta m_y^0 + \delta\dot{x}^0}{2\omega_0}\sin 2\omega_0 t + \\
&\quad + \delta x^0 + \frac{\delta\dot{z}_0}{2\omega_0}(\cos 2\omega_0 t - 1), \\
\delta z &= \frac{1}{2\omega_0}\int_0^t (\Delta n_z + 3\omega_0^2\,\Delta r)\sin 2\omega_0(t-\tau)\,d\tau - \\
&\quad - \frac{1}{2\omega_0}\int_0^t \Delta n_x\,[\cos 2\omega_0(t-\tau) - 1]\,d\tau - \\
&\quad - \frac{\delta\dot{x}^0 + r\,\Delta m_y^0}{2\omega_0}(\cos 2\omega_0 t - 1) + \delta z^0 + \frac{\delta\dot{z}^0}{2\omega_0}\sin 2\omega_0 t.
\end{aligned}\right\} \qquad (7.179)$$

By the second equation in (7.179) and the last equation in (7.176), we have

$$D_{\delta z_2} = D^{\delta z_1^0}_{\delta z_2} + D^{\delta \dot z_1^0}_{\delta z_2} + D^{\delta \dot x_1^0}_{\delta z_2} + D^{\Delta \eta_x}_{\delta z_2} + D^{\Delta \eta_z}_{\delta z_2} + D^{\Delta r}_{\delta z_2} + D^{\Delta m_y^0}_{\delta z_2}. \tag{7.180}$$

Here

$$\left.\begin{aligned} D^{\delta z_1^0}_{\delta z_2} &= D_{\delta z_1^0}, \qquad D^{\delta \dot z_1^0}_{\delta z_2} = \frac{1}{4\omega_0^2} D_{\delta \dot z_1^0} \sin^2 2\omega_0 t, \\ D^{\delta \dot x_1^0}_{\delta z_2} &= \frac{1}{4\omega_0^2} D_{\delta \dot x_1^0} (\cos 2\omega_0 t - 1)^2, \\ D^{\Delta m_y^0}_{\delta z_2} &= \frac{r_*}{4\omega_0^2} D_{\Delta m_y} (\cos 2\omega_0 t - 1)^2. \end{aligned}\right\} \tag{7.181}$$

The other three terms in (7.180) are expressed in integral form

$$\left.\begin{aligned} D^{\Delta \eta_x}_{\delta z_2} &= \frac{1}{4\omega_0^2} \int_0^t \int_0^t K_{\Delta \eta_x}(\tau, \tau') [\cos 2\omega_0 (t-\tau) - 1] \times \\ &\quad \times [\cos 2\omega_0 (t-\tau') - 1]\, d\tau\, d\tau', \\ D^{\Delta \eta_z}_{\delta z_2} &= \frac{1}{4\omega_0^2} \int_0^t \int_0^t K_{\Delta \eta_z}(\tau, \tau') \sin 2\omega_0 (t-\tau) \times \\ &\quad \times \sin 2\omega_0 (t-\tau')\, d\tau\, d\tau', \\ D^{\Delta r}_{\delta z_2} &= \frac{4\omega_0^2}{4} \int_0^t \int_0^t K_{\Delta r}(\tau, \tau') \sin 2\omega_0 (t-\tau) \times \\ &\quad \times \sin 2\omega_0 (t-\tau')\, d\tau\, d\tau'. \end{aligned}\right\} \tag{7.182}$$

These integrals are reduced to previously considered cases. Thus, the integral in the expression for $D^{\Delta \eta_x}_{\delta z_2}$ differs from the second integral in (7.128) only in that the ω_0 in the integrand has been replaced with $2\omega_0$. By analogy with (7.134) we thus have

$$\begin{aligned} D^{\Delta \eta_x}_{\delta z_2} = \frac{D_{\Delta \eta_x}}{4\omega_0^2 (\alpha_x^2 + 4\omega_0^2)} \Bigg[& \frac{3\alpha_x^2 + 4\omega_0^2}{\alpha_x} t - \frac{2\alpha_x^4 + 8\alpha_x^2 \omega_0^2 + 32\omega_0^4}{\alpha_x^2 (\alpha_x^2 + 4\omega_0^2)} + \\ & + \frac{8\omega_0^2}{\alpha_x^2} e^{-\alpha_x t} + \frac{\alpha_x}{4\omega_0} \sin 4\omega_0 t - \frac{2\alpha_x^2 + 4\omega_0^2}{\alpha_x \omega_0} \sin 2\omega_0 t + \\ & + 2\cos 2\omega_0 t + \sin^2 2\omega_0 t + \\ & + \frac{16\omega_0^3}{\alpha_x (\alpha_x^2 + 4\omega_0^2)} \left(\sin 2\omega_0 t - \frac{\alpha_x}{2\omega_0} \cos 2\omega_0 t \right) e^{-\alpha_x t} \Bigg]. \end{aligned} \tag{7.183}$$

Similarly comparison of the second and third integrals in (7.182) with the integral in (7.78) gives

$$\left.\begin{aligned} D^{\Delta \eta_z}_{\delta z_2} &= \frac{D_{\Delta \eta_z}}{\alpha_z^2 + 4\omega_0^2} \left\{ \frac{\alpha_z}{4\omega_0^2} \left(t - \frac{1}{4\omega_0} \sin 4\omega_0 t \right) + \right. \\ &\left. + \frac{2}{\alpha_z^2 + 4\omega_0^2} \left[1 - \left(\frac{\alpha_z}{2\omega_0} \sin 2\omega_0 t + \cos 2\omega_0 t \right) e^{-\alpha_z t} \right] - \frac{1}{4\omega_0^2} \sin^2 2\omega_0 t \right\}, \\ D^{\Delta r}_{\delta z_2} &= \frac{9\omega_0^4 D_{\Delta r}}{\alpha_r^2 + 4\omega_0^2} \left\{ \frac{\alpha_r}{4\omega_0^2} \left(t - \frac{1}{4\omega_0} \sin 4\omega_0 t \right) + \right. \\ &\left. + \frac{2}{\alpha_r^2 + 4\omega_0^2} \left[1 - \left(\frac{\alpha_r}{2\omega_0} \sin 2\omega_0 t + \cos 2\omega_0 t \right) e^{-\alpha_r t} \right] - \frac{1}{4\omega_0^2} \sin^2 2\omega_0 t \right\}. \end{aligned}\right\} \tag{7.184}$$

Eqs. (7.180), (7.181), (7.183), (7.184) completely determine the dispersion $D_{\delta z_2}$ as a function of time, if the correlation functions of the random instrumental errors Δn_x, Δn_z, Δm_y, Δr are given by Eqs. (7.124), (7.171). For small t, we have the approximation

$$D_{\delta z_2} = D_{\delta z^0} + D_{\delta \dot{z}^0} t^2 + \\ + \left(\omega_0^2 D_{\delta \dot{x}^0} + r^2 D_{\Delta m_y} \omega_0^2 + \frac{9\omega_0^4}{4} D_{\Delta r} + \frac{1}{4} D_{\Delta n_z}\right) t^4 + \frac{\omega_0^2}{9} D_{\Delta n_x} t^6. \tag{7.185}$$

For large t,

$$D_{\delta z_2} = \left(\frac{3D_{\Delta n_x}}{4\omega_0^2 a_x} + \frac{D_{\Delta n_z}}{4\omega_0^2 a_z} + \frac{9\omega_0^2 D_{\Delta r}}{4a_r}\right) t. \tag{7.186}$$

Comparison of (7.186) with (7.157) shows that in prolonged operation the random errors δz_2 caused by the instrumental errors Δn_x and Δn_z of a three-accelerometer system with altimeter correction have smaller dispersions than in an autonomous system. Although in either case these dispersions grow in proportion to time, the proportionality coefficient is smaller in the aided system with altimeter correction. In this case, however, the dispersion $D_{\delta z_2}$ acquires an additional term associated with the altimeter random error (the last term in (7.186)).

Let us calculate the dispersions $D_{\theta_{1y}}$ and $D_{\delta x_2}$. To find $D_{\theta_{1y}}$ we require the dispersion $D_{\delta x}$ of the solution of the first equation in (7.176). Once this is known, using the eighth equality in (7.168) we write

$$D_{\theta_{1y}} = \frac{D_{\delta x}}{r^2}. \tag{7.187}$$

The solution δx of the first equation in (7.176) has been found before. This is the first equation in (7.179). By this equation

$$D_{\delta x} = D_{\delta x}^{\delta x^0} + D_{\delta x}^{\delta \dot{x}^0} + D_{\delta x}^{\delta \dot{z}^0} + D_{\delta x}^{\Delta n_x} + D_{\delta x}^{\Delta n_z} + D_{\delta x}^{\Delta r} + D_{\delta x}^{\Delta m_y}, \tag{7.188}$$

where the first three terms are

$$\left.\begin{aligned} &D_{\delta x}^{\delta x^0} = D_{\delta x^0}, \quad D_{\delta x}^{\delta \dot{x}^0} = \frac{1}{4\omega_0^2} D_{\delta \dot{x}^0} \sin^2 2\omega_0 t, \\ &D_{\delta x}^{\delta \dot{z}^0} = \frac{1}{4\omega_0^2} D_{\delta \dot{z}^0} (\cos 2\omega_0 t - 1)^2. \end{aligned}\right\} \tag{7.189}$$

The last four terms are given in integral form:

$$\left.\begin{aligned} D_{\delta x}^{\Delta n_x} &= \frac{1}{4\omega_0^2} \int_0^t \int_0^t K_{\Delta n_x}(\tau, \tau') \sin 2\omega_0 (t-\tau) \times \\ &\qquad \times \sin 2\omega_0 (t-\tau')\, d\tau\, d\tau', \\ D_{\delta x}^{\Delta n_z} &= \frac{1}{4\omega_0^2} \int_0^t \int_0^t K_{\Delta n_z}(\tau, \tau') [\cos 2\omega_0 (t-\tau) - 1] \times \\ &\qquad \times [\cos 2\omega_0 (t-\tau') - 1]\, d\tau\, d\tau', \\ D_{\delta x}^{\Delta r} &= \frac{9\omega_0^2}{4} \int_0^t \int_0^t K_{\Delta r}(\tau, \tau') [\cos 2\omega_0 (t-\tau) - 1] \times \\ &\qquad \times [\cos 2\omega_0 (t-\tau') - 1]\, d\tau\, d\tau', \end{aligned}\right\} \tag{7.190}$$

$$D_{\delta x}^{\Delta m_y} = \int_0^t \int_0^t K_{\Delta m_y}(\tau, \tau')\, d\tau\, d\tau' + \\ + \frac{r^2}{4\omega_0^2} D_{\Delta m_y} \sin^2 2\omega_0 t - \\ - \frac{r^2}{\omega_0} \sin 2\omega_0 t \int_0^t K_{\Delta m_y}(\tau, 0)\, d\tau. \Bigg\} \tag{7.190}$$

The first integrals are analogous to the integrals in (7.182). Therefore, by analogy with (7.183), (7.184), we find

$$D_{\delta x}^{\Delta n_x} = \frac{D_{\Delta n_x}}{\alpha_x^2 + 4\omega_0^2} \left\{ \frac{\alpha_x}{4\omega_0^2} \left(t - \frac{1}{4\omega_0} \sin 4\omega_0 t \right) + \right. \\ + \frac{2}{\alpha_x^2 + 4\omega_0^2} \left[1 - \left(\frac{\alpha_x}{2\omega_0} \sin 2\omega_0 t + \cos 2\omega_0 t \right) e^{-\alpha_x t} \right] - \\ \left. - \frac{1}{4\omega_0^2} \sin^2 2\omega_0 t \right\},$$

$$D_{\delta x}^{\Delta n_z} = \frac{D_{\Delta n_z}}{4\omega_0^2(\alpha_z^2 + 4\omega_0^2)} \left[\frac{3\alpha_z^2 + 4\omega_0^2}{\alpha_z} t - \frac{2\alpha_z^4 + 8\alpha_z^2\omega_0^2 + 32\omega_0^4}{\alpha_z^2(\alpha_z^2 + 4\omega_0^2)} + \right. \\ + \frac{8\omega_0^2}{\alpha_z^2} e^{-\alpha_z t} + \frac{\alpha_z}{4\omega_0} \sin 4\omega_0 t - \frac{2\alpha_z^2 + 4\omega_0^2}{\alpha_z \omega_0} \sin 2\omega_0 t + \\ + 2\cos 2\omega_0 t + \sin^2 2\omega_0 t + \frac{16\omega_0^4}{\alpha_z(\alpha_z^2 + 4\omega_0^2)} \left(\sin 2\omega_0 t - \right. \\ \left. \left. - \frac{\alpha_z}{2\omega_0} \cos 2\omega_0 t \right) e^{-\alpha_z t} \right], \tag{7.191}$$

$$D_{\delta x}^{\Delta r} = \frac{9\omega_0^2 D_{\Delta r}}{4(\alpha_r^2 + 4\omega_0^2)} \left[\frac{3\alpha_r^2 + 4\omega_0^2}{\alpha_r} t - \right. \\ - \frac{2\alpha_r^4 + 8\alpha_r^2\omega_0^2 + 32\omega_0^4}{\alpha_r^2(\alpha_r^2 + 4\omega_0^2)} + \frac{8\omega_0^2}{\alpha_r^2} e^{-\alpha_r t} + \frac{\alpha_r}{4\omega_0^2} \sin 4\omega_0 t - \\ - \frac{2\alpha_r^2 + 4\omega_0^2}{\alpha_r \omega_0} \sin 2\omega_0 t + 2\cos 2\omega_0 t + \sin^2 2\omega_0 t + \\ \left. + \frac{16\omega_0^4}{\alpha_r(\alpha_r^2 + 4\omega_0^2)} \left(\sin 2\omega_0 t - \frac{\alpha_r}{2\omega_0} \cos 2\omega_0 t \right) e^{-\alpha_r t} \right].$$

From the fourth equation in (7.190) we have

$$D_{\delta x}^{\Delta m_y} = r^2 D_{\Delta m_y} \left[\frac{2t}{\beta_y} + \frac{1}{4\omega_0^2} \sin^2 2\omega_0 t + \left(e^{-\beta_y t} - 1 \right) \left(\frac{2}{\beta_y^2} + \frac{1}{\omega_0 \beta_y} \sin 2\omega_0 t \right) \right]. \tag{7.192}$$

Eqs. (7.187)—(7.189), (7.191), (7.192) define the dispersion $D_{\theta_{1y}}$ at any time. For small t, we have the approximation

$$D_{\theta_{1y}} = \frac{1}{r^2} \left[D_{\delta x^0} + D_{\delta \dot{x}^0} t^2 + \left(\omega_0^2 D_{\delta \dot{z}^0} + \frac{1}{4} D_{\Delta n_x} \right) t^4 + \right. \\ \left. + \frac{\omega_0^2}{9} \left(D_{\Delta n_z} + 9\omega_0^4 D_{\Delta r} \right) t^6 \right] + \frac{2}{3} \beta_y D_{\Delta m_y} t^3. \tag{7.193}$$

For large t,

$$D_{\theta_{1y}} = \left[\frac{1}{r^2} \left(\frac{D_{\Delta n_x}}{4\omega_0^2 \alpha_x} + \frac{3D_{\Delta n_z}}{4\omega_0^2 \alpha_z} + \frac{27\omega_0^2 D_{\Delta r}}{4\alpha_r} \right) + \frac{2D_{\Delta m_y}}{\beta_y} \right] t. \tag{7.194}$$

This asymptotic expression can be compared with Eq. (7.137) for the dispersion $D_{\theta_{1y}}$ of an autonomous system. Eq. (7.194) contains an additional

term with $D_{\Delta r}$. However, the other terms are linear in time, whereas Eq. (7.137) contains quadratic and cubic terms. Altimeter correction thus essentially lowers the system errors.

It now remains to find the dispersion $D_{\delta x_2}$ of the position error along the orbit.

By the tenth equality in (7.168), $\delta x_2 = \delta x + r\theta_y$. The expressions for δx and θ_y are given by the first equation in (7.179) and by (7.123). From these equations we have

$$\delta x_2 = \frac{1}{2\omega_0}\int_0^t \Delta n_x \sin 2\omega_0 (t-\tau)\, d\tau + \\ + \frac{1}{2\omega_0}\int_0^t (\Delta n_z + 3\omega_0^2 \Delta r)\left[\cos 2\omega_0 (t-\tau) - 1\right] d\tau + \\ + \frac{\delta \dot{x}^0 + r\,\Delta m_y^0}{2\omega_0}\sin 2\omega_0 t + \delta x^0 + \frac{\delta \dot{z}^0}{2\omega_0}(\cos 2\omega_0 t - 1) + r\theta_y^0. \tag{7.195}$$

Using this expression for δx_2, we write the dispersion $D_{\delta x_2}$ as the sum

$$D_{\delta x_2} = D_{\delta x_2}^{\Delta m_y^0} + D_{\delta x_2}^{\theta_y^0} + D_{\delta x_2}^{\delta \dot{x}^0} + D_{\delta x_2}^{\delta \dot{z}^0} + D_{\delta x_2}^{\Delta n_x} + D_{\delta x_2}^{\Delta n_z} + D_{\delta x_2}^{\Delta r}.$$

Here

$$\left.\begin{aligned} D_{\delta x_2}^{\Delta m_y^0} &= \frac{r^2}{4\omega_0^2} D_{\Delta m_y} \sin^2 2\omega_0 t, \\ D_{\delta x_2}^{\theta_y^0} &= r^2 D_{\theta_y^0}. \end{aligned}\right\} \tag{7.196}$$

the next two terms are expressed by the first and the third equations in (7.189), and the three last terms are given by (7.191).

For small t, we have

$$D_{\delta x_2} = D_{\delta x^0} + r^2 D_{\theta_y^0} + \left(D_{\delta \dot{x}^0} + r^2 D_{\Delta m_y}\right) t^2 + \\ + \left(\omega_0^2 D_{\delta \dot{z}^0} + \frac{1}{4} D_{\Delta n_x}\right) t^4 + \left(\frac{1}{9} D_{\Delta n_z} + \omega_0^4 D_{\Delta r}\right) \omega_0^2 t^6. \tag{7.197}$$

For large t, asymptotically

$$D_{\delta x_2} = \frac{1}{4\omega_0^2}\left(\frac{D_{\Delta n_x}}{\alpha_x} + \frac{3D_{\Delta n_z}}{\alpha_z} + \frac{27\omega_0^4 D_{\Delta r}}{\alpha_r}\right) t. \tag{7.198}$$

Comparison of the asymptotic expression with the first relation in (7.164) shows that altimeter correction substantially reduces the random error δx_2, just as we have seen for the error θ_{1y}.

We have so far considered a three-accelerometer inertial system with altimeter correction. We will now proceed with an analysis of two-accelerometer systems. Again we will only consider the case of a stationary object and the case of a satellite in circular orbit.

If the object with the inertial system is stationary in the axes $O_1\xi_*\eta_*\zeta_*$, the error equations of the inertial navigation system are derived from Eqs. (2.28), (1.96), (1.108), (1.118) putting $\omega_x = \omega_y = \omega_z = \dot{r} = 0$. These equations

take the form

$$\left.\begin{aligned}
&\delta\ddot{x} + \omega_0^2 \delta x = \Delta n_x - r\,\Delta\dot{m}_y,\\
&\delta\ddot{y} + \omega_0^2 \delta y = \Delta n_y + r\,\Delta\dot{m}_x;\\
&\dot{\theta}_x = \Delta m_x,\quad \dot{\theta}_y = \Delta m_y,\quad \dot{\theta}_z = \Delta m_z;\\
&\theta_{1x} = -\frac{\delta y}{r},\quad \theta_{1y} = \frac{\delta x}{r},\quad \theta_{1z} = -\theta_z;\\
&\delta x_2 = \delta x + r\theta_y,\quad \delta y_2 = \delta y - r\theta_x.
\end{aligned}\right\} \tag{7.199}$$

Eqs. (7.199) differ from the error equations (7.168) of a three-accelerometer system only in that they do not contain the third and the last equations from (7.168) (the equations for δz, δz_2). Since the errors δz, δz_2 do not enter the remaining equations, the analysis of Eqs. (7.199) leads to the same results as Eqs. (7.168).

Consider the case of a satellite in circular orbit. The error equations of the two-accelerometer system are derived from Eqs. (2.28), (1.96), (1.108), (1.118) where we put $\omega_x = \omega_{0x} = \dot{r} = 0$, $\omega_y = \omega_{0y} = \sqrt{\frac{\mu}{r}} = \text{const}$, $r = \text{const}$. This gives

$$\left.\begin{aligned}
&\delta\ddot{x} = -2\omega_0\,\Delta\dot{r} + \Delta n_x - r\,\Delta\dot{m}_y,\\
&\delta\ddot{y} + \omega_0^2 \delta y = \Delta n_y + r\,\Delta\dot{m}_x - r\omega_0\,\Delta m_z;\\
&\dot{\theta}_x + \omega_0\theta_z = \Delta m_x,\\
&\dot{\theta}_y = \Delta m_y,\\
&\dot{\theta}_z - \omega_0\theta_z = \Delta m_z;\\
&\theta_{1x} = -\frac{\delta y}{r},\quad \theta_{1y} = \frac{\delta x}{r},\quad \theta_{1z} = -\theta_z;\\
&\delta x_2 = \delta x + r\theta_y,\quad \delta y_2 = \delta y - r\theta_x.
\end{aligned}\right\} \tag{7.200}$$

This system again does not contain the third and the last equations from (7.176) (the equations for δz and δz_2). The structure of the first equations, however, is different.

The results obtained for Eqs. (7.176) are therefore only partially applicable to Eqs. (7.200). The dispersions D_{θ_x}, D_{θ_y}, D_{θ_z}, $D_{\theta_{1x}}$, $D_{\theta_{1z}}$, $D_{\delta y_2}$, $D_{\delta y_2}$ for the case of Eqs. (7.200) evidently have the same values as for the case of Eqs. (7.176). The dispersions $D_{\delta x}$, $D_{\theta_{1y}}$, $D_{\delta x_2}$ on the other hand, are different and have to be calculated anew.

The first equation in (7.200) was solved in §2.3. Its solution is given by the first equation in (2.75)

$$\delta x = \delta x^0 + \delta\dot{x}^0 t + \int_0^t (\Delta n_x - r\,\Delta\dot{m}_y - 2\omega_0\,\Delta\dot{r})(t-\tau)\,d\tau. \tag{7.201}$$

By-parts integration gives

$$\delta x = \delta x^0 + \delta\dot{x}^0 t + \int_0^t \Delta n_x (t-\tau)\,d\tau - \int_0^t (r\,\Delta m_y + 2\omega_0\,\Delta r)\,d\tau + r\,\Delta m_y t + 2\omega_0\,\Delta r^0 t. \tag{7.202}$$

The dispersion $D_{\delta x}$ of the random error δx can be written as the sum

$$D_{\delta x} = D_{\delta x}^{\delta x} + D_{\delta x}^{\delta\dot{x}} + D_{\delta x}^{\Delta n_x} + D_{\delta x}^{\Delta r} + D_{\delta x}^{\Delta m_y}. \tag{7.203}$$

The first two terms of this sum are

$$D_{\delta x}^{\delta x^0} = D_{\delta x^0}, \quad D_{\delta x}^{\delta \dot{x}^0} = D_{\delta \dot{x}^0} t^2. \tag{7.204}$$

The last three terms are given in integral form

$$\left.\begin{aligned} D_{\delta x}^{\Delta n_x} &= \int_0^t \int_0^t K_{\Delta n_x}(\tau, \tau')(t-\tau)(t-\tau')\,d\tau\,d\tau', \\ D_{\delta x}^{\Delta m_y} &= r^2\left[D_{\Delta m_y} t^2 + \int_0^t \int_0^t K_{\Delta m_y}(\tau, \tau')\,d\tau\,d\tau' - \right. \\ &\qquad \left. - 2t\int_0^t K_{\Delta m_y}(\tau, 0)\,d\tau\right], \\ D_{\delta x}^{\Delta r} &= 4\omega_0^2\left[D_{\Delta r} t^2 + \int_0^t \int_0^t K_{\Delta r}(\tau, \tau')\,d\tau\,d\tau' - \right. \\ &\qquad \left. - 2t\int_0^t K_{\Delta r}(\tau, 0)\,d\tau\right]. \end{aligned}\right\} \tag{7.205}$$

Inserting for the correlation functions $K_{\Delta n_x}$, $K_{\Delta r}$, $K_{\Delta m_y}$ of the random instrumental errors Δn_x, Δr, Δm_y their expressions from (7.124), (7.171) and integrating, we obtain

$$\left.\begin{aligned} D_{\delta x}^{\Delta n_x} &= D_{\Delta n_x}\left\{\frac{2}{3\alpha_x} t^3 - \frac{1}{\alpha_x^2} t^2 + \frac{2}{\alpha_x^4}\left[1 - (\alpha_x t + 1) e^{-\alpha_x t}\right]\right\}, \\ D_{\delta x}^{\Delta m_y} &= r^2 D_{\Delta m_y}\left[t^2 + \frac{2}{\beta_y^2}\left(e^{-\beta_y t} - 1\right) + \frac{2t}{\beta_y} e^{-\beta_y t}\right], \\ D_{\delta x}^{\Delta r} &= 4\omega_0^2 D_{\Delta r}\left[t^2 + \frac{2}{\alpha_r^2}\left(e^{-\alpha_r t} - 1\right) + \frac{2t}{\alpha_r} e^{-\alpha_r t}\right]. \end{aligned}\right\} \tag{7.206}$$

Eqs. (7.203), (7.204), (7.206) give the dispersion $D_{\delta x}$ of the random error δx and the dispersion $D_{\theta_{1y}}$, since $D_{\theta_{1y}} = D_{\delta x}/r^2$. For small t, we have the simple approximation

$$D_{\theta_{1y}} = \frac{1}{r^2}\left(D_{\delta x^0} + D_{\delta \dot{x}^0} t^2 + \frac{1}{4} D_{\Delta n_x} t^4 + \frac{8}{3}\omega_0^2 \alpha_r D_{\Delta r} t^3\right) + \frac{2}{3}\beta_y D_{\Delta m_y} t^3. \tag{7.207}$$

For large t,

$$D_{\theta_{1y}} = \left[\frac{1}{r^2}\left(D_{\delta \dot{x}^0} + 4\omega_0^2 D_{\Delta r}\right) + D_{\Delta m_y}\right] t^2 + \frac{2}{3r^2\alpha_x} D_{\Delta n_x} t^3. \tag{7.208}$$

Comparison of the asymptotic expressions (7.208), (7.194), (7.137) for the dispersion $D_{\theta_{1y}}$ in two-accelerometer and three-accelerometer systems with altimeter correction and in autonomous inertial systems shows that the two-accelerometer system is characterized by much higher dispersions $D_{\theta_{1y}}$ than the three-accelerometer system. Eq. (7.208) is much closer in its structure to Eq. (7.137) for the dispersion $D_{\theta_{1y}}$ in an autonomous system than to Eq. (7.194) for this dispersion in a three-accelerometer system.

Let us calculate the dispersion $D_{\delta x_2}$. Using the equality $\delta x_2 = \delta x + r\theta_y$ and Eqs. (7.202), (7.123), we find

$$\delta x_2 = \delta x^0 + r\theta_y^0 + (\delta \dot{x}^0 + r\,\Delta m_y^0 + 2\omega_0\,\Delta r^0)\,t + \int_0^t (t-\tau)\,\Delta n_x\,d\tau - 2\omega_0 \int_0^t \Delta r\,d\tau. \tag{7.209}$$

We thus have

$$D_{\delta x_2} = D^{\delta x^0}_{\delta x_2} + D^{\delta \dot{x}^0}_{\delta x_2} + D^{\theta^0_y}_{\delta x_2} + D^{\Delta r}_{\delta x_2} + D^{\Delta n_x}_{\delta x_2} + D^{\Delta m^0_y}_{\delta x_2}. \tag{7.210}$$

Here the first two terms are given by Eqs. (7.204) and

$$D^{\theta^0_y}_{\delta x_2} = r^2 D_{\theta^0_y}; \tag{7.211}$$

the fourth and the fifth term are given by the third and the first equation in (7.206) and the dispersion $D^{\Delta m^0_y}_{\delta x_2}$ is calculated from the equality

$$D^{\Delta m^0_y}_{\delta x_2} = r^2 D_{\Delta m_y} t^2. \tag{7.212}$$

For small t, we have from (7.210)

$$D_{\delta x_2} = D_{\delta x^0} + r^2 D_{\theta^0_y} + \left(D_{\delta \dot{x}^0} + r^2 D_{\Delta m_y}\right) t^2 + \frac{8}{3}\omega_0^2 \alpha_r D_{\Delta r} t^3 + \frac{1}{4} D_{\Delta n_x} t^4. \tag{7.213}$$

For large t, the simplified asymptotic expression is

$$D_{\delta x_2} = \left(D_{\delta \dot{x}^0} + r^2 D_{\Delta m_y} + 4\omega_0^2 D_{\Delta r}\right) t^2 + \frac{2}{3\alpha_r} D_{\Delta n_x} t^3. \tag{7.214}$$

Comparison of this expression with Eqs. (7.198), (7.164a) leads to the same conclusions as the comparison of Eqs. (7.208), (7.194), (7.137) for the dispersion $D_{\theta_{1y}}$.

7.3.2. Combined altimeter and Doppler correction

Consider the effect of random instrumental errors in a system where an altimeter and a Doppler velocity meter supply simultaneous guidance information. We will only analyze the simplified error equations of Chapter 5, i.e., Eqs. (5.297), (5.299), Eqs. (5.308), (5.309), Eqs. (5.311), Eqs. (5.313), and finally Eqs. (5.335), (5.337), (5.340). Since the simplified equations of three-accelerometer systems (5.297), (5.299) and (5.308), (5.309) are analogous to the simplified equations of two-accelerometer systems (5.311) and (5.313), we need only consider the two-accelerometer case. The results of this analysis are then readily extended to Eqs. (5.297), (5.299) and (5.308), (5.309).

We have seen in Chapter 5 that the second group of error equations in the case (5.311) is the same as for an autonomous inertial system. Using the symmetry of these equations and Eqs. (5.311), we conclude that only the following equations should be considered:

$$\left.\begin{aligned} &\delta\ddot{x} + k_1 \delta\dot{x} + (1 + k_2)\omega_0^2 \delta x = \\ &\quad = (1 + k_2)\Delta n_x - r\,\Delta\dot{m}_y + k_1(\Delta V_{Dx} - r\,\Delta m_y) - k_2 \Delta\dot{V}_{Dx}; \\ &\theta_{1y} = \frac{\delta x}{r}, \quad \theta_{1z} = -\theta_z; \\ &\dot{\theta}_y = \Delta m_y, \quad \dot{\theta}_z = \Delta m_z; \\ &\delta x_2 = \delta x + r\theta_y. \end{aligned}\right\} \tag{7.215}$$

Similarly, supplementing the first equation in (5.313) by the last five relations in (7.215) we obtain a system ($k_1'\omega_0^2 = k_1$):

$$\left.\begin{aligned} \delta\ddot{x} + k_1\delta\dot{x} + (1+k_2)\omega_0^2\delta x &= (1+k_2)\Delta n_x + \\ &+ \frac{k_1}{\omega_0^2}\Delta\dot{n}_x - r\Delta\dot{m}_y - k_2\Delta\dot{V}_{Dx} - \frac{k_1}{\omega_0^2}\Delta\ddot{V}_{Dx}; \\ \theta_{1y} &= \frac{\delta x}{r}, \quad \theta_{1z} = -\theta_z; \\ \dot{\theta}_y &= \Delta m_y, \quad \dot{\theta}_z = \Delta m_z; \\ \delta x_2 &= \delta x + r\theta_y. \end{aligned}\right\} \tag{7.216}$$

Now supplementing Eqs. (5.335), (5.337), (5.340) by the necessary relations and using their symmetry, we obtain the partial system

$$\left.\begin{aligned} \delta\ddot{x} + k_1\delta\dot{x} + \omega_0^2(1+k_2)\delta x &= (1+k_2)\Delta n_x - \\ &- r\Delta\dot{m}_y - k_2\Delta\dot{V}_{Dx} + k_1(\Delta V_{Dx} - r\Delta m_y); \\ \theta_{1y} &= \frac{\delta x}{r}, \quad \theta_{1z} = -\theta_z; \\ \dot{\theta}_y &= \Delta m_y + \frac{k_2}{r}\Delta V_{Dx}, \quad \theta_z = \Delta m_z; \\ \delta x_2 &= \frac{1}{1+k_2}(\delta x + r\theta_y). \end{aligned}\right\} \tag{7.217}$$

Note that the equations for θ_{1z}, θ_z are the same in Eqs. (7.215), (7.216), (7.217), being analogous to the corresponding equations for an autonomous inertial system. They were considered in the previous section and are of no interest at the moment.

Consider Eqs. (7.215). Our problem is to calculate the dispersions $D_{\theta_{1y}}$, $D_{\delta x_2}$ of the random orientation and position errors. We assume, as before, that the instrumental errors Δn_x, Δm_y, ΔV_{Dx} are stationary uncorrelated random processes. The correlation functions are defined by the equalities

$$\left.\begin{aligned} K_{\Delta n_x} &= D_{\Delta n_x}e^{-\alpha|t-t'|}, \quad K_{\Delta m_y} = D_{\Delta m_y}e^{-\beta|t-t'|}, \\ K_{\Delta V_{Dx}} &= D_{\Delta V_{Dx}}e^{-\gamma|t-t'|}. \end{aligned}\right\} \tag{7.218}$$

From the second relation in (7.215) we get $D_{\theta_{1y}} = D_{\delta x}/r^2$. Since the solution of the homogeneous equation for δx is damped (and an appropriate choice of k_1 and k_2 will ensure a sufficiently fast damping), only the steady-state value of the dispersion $D_{\delta x}$ need be calculated, using Eq. (7.48).

The dispersion $D_{\delta x}$ can be written as the sum

$$D_{\delta x} = D_{\delta x}^{\Delta n_x} + D_{\delta x}^{\Delta m_y} + D_{\delta x}^{\Delta V_{Dx}}. \tag{7.219}$$

Since the spectral density is

$$S_{\Delta n_x}(\omega) = \frac{D_{\Delta n_x}}{\pi}\frac{\alpha}{\alpha^2+\omega^2} \tag{7.220}$$

and similar expressions are obtained for $S_{\Delta m_y}$ and $S_{\Delta V_{Dx}}$, the terms on the

right in Eq. (7.219) are given, in accordance with (7.48), by

$$\left.\begin{aligned} D_{\delta x}^{\Delta n_x} &= \frac{D_{\Delta n_x}(1+k_2)^2\alpha}{\pi}\int_{-\infty}^{\infty}\frac{d\omega}{(\alpha^2+\omega^2)\left|(j\omega)^2+k_1 j\omega+(1+k_2)\omega_0^2\right|^2},\\ D_{\delta x}^{\Delta n_y} &= \frac{D_{\Delta n_y}\beta r^2}{\pi}\int_{-\infty}^{\infty}\frac{|-j\omega-k_1|^2\,d\omega}{(\beta^2+\omega^2)\left|(j\omega)^2+k_1 j\omega+(1+k_2)\omega_0^2\right|^2},\\ D_{\delta x}^{\Delta V_{Dx}} &= \frac{D_{\Delta V_{Dx}}\gamma}{\pi}\int_{-\infty}^{\infty}\frac{|k_1-k_2 j\omega|^2\,d\omega}{(\gamma^2+\omega^2)\left|(j\omega)^2+k_1 j\omega+(1+k_2)\omega_0^2\right|^2}. \end{aligned}\right\} \tag{7.221}$$

The integrals on the right are written in the form

$$J_n = \int_{-\infty}^{\infty}\frac{g_n(j\omega)\,d\omega}{h_n(j\omega)\,h_n(-j\omega)}, \tag{7.222}$$

where in general

$$\left.\begin{aligned} h_n(x) &= a_0x^n + a_1x^{n-1} + \ldots + a_n,\\ g_n(x) &= b_0x^{2n-2} + b_1x^{2n-4} + \ldots + b_{n-1} \end{aligned}\right| \tag{7.223}$$

(all the roots of $h_n(x) = 0$ are in the left half-plane).

The integrals (7.222) are rational functions of the coefficients of the polynomials $g_n(x)$ and $h_n(x)$.* Thus, for $n = 1, 2, 3, 4$, we have**

$$\left.\begin{aligned} J_1 &= \frac{\pi b_1}{a_0a_1}, \quad J_2 = \frac{\pi\left(-b_0+\frac{a_2b_1}{a_1}\right)}{a_0a_1},\\ J_3 &= \frac{\pi\left(-a_2b_0+a_1b_1-\frac{a_2a_1b_2}{a_1}\right)}{a_2(a_0a_3-a_1a_2)},\\ J_4 &= \frac{\left[b_0(-a_1a_4+a_2a_3)-a_3a_1b_1+a_3a_1b_2+\frac{a_3b_3}{a_4}(a_0a_3-a_1a_2)\right]}{a_0\left(a_0a_3^2+a_1^2a_4-a_1a_2a_3\right)}. \end{aligned}\right\} \tag{7.224}$$

For the first integral in (7.221), $n = 3$ and

$$\left.\begin{aligned} a_0 &= 1, \quad a_1 = k_1+\alpha, \quad a_2 = (1+k_2)\omega_0^2 + k_1\alpha,\\ a_3 &= \alpha\omega_0^2(1+k_2), \quad b_0 = 0, \quad b_1 = 0, \quad b_2 = 1. \end{aligned}\right| \tag{7.225}$$

Inserting these values in the third equation in (7.224), we obtain

$$D_{\delta x}^{\Delta n_x} = D_{\Delta n_x}\frac{(k_1+\alpha)(1+k_2)}{\omega_0^2k_1\left[\omega_0^2(1+k_2)+\alpha(k_1+\alpha)\right]}. \tag{7.226}$$

The only difference in the coefficients a_i of the integrals (7.221) is that the α of (7.225) has been replaced with β. The coefficients b_i are given by

$$b_0 = 0, \quad b_1 = -1, \quad b_2 = k_1^2. \tag{7.227}$$

* Bulgakov, B.V. Kolebaniya (Oscillations). — Gostekhizdat. 1954

** The values of the integrals J_n for $n = 1, \ldots, 7$ can be found in Pugachev, V.S. Teoriya sluchaynykh funktsii i ee primenenie k zadacham avtomaticheskogo upravleniya (Theory of Random Functions and Its Application to Automatic Control). — Gostekhizdat. 1957.

All this gives

$$D_{\delta x}^{\Delta m_y} = r^2 D_{\Delta m_y} \frac{\beta + \frac{k_1^2 (k_1 + \beta)}{\omega_0^2 (1 + k_2)}}{k_1 \left[\omega_0^2 (1 + k_2) + \beta (k_1 + \beta)\right]}. \tag{7.228}$$

Similarly we find

$$D_{\delta x}^{\Delta V_{Dx}} = D_{\Delta V_{Dx}} \frac{k_2^2 \gamma + \frac{k_1^2 (k_1 + \gamma)}{\omega_0^2 (1 + k_2)}}{k_1 \left[\omega_0^2 (1 + k_2) + \gamma (k_1 + \gamma)\right]}. \tag{7.229}$$

Eqs. (7.219), (7.226), (7.228), (7.229) give the steady-state dispersion $D_{\delta x}$, and using the equality

$$D_{\theta_{1y}} = \frac{1}{r^2} D_{\delta x} \tag{7.229a}$$

we find the steady-state dispersion $D_{\theta_{1y}}$.

Let us now calculate the dispersion $D_{\delta x_2}$ of the total position error along the axis x. Using the last equality in (7.215), we have

$$D_{\delta x_2} = D_{\delta r} + r^2 D_{\theta_y} + 2r K_{\delta x, \theta_y}(t, t). \tag{7.230}$$

$D_{\Delta r}$ has been calculated above (for sufficiently large t). D_{θ_y} has the same value as in an autonomous inertial system, and is thus given by (7.72). Since the first term in (7.230) is the known function $D_{\delta x}$, we may write instead of Eqs. (7.72)

$$D_{\theta_y} = D_{\theta_y^0} + 2D_{\Delta m_y}\left(\frac{t}{\beta} - \frac{1}{\beta^2}\right). \tag{7.231}$$

It thus remains to find $K_{\delta x, \theta_y}(t, t)$. The random instrumental errors and the initial conditions are assumed to be uncorrelated, and therefore by the first and the fourth equation in (7.215) we have

$$K_{\delta x, \theta_y}(t, t) = -\frac{r}{\nu} \int_0^t \int_0^t \left[k_1 K_{\Delta m_y}(\tau, \tau') + K_{\Delta \dot{m}_y, \Delta m_y}(\tau, \tau')\right] e^{-\varepsilon(t-\tau)} \sin \nu (t - \tau)\, d\tau\, d\tau', \tag{7.232}$$

where

$$\varepsilon = \frac{k_1}{2}, \quad \nu = \sqrt{\omega_0^2 (1 + k_2) - \frac{k_1^2}{4}}. \tag{7.233}$$

Inserting $K_{\Delta m_y}$ and $K_{\Delta \dot{m}_y, \Delta m_y}$ from (7.218), (7.99) in the integrand on the right in Eqs. (7.232), we find

$$K_{\delta x, \theta_y}(t, t) = -\frac{r}{\nu} D_{\Delta m_y} \int_0^t \int_0^t \left[k_1 - \beta \operatorname{sign}(\tau - \tau')\right] \times e^{-\beta|\tau - \tau'| - \varepsilon(t-\tau)} \sin \nu (t - \tau)\, d\tau\, d\tau'. \tag{7.234}$$

Developing the double integral on the right, we find

$$K_{\delta x,\,\theta_y}(t,\,t)=\frac{r}{\beta}D_{\Delta m_y}\left\{\frac{2k_1}{\varepsilon^2+\nu^2}-\frac{k_1+\beta}{(\varepsilon+\beta)^2+\nu^2}-\right.$$
$$-\frac{k_1-\beta}{(\varepsilon-\beta)^2+\nu^2}e^{-\beta t}+\frac{k_1+\beta}{\nu[(\varepsilon+\beta)^2+\nu^2]}\Big[-(\varepsilon+\beta)\sin\nu t+$$
$$+\nu\cos\nu t\Big]e^{-(\varepsilon+\beta)t}+\Big[\Big(-\frac{2k_1\varepsilon}{\nu(\varepsilon^2+\nu^2)}+$$
$$+\frac{(k_1-\beta)(\varepsilon-\beta)}{\nu[(\varepsilon-\beta)^2+\nu^2]}\Big)\sin\nu t+$$
$$\left.+\Big(-\frac{2k_1}{\varepsilon^2+\nu^2}+\frac{k_1-\beta}{(\varepsilon-\beta)^2+\nu^2}\Big)\cos\nu t\Big]e^{-\varepsilon t}\right\}. \tag{7.235}$$

When the transients are damped

$$K_{\delta x,\,\theta_y}(t,\,t)=\frac{rD_{\Delta m_y}}{\beta}\left[\frac{2k_1}{\varepsilon^2+\nu^2}-\frac{k_1+\beta}{(\varepsilon^2+\beta^2)+\nu^2}\right]. \tag{7.236}$$

The dispersion $D_{\delta x_2}$ is now written as the sum

$$D_{\delta x_2}=D_{\delta x}^{\Delta n_x}+D_{\delta x}^{\Delta V_{Dx}}+D_{\delta x_2}^{\Delta m_y}, \tag{7.237}$$

where the first two terms are calculated from (7.226), (7.229), and the last term is given by

$$D_{\delta x_2}^{\Delta m_y}=r^2D_{\theta_y}+2rK_{\delta x,\,\theta_y}(t,\,t)+D_{\delta x}^{\Delta m_y}, \tag{7.238}$$

so that using (7.228), (7.231), (7.236), (7.233) we obtain

$$\left.\begin{aligned}D_{\delta x_2}^{\Delta m_y}=r^2D_{\theta_y^0}+2r^2D_{\Delta m_y}\Big\{&\frac{t}{\beta}-\frac{1}{\beta^2}-\frac{2k_1}{\beta(1+k_2)\omega_0^2}+\\&-\frac{k_1+\beta}{\beta\left[\omega_0^2(1+k_2)+\beta^2-k_1\beta\right]}+\frac{\beta+\dfrac{k_1^2(k_1+\beta)}{\omega_0^2(1+k_2)}}{k_1\left[\omega_0^2(1+k_2)+\beta(k_1+\beta)\right]}.\end{aligned}\right\} \tag{7.239}$$

If

$$\alpha,\ \beta,\ \gamma\gg k_1,\ \omega_0(1+k_2)\quad\text{and}\quad k_2\gg 1, \tag{7.240}$$

Eqs. (7.226), (7.228), (7.229) can be greatly simplified. The expressions for the dispersions $D_{\theta_{1y}}$, $D_{\delta x_2}$ are also simplified. We obtain the approximate expressions

$$\left.\begin{aligned}D_{\theta_{1y}}&=\frac{D_{\Delta n_x}}{r^2}\frac{k_2}{\omega_0^2k_1\alpha}+\frac{D_{\Delta V_{Dx}}}{r^2}\frac{k_2^2+\dfrac{k_1^2}{k_2\omega_0^2}}{k_1\gamma}+\\&\qquad+D_{\Delta m_y}\frac{1+\dfrac{k_1^2}{k_2\omega_0^2}}{k_1\beta},\\D_{\delta x_2}&=D_{\Delta n_x}\frac{k_2}{\omega_0^2k_1\alpha}+D_{\Delta V_{Dx}}\frac{k_2^2+\dfrac{k_1^2}{\omega_0^2k_2}}{k_1\gamma}+\\&\quad+r^2D_{\theta_y^0}+2r^2D_{\Delta m_y}\left(\frac{t}{\beta}+\frac{\omega_0^2k_2-k_1^2}{\omega_0^2k_1k_2\beta}\right).\end{aligned}\right\} \tag{7.240a}$$

The second equation above shows that the dispersion $D_{\delta x_2}$ is eventually determined almost entirely by the time-proportional term which depends on the gyro errors. For large t, we have the approximation

$$D_{\delta x_2} = 2r^2 D_{\Delta m_y} \frac{t}{\beta}. \tag{7.241}$$

We have found the dispersions $D_{\theta_{1y}}$ and $D_{\delta x_2}$ for the error equations (7.215). Let us now find the corresponding dispersions for the case of Eqs. (7.216). Only the first equation is different in the two systems. Therefore, the difference in the dispersions $D_{\theta_{1y}}$ and $D_{\delta x_2}$ in these two cases is determined by differences in $D_{\delta x}$ and $K_{\delta x,\,\theta_y}(t,\,t)$.

By the first equation in (2.216), we may write for the steady-state dispersion $D_{\delta x}$

$$D_{\delta x} = D_{\delta x}^{\Delta n_x} + D_{\delta x}^{\Delta m_y} + D_{\delta x}^{\Delta V_{Dx}}, \tag{7.242}$$

and the terms of this sum are given by

$$\left.\begin{aligned}
D_{\delta x}^{\Delta n_x} &= \frac{\alpha D_{\Delta n_x}}{\pi} \int_{-\infty}^{\infty} \frac{\left|1 + k_2 + \frac{k_1}{\omega_0^2} j\omega\right|^2 d\omega}{(\alpha^2+\omega^2)\left|(j\omega)^2 + k_1 j\omega + (1+k_2)\omega_0^2\right|^2}, \\
D_{\delta x}^{\Delta m_y} &= \frac{r^2\beta D_{\Delta m_y}}{\pi} \int_{-\infty}^{\infty} \frac{|1 - j\omega|^2\, d\omega}{(\beta^2+\omega^2)\left|(j\omega)^2 + k_1 j\omega + (1+k_2)\omega_0^2\right|^2}, \\
D_{\delta x}^{\Delta V_{Dx}} &= \frac{\gamma D_{\Delta V_{Dx}}}{\pi} \int_{-\infty}^{\infty} \frac{\left|-k_2 j\omega - \frac{k_1}{\omega_0^2}(j\omega)^2\right|^2 d\omega}{(\gamma^2+\omega^2)\left|(j\omega)^2 + k_1 j\omega + (1+k_2)\omega_0^2\right|^2}.
\end{aligned}\right\} \tag{7.243}$$

The integrals in (7.243) are evaluated as integrals (7.221) above. The results give

$$\left.\begin{aligned}
D_{\delta x}^{\Delta n_x} &= D_{\Delta n_x} \frac{(k_1+\alpha)(1+k_2)\omega_0^2 + k_1^2\alpha}{\omega_0^4 k_1\left[\omega_0^2(1+k_2) + \alpha(k_1+\alpha)\right]}, \\
D_{\delta x}^{\Delta m_y} &= D_{\Delta m_y} \frac{\beta r^2}{k_1\left[\omega_0^2(1+k_2) + \beta(k_1+\beta)\right]}, \\
D_{\delta x}^{\Delta V_{Dx}} &= D_{\Delta V_{Dx}} \frac{\gamma\left\{k_1^2\left[\omega_0^2(1+k_2) + k_1\gamma\right] + k_2^2\omega_0^4\right\}}{\omega_0^4 k_1\left[\omega_0^2(1+k_2) + \gamma(k_1+\gamma)\right]}.
\end{aligned}\right\} \tag{7.244}$$

Eqs. (7.244), (7.242) and the equality

$$D_{\theta_{1y}} = D_{\delta x}/r^2$$

give the steady-state dispersion $D_{\theta_{1y}}$. If inequalities (7.240) apply, we obtain the following approximation for the steady-state dispersion $D_{\theta_{1y}}$:

$$D_{\theta_{1y}} = D_{\Delta n_x} \frac{k_2\omega_0^2 + k_1^2}{r^2\omega_0^4 k_1\alpha} + D_{\Delta m_y} \frac{1}{\beta k_1} + D_{\Delta V_{Dx}} \frac{k_1^2}{r^2\omega_0^4}. \tag{7.245}$$

The dispersion $D_{\delta x_2}$ also can be calculated from Eq. (7.230). This equation includes $D_{\delta x_1}$, D_{θ_y} and $K_{\delta x_1, \theta_y}(t, t)$. $D_{\delta x_1}$ has been found before, the function D_{θ_y} is the same as for an autonomous inertial system, i.e., it is given by (7.72). Since we are interested in the asymptotic value of $D_{\delta x_2}$, Eq. (7.72) can be replaced by its asymptotic form (7.231). The function $K_{\delta x_1, \theta_y}(t, t)$, however, should be calculated anew, since (7.236) obtained for Eqs. (7.215) does not apply to Eqs. (7.216).

From the first and fourth equations in (7.216) we have

$$K_{\delta x_1, \theta_y}(t, t) = -\frac{r}{\nu}\int_0^t\int_0^t K_{\Delta m_y, \Delta m_y}(\tau, \tau')\, e^{-\varepsilon(t-\tau)}\sin\nu(t-\tau)\,d\tau\,d\tau'. \tag{7.246}$$

where ε and ν are given by (7.233). Using (7.99) for the cross-correlation function $K_{\Delta m_y, \Delta m_y}$, we further have

$$K_{\delta x_1, \theta_y}(t, t) = \frac{r\beta}{\nu} D_{\Delta m_y}\int_0^t\int_0^t \operatorname{sign}(\tau-\tau')\, e^{-\beta|\tau-\tau'|-\varepsilon(t-\tau)}\sin\nu(t-\tau)\,d\tau\,d\tau'. \tag{7.247}$$

Taking the integral, we obtain

$$\begin{aligned} K_{\delta x_1, \theta_y}(t, t) = \frac{r}{\beta} D_{\Delta m_y}\Big\{ &-\frac{\beta}{(\varepsilon+\beta)^2+\nu^2} + \frac{\beta}{(\varepsilon-\beta)^2+\nu^2}\, e^{-\beta t} + \\ &+ \frac{\beta}{\nu\left[(\varepsilon+\beta)^2+\nu^2\right]}\left[-(\varepsilon+\beta)\sin\nu t + \nu\cos\nu t\right] e^{-(\varepsilon+\beta)t} - \\ &- \frac{\beta}{(\varepsilon-\beta)^2+\nu^2}\left(\frac{\varepsilon-\beta}{\nu}\sin\nu t + \cos\nu t\right) e^{-\varepsilon t}\Big\}. \end{aligned} \tag{7.248}$$

In the steady state,

$$K_{\delta x, \theta_y} = -\frac{r D_{\Delta m_y}}{(\varepsilon+\beta)^2+\nu^2}. \tag{7.249}$$

To complete the calculation of the dispersion $D_{\delta x_2}$, Eqs. (7.242), (7.231), (7.249), (7.244) should be inserted in (7.230). Using inequalities (7.240), we find approximately

$$\begin{aligned} D_{\delta x_2} = D_{\Delta n_x}\frac{k_2\omega_0^2+k_1^2}{\omega_0^4 k_1 a} + D_{\Delta V_{Dx}}\frac{k_1^2(k_2\omega_0^2+k_1\gamma)+k_2\omega_0^4}{\omega_0^4 k_1\gamma} + \\ + D_{\Delta m_y}\frac{r^2}{\beta}\left(\frac{1}{k_1}+2t\right) + r^2 D_{\theta_y^0}. \end{aligned} \tag{7.250}$$

For large t, when the dispersion $D_{\delta x_2}$ is determined by the terms linear in time, Eq. (7.250) reduces to the approximate relation (7.241).

Let us now consider Eqs. (7.217). It is readily seen that only the fourth and the last equation are different from those in Eqs. (7.215). The first equations are identical in both systems. The dispersion $D_{\delta x_1}$, and thus also D_{ϑ_y}, therefore have the same value as for the case of Eqs. (7.215). They are given by Eqs. (7.219), (7.226), (7.228), (7.229), (7.229a). In the case of Eqs. (7.217) the problem is thus to find the dispersion $D_{\delta x_2}$.

By the last equality in (7.217)

$$D_{\delta x_2} = \frac{1}{(1+k_2)^2} D_{\delta x} + \frac{r^2}{(1+k_2)^2} D_{\vartheta_y} + \frac{2r}{1+k_2} K_{\delta x, \theta_y}(t, t). \tag{7.251}$$

We have already noted that the dispersion D_{Δ_T} is the same as for Eqs. (7.215). The dispersion D_{ϑ_y} is also easily found. Indeed, comparing the fifth equation in (7.215) and (7.217), using Eqs. (7.72), (7.218), and seeing that the instrumental errors ΔV_{Dx} and Δm_y are uncorrelated we find

$$D_{\vartheta_y} = D_{\vartheta_y^0} + 2D_{\Delta m_y}\left[\frac{t}{\beta} + \frac{1}{\beta^2}(e^{-\beta t} - 1)\right] + \\ + D_{\Delta V_{Dx}} \frac{2k_2^2}{r^2}\left[\frac{t}{\gamma} + \frac{1}{\gamma^2}(e^{-\gamma t} - 1)\right]. \tag{7.252}$$

For large t, $e^{-\beta t}$ and $e^{-\gamma t}$ are ignorable and we have asymptotically

$$D_{\vartheta_y} = D_{\vartheta_y^0} + \frac{2}{\beta} D_{\Delta m_y}\left(t - \frac{1}{\beta}\right) + \frac{2k_2^2}{r^2\gamma} D_{\Delta V_{Dx}}\left(t - \frac{1}{\gamma}\right). \tag{7.253}$$

It now remains to find the function $K_{\delta x, \vartheta_y}(t, t)$. From the first and the fourth equation in (7.217) we get

$$K_{\delta x, \vartheta_y}(t, t) = -\frac{r}{\nu}\int_0^t\int_0^t \left[k_1 K_{\Delta m_y}(\tau, \tau') + \right. \\ + K_{\Delta \dot{m}_y, \Delta m_y}(\tau, \tau')\left] e^{-\varepsilon(t-\tau)} \sin\nu(t-\tau)\, d\tau\, d\tau' + \right. \\ + \frac{k_2}{r\nu}\int_0^t\int_0^t \left[k_1 K_{\Delta V_{Dx}}(\tau, \tau') - \right. \\ \left. - k_2 K_{\Delta \dot{V}_{Dx}, \Delta V_{Dx}}(\tau, \tau')\right] e^{-\varepsilon(t-\tau)} \sin\nu(t-\tau)\, d\tau\, d\tau'. \tag{7.254}$$

The first integral on the right has been calculated before. It is given by the expression on the right in (7.235). The second integral is similarly evaluated and we find

$$q = \frac{k_2}{r\nu}\int_0^t\int_0^t \left[k_1 K_{\Delta V_{Dx}}(\tau, \tau') - \right. \\ \left. - k_2 K_{\Delta \dot{V}_{Dx}, \Delta V_{Dx}}(\tau, \tau')\right] e^{-\varepsilon(t-\tau)} \sin\nu(t-\tau)\, d\tau\, d\tau' = \\ = \frac{k_2}{r\gamma} D_{\Delta V_{Dx}}\left\{\frac{2k_1}{\varepsilon^2+\nu^2} - \frac{k_1 - k_2\gamma}{(\varepsilon+\gamma)^2+\nu^2} - \right. \\ - \frac{k_1 + k_2\gamma}{(\varepsilon-\gamma)^2+\nu^2} e^{-\gamma t} + \frac{k_1 - k_2\gamma}{\nu[(\varepsilon+\gamma)^2+\nu^2]} \times \\ \times [-(\varepsilon+\gamma)\sin\nu t + \nu\cos\nu t]\, e^{-(\varepsilon+\gamma)t} + \\ + \left[\left(-\frac{2k_1\varepsilon}{\nu(\varepsilon^2+\nu^2)} + \frac{(k_1+k_2\gamma)(\varepsilon-\gamma)}{\nu[(\varepsilon-\gamma)^2+\nu^2]}\right)\sin\nu t + \right. \\ \left. + \left(-\frac{2k_1}{\varepsilon^2+\nu^2} + \frac{k_1+k_2\gamma}{(\varepsilon-\gamma)^2+\nu^2}\right)\cos\nu t\right] e^{-\varepsilon t}\Big\}. \tag{7.255}$$

For large t, we have asymptotically

$$q = \frac{k_2}{r\gamma}\left[\frac{2k_1}{\varepsilon^2+\nu^2} - \frac{k_1 - k_2\gamma}{(\varepsilon-\gamma)^2+\nu^2}\right]. \tag{7.256}$$

To find $D_{\delta x_2}$ in the case of Eqs. (7.217), we should insert for $D_{\delta x}$ in Eq. (7.251) its expression from (7.219), (7.226), (7.228), (7.229), for D_{ϑ_y} its expression from (7.253), and for $K_{\delta x, \vartheta_y}(t, t)$ its expression from (7.254),

(7.235), (7.255). If the inequalities (7.240) are applicable, all these manipulations lead to the approximate formula

$$D_{\delta x_2} = \frac{D_{\Delta n_x}}{\omega_0^2 k_1 k_2 a} + \frac{D_{\Delta V_{Dx}}}{\gamma}\left[\frac{\omega_0^2 k_2^3 + k_1^2}{\omega_0^2 k_1 k_2} + 2\left(\frac{2k_1}{\omega_0^2 k_2} - \frac{k_2}{\gamma}\right) + \right.$$
$$\left. + 2\left(t + \frac{1}{\gamma}\right)\right] + \frac{r^2}{k_2^2} D_{\theta_y^0} + \frac{2r^2 D_{\Delta m_y}}{\beta k_2} \times$$
$$\times \left[\frac{1}{k_2}\left(t + \frac{\omega_0^2 k_2 - k_1^2}{\omega_0^2 k_1 k_2 \beta}\right) + \frac{2k_1}{\omega_0^2 k_2} - \frac{1}{\beta}\right]. \qquad (7.257)$$

For sufficiently large t, the dispersion $D_{\delta x_2}$ is clearly determined by the terms linear in time. We thus have approximately

$$D_{\delta x_2} = 2\left(\frac{D_{\Delta V_{Dx}}}{\gamma} + \frac{r^2 D m_y}{\beta k_2^2}\right) t. \qquad (7.258)$$

Comparison of Eq. (7.258) with Eq. (7.249) shows that the correction routine corresponding to error equations (7.217) should be preferred to correction routine corresponding to error equations (7.215), (7.216) when

$$D_{\Delta V_{Dx}} \ll r^2 D_{\Delta m_y}. \qquad (7.259)$$

Otherwise, correction routines corresponding to Eqs. (7.215), (7.216) are more effective.

Let us compare the dispersions (7.240a) obtained from Eqs. (7.215) with the dispersions (7.245), (7.250) obtained from (7.216). Comparison of the first equation in (7.240a) with (7.245) shows that the dispersions of orientation errors due to random accelerometer and Doppler errors are always greater in a system corresponding to Eqs. (7.216). The dispersion of orientation errors due to random gyro errors is conversely greater in systems corresponding to Eqs. (7.215). The choice between the two systems should thus be made considering the relative values of the random errors.

When choosing the correction routine, we should consider the expectation values of the random errors, as well as their dispersions. Thus if the expectation value of a Doppler error is considerable, a system functioning according to Eqs. (7.216) probably should be preferred to system (7.215) with smaller dispersions $D_{\delta x_2}^{\Delta V_{Dx}}$, $D_{\theta_{iy}}^{\Delta V_{Dx}}$, since the right-hand sides of (7.216) contain only the derivatives of ΔV_{Dx}.

Comparison of the second equation in (7.240a) with (7.250) shows that the effect of accelerometer and Doppler errors on the dispersion $D_{\delta x_2}$ of the position errors is the same as on the dispersion of orientation errors. With regard to gyro errors, the situation is somewhat different. The time-proportional terms are the same in the case of Eqs. (7.215) and (7.216). Thus, if the gyro errors are fairly large, and the system functions for a long time, the gyro errors (drifts) predominate and the systems corresponding to Eqs. (7.215) and (7.216) are virtually equivalent with regard to position error dispersions.

Note that the above comparison is based on the approximate expressions which are derived assuming inequalities (7.240). If these inequalities do not apply, we should compare the exact expressions for the dispersions. Another

point to remember is that Doppler correction is intended to ensure sufficiently effective damping, and not only to reduce the steady-state errors.

7.3.3. Simultaneous altimeter and astro-Doppler correction

In addition to altimeter and Doppler correction, we now introduce astrocorrection and consider the combined effect of the different random errors.

We will only deal with the case of linear astrocorrection. When astrocorrection is introduced, only the left-hand sides of the fourth and the fifth equation in (7.215) are changed, and also the right-hand side of the first equation (see (6.66)). Since the equations for θ_y and θ_z are identical, it suffices to consider only the following set of equations:

$$\left.\begin{aligned} \delta\ddot{x} + k_1\delta\dot{x} + (1+k_2)\omega_0^2\delta x = \\ = (1+k_2)\Delta n_x - r(\Delta\dot{m}_y + k\dot{\theta}_y) + \\ + k_1[\Delta V_{Dx} - r(\Delta m_y + k\theta_y)] - k_2\Delta\dot{V}_{Dx}, \\ \theta_{1y} = \frac{\delta x}{r}, \\ \dot{\theta}_y + k\theta_y = \Delta m_y, \\ \delta x_2 = \delta x + r\theta_y. \end{aligned}\right\} \tag{7.260}$$

Eqs. (7.216) are thus replaced by the following equations:

$$\left.\begin{aligned} \delta\ddot{x} + k_1\delta\dot{x} + (1+k_2)\omega_0^2\delta x = \\ = (1+k_2)\Delta n_x + \frac{k_1}{\omega_0^2}\Delta\dot{n}_x - r(\Delta\dot{m}_y - k\dot{\theta}_y) - k_2\Delta\dot{V}_{Dx} - \frac{k_1}{\omega_0^2}\Delta\ddot{V}_{Dx}, \\ \theta_{1y} = \frac{\delta x}{r}, \\ \dot{\theta}_y + k\theta_y = \Delta m_y, \\ \delta x_2 = \delta x + r\theta_y. \end{aligned}\right\} \tag{7.261}$$

Finally, Eqs. (7.217) are replaced by (see Eqs. (6.166)—(6.168))

$$\left.\begin{aligned} \delta\ddot{x} + k_1\delta\dot{x} + (1+k_2)\omega_0^2\delta x - rk\dot{\theta}_y - rk_1k\theta_y = \\ = (1+k_2)\Delta n_x - r\Delta\dot{m}_y + \\ + k_1(\Delta V_{Dx} - r\Delta m_y) - k_2\Delta\dot{V}_{Dx}, \\ (1+k_2)\dot{\theta}_y + k\theta_y + k_2\frac{\delta\dot{x}}{r} = \Delta m_y + \frac{k_2}{r}\Delta V_{Dx}, \\ \dot{\theta}_z + k\theta_z = \Delta m_z, \\ \theta_{1y} = \frac{\delta x}{r}, \quad \theta_{1z} = -\theta_z, \\ \delta x_2 = \delta x + r\theta_y. \end{aligned}\right\} \tag{7.262}$$

For $k = 0$, substitution of variables (6.169), reduces Eqs. (7.262) to Eqs. (7.217).

Consider Eqs. (7.260). Let us calculate the steady-state dispersions D_θ, $D_{\delta x}$, D_{θ_1}, $D_{\delta x_2}$ of the errors θ, δx, θ_1, δx_2. The spectral densities of the instrumental errors Δn_x, Δm_y, ΔV_{Dx} are taken in the form (7.220).

From the third equation in (7.260) and the second equality in (7.218) we have

$$D_{\dot\theta_y} = \frac{\beta D_{\Delta m_y}}{\pi} \int_{-\infty}^{\infty} \frac{d\omega}{(\beta^2+\omega^2)\,|j\omega+k|^2}. \tag{7.263}$$

To evaluate the integral on the right, we use representation (7.222). Here $n=2$, and $a_0=1$, $a_1=k-\beta$, $a_2=k\beta$, $b_0=0$, $b_1=1$. By the second equation in (7.224), we have

$$D_{\dot\theta_y} = \frac{D_{\Delta m_y}}{k(k+\beta)}. \tag{7.264}$$

As before,

$$D_{\Delta_{1y}} = \frac{1}{r^2} D_{\delta x}. \tag{7.265}$$

The dispersion $D_{\delta x}$ can be written as the sum

$$D_{\delta x} = D_{\delta x}^{\Delta n_x} + D_{\delta x}^{\Delta V_{Dx}} + D_{\delta x}^{\Delta m_y}, \tag{7.266}$$

where on the right we have the sum of dispersions of the independent random errors Δn_x, ΔV_{Dx}, Δm_y which enter the right-hand sides of Eqs. (7.260). Clearly the dispersions $D_{\delta x}^{\Delta n_x}$ and $D_{\delta x}^{\Delta V_{Dx}}$ have the same values as the $D_{\delta x}^{\Delta n_x}$ and $D_{\delta x}^{\Delta V_{Dx}}$ obtained for Eqs. (7.215). The latter dispersions are given by Eqs. (7.226) and (7.229).

To find the term $D_{\delta x}^{\Delta m_y}$ note that by putting in the right-hand side of the first equation in (7.260)

$$\Delta n_x = 0, \qquad \Delta V_{Dx} = 0,$$

and writing $\dot\theta_y$ for the difference $\Delta m_y - k\theta_y$ by the third equation, we reduce the first equation in (7.260) to the form

$$\delta\ddot{x} + k_1\delta\dot{x} + (1+k_2)\omega_0^2\delta x = -r(\ddot\theta_y + k_1\dot\theta_y). \tag{7.267}$$

The transfer function

$$\Phi(j\omega) = \frac{\delta x}{\Delta m_y}$$

therefore can be written in the form

$$\Phi(j\omega) = -\frac{rj\omega(j\omega+k_1)}{[(j\omega)^2 + k_1 j\omega + (1+k_2)\omega_0^2](j\omega+k)}. \tag{7.268}$$

We made use of the equality

$$\frac{\theta_y}{\Delta m_y} = \frac{1}{j\omega+k}.$$

Now, seeing that

$$S_{\Delta m_y}(\omega) = \frac{D_{\Delta m_y}}{\pi}\frac{\beta}{\beta^2+\omega^2}, \tag{7.269}$$

we obtain by (7.48)

$$D_{\delta x}^{\Delta m_y} = \frac{r^2 \beta D_{\Delta m_y}}{\pi} \int_{-\infty}^{\infty} \left| \frac{-(j\omega)^2 - k_1 j\omega}{\left[(j\omega)^2 + k_1 j\omega + \omega_0^2 (1+k_2)\right](j\omega + k)} \right|^2 \frac{d\omega}{\beta^2 + \omega^2}. \tag{7.270}$$

The integral is taken using the representation (7.222). In this case $n = 4$ and

$$\left.\begin{aligned} a_0 &= 1, \\ a_1 &= k + \beta + k_1, \\ a_2 &= \omega_0^2 (1 + k_2) + k\beta + k_1 (k + \beta), \\ a_3 &= k_1 k\beta + \omega_0^2 (1 + k_2)(k + \beta), \\ a_4 &= k\beta\omega_0^2 (1 + k_2), \\ b_0 &= 0, \\ b_1 &= 1, \\ b_2 &= -k_1^2, \\ b_3 &= 0. \end{aligned}\right\} \tag{7.271}$$

Inserting these values of the coefficients a_i, b_i in the fourth equation in (7.224) we obtain after obvious simplifications

$$\begin{aligned} D_{\delta x}^{\Delta m_y} = \frac{r^2 \beta D_{\Delta m_y}}{(k+\beta)} &\left[\omega_0^2 (1+k_2)(k+\beta) + k\beta + k_1 (k_1 + k + \beta)\right] \times \\ &\times \left\{k\beta + \left[k\beta + k_1 (k + \beta + k_1)\right] + \omega_0^4 (1+k_2)^2 + \right. \\ &\left. + \omega_0^2 (1+k_2)\left[k_1 (\beta + k) + k^2 + \beta^2\right]\right\}^{-1}. \end{aligned} \tag{7.272}$$

Eqs. (7.266), (7.226), (7.229), (7.272) give the steady-state dispersion $D_{\delta x_1}$ for any values of the correction parameters. If the inequalities (7.240) are applicable and since usually

$$k \gg k_1, \quad \omega_0 \sqrt{1 + k_2}, \tag{7.273}$$

we obtain the following approximate expression for the dispersion $D_{\delta x}$:

$$D_{\delta x} = D_{\Delta n_x} \frac{k_2}{\omega_0^2 a k_1} + D_{\Delta m_y} \frac{r^2}{k(k+\beta)} + D_{\Delta V_{Dx}} \frac{k_1^2 + k_2^2 \omega_0^2}{\omega_0^2 k_1 \gamma k_2}. \tag{7.274}$$

Let us now calculate the dispersion $D_{\delta x_2}$. As in (7.266), we write

$$D_{\delta x_2} = D_{\delta x_2}^{\Delta n_x} + D_{\delta x_2}^{\Delta V_{Dx}} + D_{\delta x_2}^{\Delta m_y}. \tag{7.275}$$

Here the first two terms on the right are respectively equal to $D_{\delta x}^{\Delta n_x}$ and $D_{\delta x}^{\Delta V_{Dx}}$ in the sum (7.266), which are given by (7.226) and (7.229):

$$D_{\delta x_2}^{\Delta n_x} = D_{\delta x}^{\Delta n_x},$$

$$D_{\delta x_2}^{\Delta V_{Dx}} = D_{\delta x}^{\Delta V_{Dx}}.$$

To find the third term in (7.275), we use the equality

$$D_{\delta x_2}^{\Delta m_y} = \frac{r^2 \beta D_{\Delta m_y}}{\pi} \int_{-\infty}^{\infty} \frac{\omega_0^4 (1+k_2)^2 \, d\omega}{\left|\left[(j\omega)^2 + k_1 j\omega + (1+k_2)\omega_0^2\right](j\omega + k)\right|^2 (\beta^2 + \omega^2)}. \tag{7.276}$$

which is obtained along the same lines as Eq. (7.270).

The integral in (7.276) is calculated using the fourth relation in (7.224). The coefficients a_i are given by the first four equalities in (7.271) and the coefficients b_i are respectively

$$b_0 = 0, \quad b_1 = 0, \quad b_2 = 0, \quad b_3 = \omega_0^2(1 + k_2). \tag{7.277}$$

We thus find

$$D_{\delta x}^{\Delta m_y} = \frac{r^2\omega_0^2(1+k_2)D_{\Delta m}}{k_1 k(k+\beta)} \times$$
$$\times \{k_2\omega_0^2(1+k_2) + (k+\beta)[k\beta - k_1(k_1+k+\beta)]\} \times$$
$$\times \{\omega_0^2(1+k_2)^2 + \omega_0^2(1-k_2)[k^2+\beta^2+k_1(k+\beta)] +$$
$$+ k\beta[k\beta + k_1(k_1+k+\beta)]\}^{-1}. \tag{7.278}$$

Eqs. (7.272), (7.226), (7.229), (7.276) determine the dispersion $D_{\delta x}$ for arbitrary correction coefficients. If the coefficients are restrained in the same way as in the derivation of (7.271), we have the approximation

$$D_{\delta x} = D_{\Delta n_y}\frac{k_2}{\omega_0^2 k_1} + D_{\Delta m}\frac{r^2\omega_0^2 k_2}{k_1 k^2\beta} + D_{\Delta V_{Dx}}\frac{k_1^2 + \omega_0^2 k_2}{\omega_0^2 k_1 k_2 V}. \tag{7.279}$$

Consider Eqs. (7.261). The third equation in (7.261) coincides with the third equation in (7.260). The dispersion D_{θ_y} is therefore expressed as before by (7.264). It thus remains to calculate the dispersions $D_{\delta x}$, $D_{\theta_{1y}}$, and $D_{\delta x_1}$.

The dispersion $D_{\delta x}$ is written as the sum

$$D_{\delta x} = D_{\delta x}^{\Delta n_x} + D_{\delta x}^{\Delta V_{Dx}} + D_{\delta x}^{\Delta m_y}. \tag{7.280}$$

As before,

$$D_{\theta_{1y}} = \frac{1}{r^2} D_{\delta x}.$$

Comparison of (7.261) and (7.216) shows that $D_{\delta x}^{\Delta n_x}$ and $D_{\delta x}^{\Delta V_{Dx}}$ are given by the first and the third equation in (7.244). For $D_{\delta x}^{\Delta m_y}$ we have

$$D_{\delta x}^{\Delta m_y} = \frac{r^2 D_{\Delta m_y}}{\pi}\int_{-\infty}^{\infty}\left|\frac{-j\omega}{[(j\omega)^2 + k_1 j\omega + \omega_0^2(1+k_2)](j\omega+k)}\right|^2 \frac{d\omega}{\beta^2+\omega^2}. \tag{7.281}$$

The integral on the right is written in the form (7.222). For the coefficients a_i of the polynomial $h_4(x)$ we have the same values as in (7.271); the coefficients b_i are respectively

$$b_0 = 0, \quad b_1 = 1, \quad b_2 = 0, \quad b_3 = 0. \tag{7.282}$$

Inserting these coefficients a_i and b_i in the fourth equation in (7.224), we obtain after some simplifications

$$D_{\delta x_2}^{\Delta m_y} = \frac{r^2\beta D_{\Delta m_y}}{k_1(k+\beta)}\left[\omega_0^2(1+k_2)(k+\beta)+k_1k\beta\right]\times$$
$$\times\left\{\omega_0^4(1+k_2)^2+\omega_0^2(1+k_2)\left[k_1(k+\beta)+k^2+\beta^2\right]+\right.$$
$$\left.+k\beta\left[k\beta+k_1(k_1+k+\beta)\right]\right\}^{-1}. \qquad (7.283)$$

If the astrocorrection coefficient is finite but fairly large (i.e. it satisfies (7.273)) and inequalities (7.240) are applicable, we obtain the following approximation from (7.283):

$$D_{\delta x}^{\Delta m_y} = \frac{r^2 D_{\Delta m_y}}{k(k+\beta)}. \qquad (7.284)$$

From (7.280), the first and the third equations in (7.244) using (7.240), and from (7.284) we now have the approximation

$$D_{\delta x} = D_{\Delta n_x}\frac{k_2\omega_0^2+k_1^2}{\omega_0^4 k_1 a} + D_{\Delta m_y}\frac{r^2}{k(k+\beta)} + D_{\Delta V_{Dx}}\frac{k_1^2}{\omega_0^4}. \qquad (7.285)$$

Comparison of (7.274) with (7.285) shows that given equal numerical values of the parameters, the coefficients before $D_{\Delta n_x}$ and $D_{\Delta V_{Dx}}$ are larger in Eq. (7.285) and the coefficients before $D_{\Delta m_y}$ are the same in both cases. The approximate dispersion of the accelerometer and Doppler errors δx for a system with error equations (7.261) is thus larger than for error equations (7.260), whereas the dispersion of the gyro errors is the same for error equations (7.261) and (7.262).

Let us calculate the dispersion $D_{\delta x_2}$ of the total position error δx_2 corresponding to Eqs. (7.261). We have

$$D_{\delta x_2} = D_{\delta x_1}^{\Delta n_x} + D_{\delta x_2}^{\Delta V_{Dx}} + D_{\delta x_2}^{\Delta m_y}. \qquad (7.286)$$

Here

$$D_{\delta x_2}^{\Delta n_x} = D_{\delta x}^{\Delta n_x},$$
$$D_{\delta x_2}^{\Delta V_{Dx}} = D_{\delta x}^{\Delta V_{Dx}}, \qquad (7.287)$$

where $D_{\delta x}^{\Delta n_x}$ and $D_{\delta x}^{\Delta V_{Dx}}$ are given by the first and the third equalities in (7.244) and

$$D_{\delta x_2}^{\Delta m_y} = \frac{r^2\beta D_{\Delta m_y}}{\pi}\int_{-\infty}^{\infty}\frac{\left|k_1 j\omega+\omega_0^2(1+k_2)\right|^2 d\omega}{\left|\left[(j\omega)^2+k_1 j\omega+\omega_0^2(1+k_2)\right](j\omega+k)\right|^2(\beta^2+\omega^2)}. \qquad (7.288)$$

The integral is again evaluated using the fourth equation in (7.224). As before, the coefficients a_i are given by the first four equalities in (7.271) and the coefficients b_i are

$$b_0 = 0, \quad b_1 = 0, \quad b_2 = -k_1^2,$$
$$b_3 = \omega_0^4(1+k_2)^2. \qquad (7.289)$$

Thus,

$$D_{\delta x_2}^{\Delta m_y} = \frac{r^2 D_{\Delta m_y}}{k_1 k (k+\beta)} \times$$
$$\times \{k_1 \omega_0^4 (1+k_2)^2 + \omega_0^2 (1+k_2)(k+\beta) \times$$
$$\times [k\beta + k_1 (k_1 + k + \beta)] + k_1^2 k\beta (k_1 + k + \beta)\} \times$$
$$\times \{\omega_0^4 (1+k_2)^2 + \omega_0^2 (1+k_2)[k^2 + \beta^2 + k_1 (k+\beta)] +$$
$$+ k\beta [k\beta + k_1 (k_1 + k + \beta)]\}^{-1}. \qquad (7.290)$$

Eqs. (7.286), (7.290) together with the first and the third equation in (7.244) give the dispersion $D_{\delta x_2}$ for error equations (7.262) with arbitrary correction coefficients k_1, k_2, k. Simplifications as in (7.279) lead to the approximate relation

$$D_{\delta x_2} = D_{\Delta n_x} \frac{k_2 \omega_0^2 + k_1^2}{\omega_0^4 k_1 \alpha} + D_{\Delta m_y} \frac{r^2 (\omega_0^2 k_2 + k_1^2)}{k_1 k^2 \beta} + D_{\Delta V_{Dx}} \frac{k_1}{\omega_0^2}. \qquad (7.291)$$

Comparison of (7.291) with (7.279) shows that accelerometer and gyro errors may give larger dispersions of δx_2 in the case of Eqs. (7.261), whereas Doppler errors conversely may lead to larger dispersions in the case of Eqs. (7.260).

It now remains to analyze the error equations (7.262).

Note that the third equation in (7.262) is analogous to the third equation in (7.260) and in (7.261), and the previous results are therefore applicable.

We thus only have to consider the first, the second, the fourth, and the sixth equation in (7.262). They should give the dispersion $D_{\theta_{1y}}$ of the orientation errors θ_{1y} and the dispersion $D_{\delta x_2}$ of the position errors δx_2. The coefficients k_1, k_2, k can always be chosen so that the characteristic equation of (7.262) has roots with negative real parts, so that $D_{\theta_{1y}}$ and $D_{\delta x_2}$ will be calculated using (7.48).

First we consider the dispersion $D_{\theta_{1y}}$, which can be written as the sum

$$D_{\theta_{1y}} = D_{\theta_{1y}}^{\Delta n_x} + D_{\theta_{1y}}^{\Delta V_{Dx}} + D_{\theta_{1y}}^{\Delta m_y}. \qquad (7.292)$$

The terms of this sum are determined in accordance with the first, second, and fourth equations in (7.262):

$$\left.\begin{aligned} D_{\theta_{1y}}^{\Delta n_x} &= \frac{\alpha D_{\Delta n_x}}{\pi r^2} \int_{-\infty}^{\infty} \left| \frac{(1+k_2) j\omega + k}{\Delta (j\omega)} \right|^2 \frac{d\omega}{\alpha^2 + \omega^2}, \\ D_{\theta_{1y}}^{\Delta V_{Dx}} &= \frac{\gamma D_{\Delta V_{Dx}}}{\pi r^2} \int_{-\infty}^{\infty} \left| \frac{-k_2 (j\omega)^2 + k_1 j\omega + k_1 k}{\Delta (j\omega)} \right| \frac{d\omega}{\gamma^2 + \omega^2}, \\ D_{\theta_{1y}}^{\Delta m_y} &= \frac{r^2 D_{\Delta m_y}}{\pi} \int_{-\infty}^{\infty} \left| \frac{(j\omega)^2 + k_1 j\omega}{\Delta (j\omega)} \right|^2 \frac{d\omega}{\beta^2 + \omega^2}, \end{aligned}\right\} \qquad (7.293)$$

where

$$\Delta (j\omega) = (j\omega)^3 + (k_1 + k)(j\omega)^2 + [\omega_0^2 (1+k_2) + k_1 k]\, j\omega + k_1 \omega_0^2. \qquad (7.294)$$

The integrals in (7.293) are evaluated using the representation (7.222) and the last equality in (7.224). For the first of these integrals we thus have

$$D_{\theta_1 y}^{\Delta n_x} = \frac{\alpha D_{\Delta n_x}}{r^2} \frac{b_0(-a_1a_4 + a_2a_3) - a_0a_3b_1 + a_0a_1b_2 + \frac{a_0b_3}{a_4}(a_0a_3 - a_1a_2)}{a_0(a_0a_3^2 + a_1^2a_4 - a_1a_2a_3)}, \tag{7.295}$$

where

$$\left.\begin{aligned} &a_0 = 1, \quad a_1 = k_1 + k + \alpha, \\ &a_2 = \alpha(k_1 + k) + k_1k + \omega_0^2(1 + k_2), \\ &a_3 = \alpha k_1 k + \omega_0^2[\alpha(1 + k_2) + k], \\ &a_4 = \alpha k \omega_0^2, \\ &b_0 = 0, \\ &b_1 = 0, \\ &b_2 = -(1 + k_2)^2, \quad b_3 = k^2. \end{aligned}\right\} \tag{7.296}$$

To evaluate the second integral in (7.293), $\alpha D_{\Delta n_x}$ on the right in (7.295) should be replaced with $\gamma D_{\Delta V_{Dx}}$, and in the first four coefficients a_i in (7.296) α is replaced with γ. The coefficients b_i are respectively taken in the form

$$\left.\begin{aligned} b_0 = 0, \quad b_1 = k_2^2, \quad b_2 = -k_1(k_1 + 2k), \\ b_3 = k_1^2 k^2. \end{aligned}\right\} \tag{7.297}$$

Finally, to evaluate the third integral in (7.279), we should insert $\beta D_{\Delta m_y}$ for $\frac{\alpha D_{\Delta n_x}}{r^2}$ on the right in (7.295) and replace α with β in the coefficients a_i. The coefficients b_i are given by the equalities

$$b_0 = 0, \quad b_1 = 1, \quad b_2 = -k_1^2, \quad b_3 = 0. \tag{7.298}$$

The dispersion $D_{\delta x_2}$ can be calculated along the same lines as $D_{\theta_1 y}$. It is written as the sum

$$D_{\delta x_2} = D_{\delta x_2}^{\Delta n_x} + D_{\delta x_2}^{\Delta V_{Dx}} + D_{\delta x_2}^{\Delta m_y}. \tag{7.299}$$

Each of the terms in this sum again can be calculated using (7.295). To obtain the first term, r^2 is omitted from the right-hand side of Eqs. (7.295) and the coefficients a_i are taken from (7.296). The coefficients b_i are given by

$$b_0 = 0, \quad b_1 = 0, \quad b_2 = -1, \quad b_3 = k^2. \tag{7.300}$$

In calculating the second term in (7.299), r^2 is again omitted from the right-hand side of Eq. (7.295) (it is taken equal to unity), and $\alpha D_{\Delta n_x}$ is replaced with $\gamma D_{\Delta V_{Dx}}$. In the coefficients a_i, α is similarly replaced with γ, whereas the coefficients b_i are given by

$$b_0 = 0, \quad b_1 = 0, \quad b_2 = -k_1^2, \quad b_3 = (\omega_0^2 k_2 + k_1 k_2)^2. \tag{7.301}$$

Finally, in evaluating the third term in (7.299), $\frac{\alpha D_{\Delta n_x}}{r^2}$ in (7.295) is replaced by $D_{\Delta m_y} r^2 \beta$, α in the coefficients a_i is replaced with β, and the coefficients b_i are taken from

$$b_0 = 0, \quad b_1 = 0, \quad b_2 = 0, \quad b_3 = \omega_0^4. \tag{7.302}$$

Let us compare the dispersions $D_{\theta_{1y}}$ and $D_{\delta x_1}$ obtained for the case (7.262) from (7.292), (7.299) with the corresponding dispersions for the case of Eqs. (7.260) and (7.261). More precisely, let us compare the dependence of these dispersions on the correction coefficients k_1, k_2, k. However, before we can make this comparison, we should agree on some criterion for the comparison of the errors. Indeed, in comparing the dispersions of the errors for Eqs. (7.260) and (7.261) the procedure was meaningful since the homogeneous equations corresponding to (7.260) and (7.261) were identical, and the comparison was therefore carried out for equal values of the correction coefficients. The homogeneous equations (7.262), on the other hand, are different from the homogeneous equations (7.260) and (7.261), and therefore we cannot confine the comparison to steady-state dispersions: the variation of the dynamic properties of the homogeneous system for different choice of the correction coefficients should also be considered.

Let the correction coefficients k_1, k_2, k for Eqs. (7.262) be designated k_1', k_2', k', to distinguish them from the corresponding coefficients of Eqs. (7.260), (7.261). The characteristic equation (6.171) of Eqs. (7.262) is written in the form

$$p^3 + (k_1' + k')\, p^2 + \left[\omega_0^2 (1 + k_2') + k_1' k'\right] p + \omega_0^2 k' = 0.$$

Let p_1', p_2', p_3' be the roots of this equation.

The characteristic equation of Eqs. (7.260) or (7.261) separates into two equations:

$$p^2 + k_1 p + \omega_0^2 (1 + k_2) = 0, \quad p + k = 0.$$

The roots of these equations are

$$p_{1,2} = -\frac{k_1}{2} \pm j \sqrt{\omega_0^2 (1 + k_2) - \frac{k_1^2}{4}},$$
$$p_3 = -k.$$

We now choose the correction coefficients k', k_1', k_2' for the case of error equations (7.262) so that the characteristic equation (7.289) has the roots

$$p_{1,2}' = p_{1,2}, \quad p_3' = -\frac{k_1}{2}.$$

This can be accomplished, say, by taking

$$k' = \frac{k_1}{2}(1 + k_2),$$
$$k_1' = k_1\left(1 - \frac{k_2}{2}\right),$$
$$k_2' = k_2\left[1 + \frac{k_1^2}{4\omega_0^2}(k_2 - 1)\right].$$

This choice of the coefficients k', k_1', k_2' ensures equal damping in (7.260)—(7.262), so that from the point of view of damping the systems are equivalent to a degree.* Comparison of dispersions is thus again meaningful.

§7.4. DYNAMICS OF A PHASE ASTROCORRECTION SYSTEM WITH RANDOM ERRORS

In §6.4 we have considered the operation of a phase astrocorrection system near the equilibrium. The problem was reduced to a system of linear equations (6.139) with variable coefficients which under conditions (6.142) were reduced by substitution of variables (6.141) to linear equations (6.145) with constant coefficients. The stability conditions of these equations were derived in the form of inequalities (6.148).

In (6.145) f_2 is the noise or error signal. Noises may be either determinate (e.g., the scattered solar radiation in the atmosphere) or random functions of time. Since the luminous flux of the stars hitting the photocell in astrocorrection systems is small if the diameter of the telescope objective is small, these systems have to function at a high random noise level, which markedly exceeds the effective signal level. The main noise component is contributed by the photocell. This is a point of considerable significance and must not be ignored when choosing the astrocorrection parameters.

Consider Eqs. (6.145) which describe the operation of an astrocorrection system, retaining only the noise function $f_2(t)$ on the right. We assume that this is a stationary centered random function of time which corresponds to a delta-correlated process (white noise). Let the correlation function be given by

$$K_{f_2}(\tau, \tau') = 2\pi s^0(\tau - \tau'), \tag{7.303}$$

where s^0 is the constant spectral density.

To simplify the further treatment, we take

$$\alpha = \frac{\pi}{2}, \quad \nu = \omega. \tag{7.304}$$

Eqs. (6.145) are then written in the form

$$\left.\begin{aligned} &u = \mu E_1 (a + f_2), \\ &\left(\frac{D^2}{\omega^2} + 2\zeta\frac{D}{\omega} + 1\right)V = u, \\ &\dot{a} - \omega b = 0, \\ &\dot{b} + \omega a - kV = 0. \end{aligned}\right\} \tag{7.305}$$

The characteristic equation (6.146) of the system in the case (7.304) takes the form

$$\Delta(p) = p^4 + 2\zeta\omega p^3 + 2\omega^2 p^2 + 2\zeta\omega^3 p + \omega^3(\omega - 2n), \tag{7.306}$$

* The systems of course are not entirely equivalent, since the third root of the characteristic equation is different in the two cases.

where

$$n = \frac{\mu E_1 k}{2}, \tag{7.307}$$

and the stability conditions (6.148) are reduced to the inequalities

$$\zeta\omega > 0, \quad \omega - 2n > 0. \tag{7.308}$$

Let us calculate the steady-state dispersions D_a and D_b of the variables a and b. In accordance with Eq. (7.48), Eqs. (7.305), and the equality $S_{f_2}(\omega) = s^0$, we have

$$\left.\begin{aligned} D_a &= \int_{-\infty}^{\infty} \frac{2ns^0\omega^2}{|\Delta(j\Omega)|^2}\, d\Omega, \\ D_b &= \int_{-\infty}^{\infty} 2ns^0\omega^2 \left|\frac{j\Omega}{\Delta(j\Omega)}\right|^2 d\Omega, \end{aligned}\right\} \tag{7.309}$$

where $\Delta(j\Omega)$ is the value of the characteristic determinant (7.306) for $p = j\Omega$.

To evaluate the integrals in (7.309), we write them in the form (7.222) and then apply the fourth equality in (7.224). For the first integral in (7.309)

$$\left.\begin{aligned} &a_0 = 1, \quad a_1 = 2\zeta\omega, \quad a_2 = 2\omega^2, \quad a_3 = 2\zeta\omega^3, \\ &a_4 = \omega^3(\omega - 2n), \\ &b_0 = 0, \quad b_1 = 0, \quad b_2 = 0, \quad b_3 = 2ns^0\omega^3, \end{aligned}\right\} \tag{7.310}$$

the coefficients a_i for the second integral are obviously the same, and the coefficients b_i are given by

$$b_0 = 0, \quad b_1 = 0, \quad b_2 = -4n^2s^0\omega^4, \quad b_3 = 0. \tag{7.311}$$

Using these coefficients, we obtain for the dispersions D_a and D_b

$$D_a = \frac{\pi s^0\omega n}{(\omega - 2n)\zeta}, \quad D_b = \frac{\pi s^0 n}{\zeta}. \tag{7.312}$$

The variables in the starting set of equations, (6.139), from which Eqs. (7.305) have been derived, are θ_x and θ_y, and not a and b. However, we are primarily concerned with the distance ρ from the center of the telescopic field to the star image. It is the distance ρ that characterizes the angular correction error. Since

$$\rho^2 = \theta_x^2 + \theta_y^2 = a^2 + b^2, \tag{7.313}$$

the dispersions (7.312) completely determine the root mean square error ρ.

Eqs. (7.312) for the dispersions D_a and D_b of the deviations a and b for $t = \infty$ are readily interpreted.

D_b is entirely independent of ω, the dependence of D_a is weak and for small n it can be ignored. This result is associated with the constant spectral density s^0 of the function $f_2(t)$. Further, for $n = 0$, we have $D_a = D_b = 0$. We have an open system in this case and the noise function f_2 does not cause any deviation in the coordinates a and b. For $\zeta = 0$ or $\omega = 2n$ the stability condition (7.308) is broken and the dispersions of the deviations a

and b increase indefinitely with time. By increasing ζ, we reduce the dispersions. This is also readily understandable if we remember that $1/\zeta$ characterizes the effective passband of the resonance filter. By decreasing the passband in the case of white noise, we naturally reduce D_a and D_b.

It follows from (7.312) that to reduce the effect of random errors, we should increase ζ and lower n. However, minimum random noise is not the only requirement to be satisfied by the correction system. Another important requirement is a high tracking rate, whereby the star image is centered in the telescopic field starting from some initial position a^0, b^0, and fast damping of transients in the system, which in the final analysis reduces to a proper choice of the roots of the characteristic equation.

Consider the characteristic equation $\Delta(p) = 0$. This is a fourth-degree equation. As we know, the roots of the fourth-degree equation

$$p^4 + a_1 p^3 + a_2 p^2 + a_3 p + a_4 = 0 \tag{7.314}$$

coincide with the roots of two quadratic equations

$$x^2 + \frac{A_{1,2} + a_1}{2} x + y + \frac{a_1 y - a_3}{A_{1,2}} = 0, \tag{7.315}$$

where

$$A_{1,2} = \pm \sqrt{8y + a_1^2 - 4a_2}, \tag{7.316}$$

and y is some real root of the cubic equation

$$8y^3 - 4a_2 y^2 + (2a_1 a_3 - 8a_4) y + a_4 (4a_2 - a_1^2) = 0. \tag{7.317}$$

In our case, a fairly accurate and simple approximate expression can be obtained for the real root of Eq. (7.317), so that the roots of Eqs. (7.315) can be found for the entire relevant range of the coefficients of the characteristic equation (7.314).

Inserting the coefficients from (7.306), (7.307) in Eq. (7.317) we obtain the auxiliary cubic equation

$$y^3 - \omega^2 y^2 + \omega^4 \left[\zeta^2 - 1 + \frac{2n}{\omega}\right] y + \omega^6 \left[1 - \zeta^2 - \frac{n}{\omega}(2 - \zeta^2)\right] = 0. \tag{7.318}$$

We will find its solution for n/ω and ζ small compared to unity. Note that for $\zeta = n = 0$ the left-hand side of Eq. (7.318) can be factored and the equation takes the form

$$(y - \omega^2)^2 (y + \omega^2) = 0. \tag{7.319}$$

As a first approximation to the solution of Eq. (7.318) we take the root

$$y = -\omega^2 \tag{7.320}$$

of Eq. (7.319). Note that although the roots $y_{2,3} = \omega^2$ are also real, it is inadvisable to use them as the first approximation, since the introduction of

a multiple root greatly enhances the difficulties of the successive approximation techniques. Using (7.320) as the first approximation root and applying Newton's method, we readily find the next approximation in the form

$$y = -\omega^2 + \omega n \tag{7.321}$$

From (7.316) we find

$$A_{1,2} = \pm j(4\omega - n). \tag{7.322}$$

and the solution of the quadratic equations (7.315) gives the following four roots of the characteristic equation (7.314) of system (7.305):

$$p_{1,2,3,4} = -\frac{\zeta\omega}{2} \pm j\left(\omega - \frac{n}{4} \pm \sqrt{\frac{\omega n}{\zeta}}\right). \tag{7.323}$$

These roots, as we see, are fairly accurate at least for $n < \omega/4$, $\zeta < 0.1 - 0.5$.

The real parts of all the roots in (7.323) are negative, being equal to $-\zeta\omega/2$. The real part of the roots $-\zeta\omega/2$ determines in the final analysis the centering time of the telescopes. Comparison of the roots (7.323) with the dispersions D_a and D_δ, (7.312), shows that by increasing ζ we simultaneously reduce both the random noise and the telescope centering time. This simultaneity, however, is observed only for small ζ.

For an arbitrary ζ and bounded n, the cubic equation (7.318) has the root

$$y = -\omega^2\left[1 + \frac{\zeta^2(n - 2\omega) - 4n}{(4 + \zeta^2)\omega + 2n}\right]. \tag{7.324}$$

For small n and ζ, this root is readily seen to take on the value (7.321). Inserting the root (7.324) into (7.316), we find

$$A_{1,2} = \pm \frac{2}{4 + \zeta^2}\sqrt{\omega^2(\zeta^6 + 8\zeta^4 - 64) - 2n\omega(\zeta^4 + 4\zeta^2 - 16)}. \tag{7.325}$$

For sufficiently large ζ this gives

$$A_{1,2} = \pm\left(2\omega\zeta - \frac{2n}{\zeta}\right), \tag{7.326}$$

and the auxiliary quadratic equations therefore take the form

$$\left.\begin{aligned} x^2 + 2\zeta\omega x + \omega^2 - 2n\omega = 0, \\ x^2 + \frac{n}{\zeta}x + \omega^2 = 0. \end{aligned}\right\} \tag{7.327}$$

Solving these equations, we obtain the roots of the characteristic equation of our system:

$$\left.\begin{aligned} p_1 = -\frac{\omega - 2n}{2\zeta}, \quad p_2 = -2\zeta\omega, \\ p_{3,4} = -\frac{n}{2\zeta} \pm j\omega. \end{aligned}\right\} \tag{7.328}$$

For large ζ, the roots are described by (7.328) up to n values of the order of ω.

As ζ increases, the root p_2 goes to minus infinity. This implies that the order of the system can be reduced, or else the resonance filter can be replaced with an aperiodic filter.

As we see from (7.328), the real parts of the roots $p_{1,3,4}$ at large ζ decrease in magnitude with increasing ζ. The increase in n, on the other hand, while increasing the magnitude of the real roots $p_{3,4}$, reduces the magnitude of the real part of the root p_1. For $\omega = 2n$ this root is zero. In this case the second stability condition (7.308) is broken. An increase in n also increases the errors due to random noise.

Thus, for small ζ, the increase in ζ simultaneously reduces the effect of noise and the time to approach equilibrium, whereas, from some point on, further increase in ζ while still reducing the effect of random noise increases the time to approach equilibrium.

Finding a compromise solution between these two opposing trends is one of the fundamental problems in the dynamics of astrocorrection systems. Note that similar problems are encountered in all other angular tracking systems, e.g., radar.

Appendix I

*SCATTERED SOLAR RADIATION IN THE ATMOSPHERE AS AN EXAMPLE OF DETERMINATE ERRORS IN PHASE ASTROCORRECTION SYSTEMS**

Besides the luminous flux of the star, the chopper in the phase indication system also transmits background light, whose power is considerably higher than the star light power. If the background illumination were uniform over the entire field of view of the telescope, it would be blocked by the resonance filter (see block diagram in Figure 6.7). The background flux, however, is nonuniform; its nonuniformity is thus modulated by the chopper and transmitted by the filter. If the background modulation signal is comparable with the effective signal, considerable distortion results and the system may fail to function properly.

Background modulation is one of the principal shortcomings of chopper systems: it limits the field of view of the system, i.e., the value of $|\theta|$ for astrocorrection.

The background modulation signal can also be reduced by using a more complex chopper. Further developments of the semicircular chopping disk described in § 6.4 comprise** a single-cycle reticle (Figure A.1a), a uniform reticle (Figure A.1b), a semidisk reticle (Figure A.1c), a dual-cycle reticle, i.e., a combination of two halves of uniform reticles with different cycles (Figure A.1d).

However, complex choppers also attenuate the effective signal. They force us to work with several harmonics of the chopper frequency and thus complicate the photocell output channel; in some cases (as with a uniform reticle, sav), scanning is inevitable. It is thus absolutely essential to carry out a theoretical analysis of the signal amplitude and background modulation amplitude at the working frequency as a function of chopper parameters and the diameter of the field of view. The theoretical analysis is especially important since direct experiments are highly delicate and complex, and their results are not always reliable.

As an example of a determinate background noise we will consider the scattered solar light in the atmosphere. The scattered light greatly interferes with the proper operation of the phase system in daytime observation of stars inside the Earth's atmosphere. The method applied to the analysis of this factor and some of the results can be readily extended to other determinate noise factors, including the effect of the nonideal telescope optics.

* Andreev, V.D. Transmission of the useful signal and background noise through the reticles of light indication systems. — Izvestiya AN SSSR, technical cybernetics, No.3. 1963. Determination of optimal parameters of light indication systems. — Ibid., No.4. 1963.

** Aroyan, G.F. Technical Spatial Filtering. — Proceedings of the IRE, No.9. 1959.

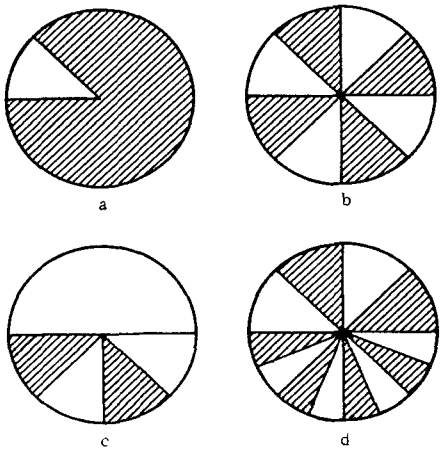

FIGURE A.1.

First let us consider the transmission of the effective signal and the background signal through various reticles.

The incident luminous flux, as we have mentioned, is the sum of the star flux and background flux. If the star is regarded as a point source and the telescope optics is ideal, the effective luminous flux P_0 reaching the photocell through a chopper (of the type shown in Figure A.1) rotating with angular frequency ω is a periodic function of time with period $T \leqslant \frac{2\pi}{\omega}$. For $0 < t \leqslant \frac{2\pi}{\omega}$, the transmitted flux can be written in the form

$$P_0 = p \sum_{j=0}^{n-1} \left[1\left(\frac{\tau_j}{\omega}\right) - 1\left(\frac{\tau_{j+1}}{\omega}\right) \right]. \tag{I.1}$$

Here p is a proportionality coefficient dependent on the brightness of the star, n is the number of transparent wedges in the reticle, τ_j and τ_{j+1} are the angles corresponding to the beginning and the end of the j-th transparent wedge, and the function $1(\tau_j/\omega)$ is defined as follows:

$$1\left(\frac{\tau_j}{\omega}\right) = \begin{cases} 0 & \text{for } t < \frac{\tau_j}{\omega}, \\ 1 & \text{for } t \geqslant \frac{\tau_j}{\omega}. \end{cases} \tag{I.2}$$

To determine the working frequencies, the function $P_0(t)$ should be expanded in a Fourier series.

Let the background brightness be

$$B = B(r, \tau), \tag{I.3}$$

where r and τ are the polar coordinates of a general direction inside the solid angle of the field of view; r is reckoned from the telescope axis and

the zero-point of the angles τ can be chosen from consideration of the form of the function B.

The background flux through the chopper is then given by

$$P_1 = \sum_{j=0}^{n-1} \int_{\omega t - \tau_j}^{\omega t - \tau_{j-1}} \int_0^{r_0} B(r, \tau) r \, dr \, d\tau. \tag{I.4}$$

Inside the field of view the background brightness B is a periodic function of τ with period 2π. $P_1(t)$ is therefore also periodic in t with a period equal to or less than $2\pi/\omega$. The determination of the background signal amplitudes for given $B(r, \tau)$ thus also reduces to calculation of the Fourier expansion coefficients of the function $P_1(t)$ in the interval $\left(0, \frac{2\pi}{\omega}\right)$ corresponding to the working frequencies, i.e., the Fourier frequencies of the effective signal $P_0(t)$. Expressing the Fourier coefficients of $P_1(t)$ in terms of the chopper parameters and the angle of view, we obtain the sought dependence.

Note that in most cases B is not given as a function of r and τ.

Commonly the background brightness B is defined as a function of the angles z and φ between the telescope axis and some two fixed directions. In observations of stars against the background of scattered sunlight, z and φ are the angles between the direction to the star and the directions to the zenith and the Sun.

The Fourier expansion of the effective signal is written in the form

$$P_0(t) = \frac{a_0^0}{2} + \sum_{k=1}^{\infty} (a_k^0 \cos k\omega t + b_k^0 \sin k\omega t). \tag{I.5}$$

To calculate the coefficients of (I.5), we will use Eqs. (6.111) inserting for p_* the function P_0 from Eqs. (I.1) and (I.2).

For a single-cycle reticle, we have

$$a_0^0 = \frac{p\tau_0}{\pi}, \quad a_k^0 = \frac{p}{\pi k} \sin k\tau_0, \quad b_k^0 = \frac{p}{\pi k}(1 - \cos k\tau_0). \tag{I.6}$$

where τ_0 is the angular width of the cycle.

For a semidisk $\tau_0 = \pi$, so that

$$a_0^0 = p, \quad a_k^0 = 0, \quad b_{2q} = 0, \quad b_{2q+1} = \frac{2p}{(2q+1)\pi}. \tag{I.7}$$

For a uniform reticle with even n

$$\tau_j = \frac{\pi}{n\omega} 2j, \quad \tau_{j+1} = \frac{\pi}{n\omega}(2j+1), \tag{I.8}$$

and thus

$$\left.\begin{aligned} a_0^0 &= p, \\ a_k^0 &= \frac{2p}{k\pi} \sum_{j=0}^{n-1} \cos(4j+1) \frac{k\pi}{2n} \sin \frac{k\pi}{2n}, \\ b_k^0 &= \frac{2p}{k\pi} \sum_{j=0}^{n-1} \sin(4j+1) \frac{k\pi}{2n} \sin \frac{k\pi}{2n}. \end{aligned}\right\} \tag{I.9}$$

Since n is even, we have

$$\left.\begin{aligned}
&\sum_{j=0}^{n-1} \sin(4j+1)\frac{k\pi}{2n}\sin\frac{k\pi}{2n} = \\
&\quad = \sum_{j=0}^{\frac{n}{2}-1} \sin(4j+1)\frac{k\pi}{2n}\sin\frac{k\pi}{2n} + \\
&\quad + \sum_{j=0}^{\frac{n}{2}-1} \sin\left[k\pi+(4j+1)\frac{k\pi}{2n}\right]\sin\frac{k\pi}{2n}, \\
&\sum_{j=0}^{n-1} \cos(4j+1)\frac{k\pi}{2n}\sin\frac{k\pi}{2n} = \\
&\quad = \sum_{j=0}^{\frac{n}{2}-1} \cos(4j+1)\frac{k\pi}{2n}\sin\frac{k\pi}{2n} + \\
&\quad + \sum_{j=0}^{\frac{n}{2}-1} \cos\left[k\pi+(4j+1)\frac{k\pi}{2n}\right]\sin\frac{k\pi}{2n}.
\end{aligned}\right\} \tag{I.10}$$

Therefore, for an even $k = 2q$:

$$\left.\begin{aligned}
a_{2q}^0 &= \frac{2p}{q\pi}\sum_{j=0}^{\frac{n}{2}-1} \cos(4j+1)\frac{k\pi}{2n}\sin\frac{k\pi}{2n}, \\
b_{2q}^0 &= \frac{2p}{q\pi}\sum_{j=0}^{\frac{n}{2}-1} \sin(4j+1)\frac{k\pi}{2n}\sin\frac{k\pi}{2n}.
\end{aligned}\right\} \tag{I.11}$$

For an odd $k = 2q+1$:

$$a_{2q+1} = b_{2q+1} = 0. \tag{I.12}$$

Similarly for a semidisk reticle we have

$$\left.\begin{aligned}
a_0^0 &= \frac{p}{2}, \\
a_k^0 &= \frac{2p}{k\pi}\sum_{j=0}^{\frac{n}{2}-1} \cos(4j+1)\frac{k\pi}{2n}\sin\frac{k\pi}{2n}, \\
b_k^0 &= \frac{2p}{k\pi}\sum_{j=0}^{\frac{n}{2}-1} \sin(4j+1)\frac{k\pi}{2n}\sin\frac{k\pi}{2n}.
\end{aligned}\right\} \tag{I.13}$$

Finally, for a dual-cycle reticle (with cycles of $\frac{\pi}{n}$ and $\frac{\pi}{m}$) we have

$$\left.\begin{aligned}
a_0^0 &= p, \\
a_k^0 &= \frac{2p}{k\pi}\left\{\sum_{j=0}^{\frac{n}{2}-1} \cos(4j+1)\frac{k\pi}{2n}\sin\frac{k\pi}{2n} + \right. \\
&\quad \left. + \sum_{j=0}^{\frac{m}{2}-1} \cos\left[k\pi+(4j+1)\frac{k\pi}{2m}\right]\sin\frac{k\pi}{2m}\right\},
\end{aligned}\right\} \tag{I.14}$$

$$b_k^0 = \frac{2p}{k\pi}\left\{\sum_{j=0}^{\frac{n}{2}-1}\sin(4j+1)\frac{k\pi}{2n}\sin\frac{k\pi}{2n} + \right.$$
$$\left. + \sum_{j=0}^{\frac{m}{2}-1}\sin\left[k\pi+(4j+1)\frac{k\pi}{2m}\right]\sin\frac{k\pi}{2m}\right\}. \qquad \text{(I.14)}$$

Further simplifications are based on the equalities*

$$\left.\begin{aligned}\sum_{j=0}^{\frac{n}{2}-1}\cos(4j+1)\frac{k\pi}{2n}\sin\frac{k\pi}{2n} &= \frac{\sin^2\frac{k\pi}{2}\sin\frac{k\pi}{2n}}{2\cos\frac{k\pi}{2n}},\\ \sum_{j=0}^{\frac{n}{2}-1}\sin(4j+1)\frac{k\pi}{2n}\sin\frac{k\pi}{2n} &= \frac{1}{2}\left(\sin^2\frac{k\pi}{2}-\frac{1}{2}\frac{\sin k\pi\sin\frac{k\pi}{2n}}{\cos\frac{k\pi}{2n}}\right).\end{aligned}\right\} \qquad \text{(I.15)}$$

For an odd $k = 2q+1$, we thus have

$$\left.\begin{aligned}\sum_{j=0}^{\frac{n}{2}-1}\cos(4j+1)\frac{(2q+1)\pi}{2n}\sin\frac{(2q+1)\pi}{2n} &= \frac{1}{2}\operatorname{tg}\frac{(2q+1)\pi}{2n},\\ \sum_{j=0}^{\frac{n}{2}-1}\sin(4j+1)\frac{(2q+1)\pi}{2n}\sin\frac{(2q+1)\pi}{2n} &= \frac{1}{2}.\end{aligned}\right\} \qquad \text{(I.16)}$$

For an even $k = 2q$, Eqs. (I.15) vanish for all q except

$$q = \frac{n}{2}(2s+1). \qquad \text{(I.17)}$$

For these q, Eq. (I.15) has the form $\frac{0}{0}$, which eventually gives

$$\left.\begin{aligned}\sum_{j=0}^{\frac{n}{2}-1}\cos(4j+1)\frac{n(2s+1)\pi}{2n}\sin\frac{n(2s+1)\pi}{2n} &= 0,\\ \sum_{j=0}^{\frac{n}{2}-1}\sin(4j+1)\frac{n(2s+1)\pi}{2n}\sin\frac{n(2s+1)\pi}{2n} &= \frac{n}{2}.\end{aligned}\right\} \qquad \text{(I.18)}$$

From (I.16) and (I.18), using (I.9)—(I.11), (I.13), (I.14), we obtain

$$\left.\begin{aligned}a_{2q+1}^0 = b_{2q+1}^0 = a_{2q}^0 &= 0,\\ b_{n(2s+1)}^0 &= \frac{2p}{\pi(2s+1)};\end{aligned}\right\} \qquad \text{(I.19)}$$

$$\left.\begin{aligned}a_{2q+1}^0 &= \frac{p}{\pi(2q+1)}\operatorname{tg}\frac{\pi(2q+1)}{2n},\\ b_{2q+1}^0 &= \frac{p}{\pi(2q+1)},\\ a_{2q}^0 = 0,\quad b_{n(2s+1)}^0 &= \frac{p}{\pi(2s+1)};\end{aligned}\right\} \qquad \text{(I.20)}$$

* Ryzhik, I.M. and I.S. Gradshtein. Tablitsy integralov, summ, ryadov i proizvedenii (Tables of Integrals, Sums, Series and Products). — Gostekhizdat. 1951.

$$\left.\begin{aligned} a^0_{2q+1} &= \frac{p}{\pi(2q+1)}\left[\operatorname{tg}\frac{\pi(2q+1)}{2n} - \operatorname{tg}\frac{\pi(2q+1)}{2m}\right], \\ b^0_{2q+1} &= a^0_{2q} = 0, \\ b^0_{n(2s+1)} &= \frac{p}{\pi(2s+1)}, \\ b^0_{m(2s+1)} &= \frac{p}{\pi(2s+1)}. \end{aligned}\right\} \tag{I.21}$$

We can now write the Fourier expansions of P_0 in explicit form: for a single-cycle reticle

$$P_0^{(1)} = \frac{p\tau_0}{2\pi} + \frac{p}{\pi}\sum_{k=1}^{\infty}\left(\frac{\sin k\tau_0}{k}\cos k\omega t + \frac{1-\cos k\tau_0}{k}\sin k\omega t\right); \tag{I.22}$$

for a semicircular chopping disk

$$P_0^{(2)} = \frac{p}{2} + \frac{2p}{\pi}\sum_{q=0}^{\infty}\frac{1}{2q+1}\sin(2q+1)\omega t; \tag{I.23}$$

for a uniform reticle

$$P_0^{(3)} = \frac{p}{2} + \frac{2p}{\pi}\sum_{q=0}^{\infty}\frac{1}{2q+1}\sin n(2q+1)\omega t; \tag{I.24}$$

for a semidisk reticle

$$P_0^{(4)} = \frac{p}{4} + \frac{p}{\pi}\sum_{q=0}^{\infty}\frac{1}{2q+1}\left[\operatorname{tg}\frac{(2q+1)\pi}{2n}\cos(2q+1)\omega t + \right.$$
$$\left. + \sin(2q+1)\omega t + \sin n(2q+1)\omega t\right]; \tag{I.25}$$

for a dual-cycle reticle

$$P_0^{(5)} = \frac{p}{2} + \frac{p}{\pi}\sum_{q=0}^{\infty}\frac{1}{2q+1}\left\{\left[\operatorname{tg}\frac{(2q+1)\pi}{2n} - \right.\right.$$
$$\left.\left. - \operatorname{tg}\frac{(2q+1)\pi}{2m}\right]\cos(2q+1)\omega t + \sin n(2q+1)\omega t + \sin m(2q+1)\omega t\right\}. \tag{I.26}$$

The working frequencies are all the frequencies contained in the spectrum of the effective signal transmitted by the chopper.

We should preferably use those frequencies which carry most of the energy of the transmitted signal reaching the photocell and whose amplitudes are fairly large relative to the background modulation amplitudes at the same frequencies.

Let us now find the background modulation amplitudes at the working frequencies. As we have noted before, they are determined by the corresponding Fourier coefficients of the function $P_1(t)$ over the interval $\left(0, \frac{2\pi}{\omega}\right)$.

The Fourier coefficients are conveniently calculated using the Fourier expansion of the function $B(r, \tau)$, which is periodic in τ with a period 2π:

$$B = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos k\tau + b_k\sin k\tau). \tag{I.27}$$

From (I.4) and (I.27) we get

$$P_1(t) = \sum_{j=0}^{n-1} \int_{\omega t+\tau_j}^{\omega t+\tau_{j+1}} \left[\frac{\bar{a}_0 r_0^2}{4} + \sum_{k=1}^{\infty} (\bar{a}_k \cos k\tau + \bar{b}_k \sin k\tau) \right] d\tau. \tag{I.28}$$

where

$$\left.\begin{aligned} \bar{a}_k(r_0) &= \frac{1}{\pi} \int_0^{r_0} \int_0^{2\pi} B(r, \tau)\, r \cos k\tau\, dr\, d\tau, \\ \bar{b}_k(r_0) &= \frac{1}{\pi} \int_0^{r_0} \int_0^{2\pi} B(r, \tau) \sin k\tau\, dr\, d\tau. \end{aligned}\right\} \tag{I.29}$$

Integrating over τ on the right in (I.29), we obtain for the slit chopper

$$P_1^{(1)} = \frac{\bar{a}_0 \tau_0 r_0^2}{4} + \sum_{k=1}^{\infty} \left\{ \frac{1}{k} [\bar{a}_k (\cos k\tau_0 - 1) + \bar{b}_k \sin k\tau_0] \sin k\omega t + \right.$$
$$\left. + \frac{1}{k} [\bar{a}_k \sin k\tau_0 - \bar{b}_k (\cos k\tau_0 - 1)] \cos k\omega t \right\}. \tag{I.30}$$

Hence for $\tau_0 = \pi$ (a semicircular chopping disk) we have

$$P_1^{(2)} = \frac{\bar{a}_0 \pi r_0^2}{4} + \sum_{j=1}^{\infty} \left[-\frac{2\bar{a}_{2q+1}}{2q+1} \sin(2q+1)\omega t + \frac{2\bar{b}_{2q+1}}{2q+1} \cos(2q+1)\omega t \right]. \tag{I.31}$$

For a uniform reticle

$$P_1^{(3)} = \frac{a_0 \pi r_0^2}{4} + \sum_{k=1}^{\infty} \frac{2 \sin \frac{k\pi}{2n}}{k} \left[\bar{a}_k \sum_{j=0}^{n-1} \cos k\left(\omega t + \frac{4j+1}{2} \frac{\pi}{n}\right) + \right.$$
$$\left. + \bar{b}_k \sum_{j=0}^{n-1} \sin k\left(\omega t + \frac{4j+1}{2} \frac{\pi}{n}\right) \right], \tag{I.32}$$

for a semidisk reticle

$$P_1^{(4)} = \frac{\bar{a}_0 \pi r_0^2}{8} + \sum_{k=1}^{\infty} \frac{2 \sin \frac{k\pi}{2n}}{k} \left[\bar{a}_k \sum_{j=0}^{\frac{n}{2}-1} \cos k\left(\omega t + \frac{4j+1}{2} \frac{\pi}{n}\right) + \right.$$
$$\left. + \bar{b}_k \sum_{j=0}^{\frac{n}{2}-1} \sin k\left(\omega t + \frac{4j+1}{2} \frac{\pi}{n}\right) \right]. \tag{I.33}$$

Similarly for a dual-cycle reticle

$$P_1^{(5)} = \frac{a_0 \pi r_0^2}{4} + \sum_{k=1}^{\infty} \left\{ \frac{2 \sin \frac{k\pi}{2n}}{k} \left[\bar{a}_k \sum_{j=0}^{\frac{n}{2}-1} \cos k\left(\omega t + \frac{4j+1}{2} \frac{\pi}{n}\right) + \right.\right.$$
$$\left.\left. + \bar{b}_k \sum_{j=0}^{\frac{n}{2}-1} \sin k\left(\omega t + \frac{4j+1}{2} \frac{\pi}{n}\right) \right] + \frac{2 \sin \frac{k\pi}{2m}}{k} \times \right.$$

$$\times\left[\bar{a}_k \sum_{j=0}^{\frac{m}{2}-1} \cos k\left(\pi+\omega t+\frac{4j+1}{2}\frac{\pi}{m}\right)+\right.$$

$$\left.+\bar{b}_k \sum_{j=0}^{\frac{m}{2}-1} \sin k\left(\pi+\omega t+\frac{4j+1}{2}\frac{\pi}{m}\right)\right]. \tag{I.34}$$

Using (I.10)—(I.12), we can write Eq. (I.32) for $P_1^{(3)}$ in the form

$$P_1^{(3)}=\frac{\bar{a}_0\pi r_0^2}{4}+\sum_{q=1}^{\infty}\frac{2\sin\frac{q\pi}{n}}{q}\left[\bar{a}_{2q}\sum_{j=0}^{\frac{n}{2}-1}\cos 2q\times\right.$$

$$\left.\times\left(\omega t+\frac{4j+1}{2}\frac{\pi}{n}\right)+\bar{b}_{2q}\sum_{j=0}^{\frac{n}{2}-1}\sin 2q\left(\omega t+\frac{4j+1}{2}\frac{\pi}{n}\right)\right]. \tag{I.35}$$

Using Eqs. (I.15), (I.16), (I.18), we readily rewrite Eqs. (I.33)—(I.35) in the form

$$P_1^{(3)}=\frac{\bar{a}_0\pi r_0^2}{4}+\sum_{q=0}^{\infty}\frac{2}{2q+1}\left[\bar{b}_{n\,(2q+1)}\cos n(2q+1)\,\omega t-\right.$$

$$\left.-\bar{a}_{n\,(2q+1)}\sin n(2q+1)\,\omega t\right]. \tag{I.36}$$

$$P_1^{(4)}=\frac{\bar{a}_0\pi r_0^2}{8}+\sum_{q=0}^{\infty}\frac{1}{2q+1}\left\{\left[\bar{a}_{2q+1}\operatorname{tg}\frac{(2q+1)\pi}{2n}+\bar{b}_{2q+1}\right]\times\right.$$

$$\times\cos(2q+1)\,\omega t+\left[\bar{b}_{2q+1}\operatorname{tg}\frac{(2q+1)\pi}{2n}-a_{2q+1}\right]\times$$

$$\times\sin(2q+1)\,\omega t+\bar{b}_{n\,(2q+1)}\cos n(2q+1)\,\omega t-$$

$$\left.-a_{n\,(2q+1)}\sin n(2q+1)\,\omega t\right\}, \tag{I.37}$$

$$P_1^{(5)}=\frac{\bar{a}_0\pi r_0^2}{4}+\sum_{q=0}^{\infty}\frac{1}{2q+1}\left\{a_{2q+1}\left[\operatorname{tg}\frac{(2q+1)\pi}{2n}-\right.\right.$$

$$-\left.\operatorname{tg}\frac{(2q+1)\pi}{2m}\right]\cos(2q+1)\,\omega t+\bar{b}_{2q+1}\left[\operatorname{tg}\frac{(2q+1)\pi}{2n}-\right.$$

$$-\left.\operatorname{tg}\frac{(2q+1)\pi}{2m}\right]\sin(2q+1)\,\omega t+\bar{b}_{n\,(2q+1)}\cos n(2q+1)\,\omega t+$$

$$+\bar{b}_{m\,(2q+1)}\cos m(2q+1)\,\omega t-\bar{a}_{n\,(2q+1)}\sin n(2q+1)\,\omega t-$$

$$\left.-\bar{a}_{m\,(2q+1)}\sin m(2q+1)\,\omega t\right\}. \tag{I.38}$$

We have thus found the Fourier expansions of the effective signal

$$P_0^{(t)}(t)=\frac{a_0^0}{2}+\sum_{k=1}^{\infty}\left(a_k^0\cos k\omega t+b_k^0\sin k\omega t\right) \tag{I.39}$$

and the background modulation signal

$$P_1^{(t)}(t)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k\cos k\omega t+b_k\sin k\omega t\right) \tag{I.39a}$$

for several characteristic reticles.

We now require the ratio of the effective signal amplitude to the background amplitude at the working frequencies, i.e., the ratio

$$\lambda_k = \frac{\sqrt{(a_k^0)^2 + (b_k^0)^2}}{\sqrt{(\bar{a}_k)^2 + (\bar{b}_k)^2}}. \tag{I.40}$$

From (I.22)—(I.26) and (I.30)—(I.38) we see that for our radial slit choppers

$$\lambda_k = \frac{p}{\pi \sqrt{\bar{a}_k^2 + \bar{b}_k^2}} = \frac{p}{\pi A_k}, \tag{I.41}$$

where p is the star brightness, $\bar{a}_k$ and $\bar{b}_k$ are related to the Fourier coefficients of the background brightness $B(r, \tau)$ by the equalities

$$\left.\begin{aligned} \bar{a}_k(r_0) &= \int_0^{r_0} a_k(r)\, r\, dr, \\ \bar{b}_k(r_0) &= \int_0^{r_0} b_k(r)\, r\, dr. \end{aligned}\right\} \tag{I.42}$$

We will now prove that Eq. (I.41) is valid for any general chopping reticle.

From (I.1) and (I.2) we have

$$\left.\begin{aligned} a_k^0 &= \frac{p}{\pi k} \sum_{j=0}^{n-1} (\sin k\tau_{j+1} - \sin k\tau_j), \\ b_k^0 &= -\frac{p}{\pi k} \sum_{j=0}^{n-1} (\cos k\tau_{j+1} - \cos k\tau_j), \end{aligned}\right\} \tag{I.43}$$

and from (I.28) we obtain

$$\left.\begin{aligned} a_k &= \frac{\bar{a}_k}{k} \sum_{j=0}^{n-1} (\sin k\tau_{j+1} - \sin k\tau_j) - \frac{\bar{b}_k}{k} \sum_{j=0}^{n-1} (\cos k\tau_{j-1} - \cos k\tau_j), \\ b_k &= \frac{\bar{a}_k}{k} \sum_{j=0}^{n-1} (\cos k\tau_{j+1} - \cos k\tau_j) + \frac{\bar{b}_k}{k} \sum_{j=0}^{n-1} (\sin k\tau_{j+1} - \sin k\tau_j). \end{aligned}\right\} \tag{I.44}$$

Clearly,

$$a_k^2 + b_k^2 = \frac{\bar{a}_k^2 + \bar{b}_k^2}{k^2} \left\{ \left[\sum_{j=0}^{n-1} (\sin k\tau_{j+1} - \sin k\tau_j) \right]^2 + \left[\sum_{j=0}^{n-1} (\cos k\tau_{j+1} - \cos k\tau_j) \right]^2 \right\}. \tag{I.45}$$

Eq. (I.41) now follows directly from (I.43) and (I.45) for any general reticle.

This expression, however, is not applicable to reticles where the transparent and opaque segments are delimited by curved lines from the disk center to its periphery, i.e., when

$$\tau_j = \tau_j(r) \tag{I.46}$$

(at least for a general background luminance and arbitrary distance of the

luminous object from the center), since for these choppers

$$\left.\begin{aligned} a_k &= \frac{1}{k}\sum_{j=0}^{n-1}\int_0^{r_0}[\sin k\tau_{j+1}(r)-\sin k\tau_j(r)]\,a_k(r)\,r\,dr- \\ &-\frac{1}{k}\sum_{j=0}^{n-1}\int_0^{r_0}[\cos k\tau_{j+1}(r)-\cos k\tau_j(r)]\,b_k(r)\,r\,dr, \\ b_k &= \frac{1}{k}\sum_{j=0}^{n-1}\int_0^{r_0}[\cos k\tau_{j+1}(r)-\cos k\tau_j(r)]\,a_k(r)\,r\,dr+ \\ &+\frac{1}{k}\sum_{j=0}^{n-1}\int_0^{r_0}[\sin k\tau_{j+1}(r)-\sin k\tau_j(r)]\,b_k(r)\,r\,dr. \end{aligned}\right\} \tag{I.47}$$

In particular, for a general "helical" chopper, when

$$\tau_j(r)=\tau_j+\alpha(r), \tag{I.48}$$

τ_j being the angle of the ordinary "nonhelical" chopping reticle, we have

$$\begin{aligned} k^2[(a_k)^2+(b_k)^2] &= \left\{\left[\sum_{j=0}^{n-1}(\sin k\tau_{j+1}-\sin k\tau_j)\right]^2+\right. \\ &\left.+\left[\sum_{j=0}^{n-1}(\cos k\tau_{j+1}-\cos k\tau_j)\right]^2\right\}\left\{\left[\int_0^{r_0}(a_k\cos k\alpha+\right.\right. \\ &\left.\left.+b_k\sin k\alpha)r\,dr\right]^2+\left[\int_0^{r_0}(b_k\cos k\alpha-a_k\sin k\alpha)\,r\,dr\right]^2\right\}. \end{aligned} \tag{I.49}$$

Now, from the Schwartz—Cauchy inequality

$$\begin{aligned} &\left[\int_0^{r_0}(a_k\cos k\alpha+b_k\sin k\alpha)\,r\,dr\right]^2+\left[\int_0^{r_0}(b_k\cos k\alpha-\right. \\ &\left.-a_k\sin k\alpha)\,r\,dr\right]^2\leqslant\int_0^{r_0}r^2\,dr\int_0^{r_0}(a_k^2+b_k^2)\,dr, \end{aligned} \tag{I.50}$$

and so we have

$$\lambda_k\geqslant\frac{p}{\pi\sqrt{\frac{r_0^3}{3}\int_0^{r_0}(a_k^2+b_k^2)\,dr}} \tag{I.51}$$

Note that in choppers with slits of the form $\tau_j=\tau_j(r)$, the coordinates τ and r of the sighted objects are not as readily separated as in a chopping reticle, where τ can be found independently of r.

Consider the case of scattered solar radiation in the atmosphere. The corresponding background brightness is*

$$B=B(z,\ \varphi)=Ay_1(z)\,y_2(\varphi), \tag{I.51a}$$

* See Kondrat'ev, K.Ya. Luchistaya energiya Solntsa (The Sun's Radiant Energy). — Gidrometeoizdat, 1951.

where A is a constant, z is the angle between the directions to the star and to the zenith, φ is the angle between the directions to the star and to the Sun.

The function $y_1(z)$ specifies the effect of the air column in the direction of the telescope axis, $y_2(\varphi)$ is the scattering diagram, which describes the distribution of scattered intensity from a unit air mass in all directions.

Before we can write the Fourier expansion of B, we have to express this function in the coordinates r and τ.

Consider the axes $O_1\xi\eta\zeta$ (Figure A.2) with the origin at the Earth's center and the axis ζ pointing to the zenith, i.e., along the line from the Earth's center O_1 to the object's current position.

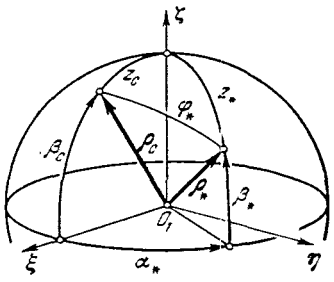

FIGURE A.2.

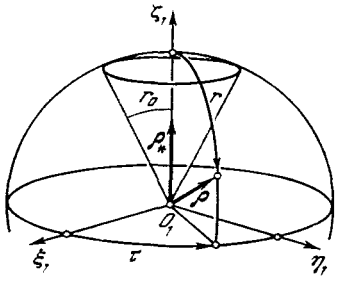

FIGURE A.3.

A general direction ρ in this system is defined by the longitude α, reckoned in the plane $\xi O_1\eta$ from the direction $O_1\xi$, and the latitude β, reckoned from the plane $\xi O_1\eta$.

The directions to the star and the Sun are defined by the vectors $\rho_*(\alpha_*, \beta_*)$ and $\rho_c(\beta_c)$, respectively (without loss of generality, the angle α_c may be taken equal to zero).

The variables z and φ entering Eq. (I.51a) are evidently expressed by the relations

$$\cos z = \rho \cdot \zeta, \quad \cos\varphi = \rho \cdot \rho_c. \tag{I.52}$$

We introduce another right orthogonal system of coordinates $O_1\xi_1\eta_1\zeta_1$ (Figure A.3) with the origin at O_1, the axis ζ_1 directed along the vector ρ_*, and the axis η_1 along the vector $\zeta \times \rho_*$ (Figure A.2).

A general direction ρ in the axes $O_1\xi_1\eta_1\zeta_1$ is determined by the longitude τ reckoned in the plane $\xi_1 O_1\eta_1$ from the direction $O_1\xi_1$, and the colatitude r reckoned from the axis $O_1\zeta_1$.

The field of view of the telescopic system in this case is the interior of a cone with a vertex angle of $2r_0$, whose axis is along ρ_*.

The relative orientation of the axes ξ, η, ζ and ξ_1, η_1, ζ_1 is described by the following matrix of direction cosines:

	ξ	η	ζ
ξ_1	$\sin\beta_* \cos\alpha_*$	$\sin\beta_* \sin\alpha_*$	$-\cos\beta_*$
η_1	$-\sin\alpha_*$	$\cos\alpha_*$	0
ζ_1	$\cos\beta_* \cos\alpha_*$	$\cos\beta_* \sin\alpha_*$	$\sin\beta_*$.

(I.53)

The vector ρ in the axes $O_1\xi_1\eta_1\zeta_1$ is described by the equality

$$\rho = \xi_1 \sin r \cos\tau + \eta_1 \sin r \sin\tau + \zeta_1 \cos r, \tag{I.54}$$

where ξ_1, η_1, ζ_1 are the unit vectors of the corresponding axes.

From Eqs. (I.52), (I.54) and matrix (I.53) we have

$$\left.\begin{aligned} \cos z &= -\sin z_* \sin r \cos\tau + \cos z_* \cos r, \\ \cos\varphi &= -\frac{(\cos z_C - \cos z_* \cos\varphi_*)\sin r \cos\tau}{\sin z_*} \\ &\quad - \sin\alpha_* \sin z_C \sin r \sin\tau + \cos\varphi_* \cos r. \\ \cos\alpha_* &= \frac{\cos\varphi_* - \cos z_* \cos z_C}{\sin z_* \sin z_C}. \end{aligned}\right\} \tag{I.55}$$

Hence

$$\left.\begin{aligned} \cos z &= -\sin z_* \sin r \cos\tau + \cos z_* \cos r, \\ \cos\varphi &= -\sin\varphi_* \sin r \cos(\tau+\varepsilon) + \cos\varphi_* \cos r, \\ \varepsilon &= \operatorname{arctg} \frac{\sin\alpha_* \sin z_C \sin z_*}{\cos z_C - \cos z_* \cos\varphi_*}. \end{aligned}\right\} \tag{I.56}$$

If we take

$$\varphi = \varphi_* + \delta\varphi, \quad z = z_* + \delta z \tag{I.57}$$

and assume $\delta\varphi$, δz and r to be small within the field of view, we readily obtain from (I.55)

$$\left.\begin{aligned} \delta z &= r\cos\tau, \\ \delta\varphi &= r\cos(\tau+\varepsilon). \end{aligned}\right\} \tag{I.58}$$

Eqs. (I.56) or (I.57) and (I.58) are the transformation relations from the variables φ and z to the variables r and τ.

We can now write the Fourier expansion of $B(r, \tau)$ in τ. Assuming z to be fairly far from $z = \frac{\pi}{2}$, so that approximately

$$y_1 = \frac{1}{\cos z}, \tag{I.59}$$

and taking for $y_2(\varphi)$ as the first approximation the Rayleigh scattering diagram*

$$y_2 = 1 + \cos^2\varphi, \tag{I.60}$$

we obtain $B(z, \varphi)$ in the form

$$B = \frac{A}{\cos z}(1 + \cos^2\varphi). \tag{I.61}$$

Recalling Eqs. (I.56) and writing $\operatorname{tg} r = \sin r = r$, $\cos r = 1$ (since r is small) we expand the function y_1 in powers of $r \operatorname{tg} z_*$.

* See the book mentioned in the footnote to p. 376.

Collecting the terms, we get

$$B = \tilde{a}_0 - \tilde{a}_2 r\cos\tau + \tilde{a}_3 r\sin\tau + \tilde{a}_4 r^2 \sum_{q=0}^{\infty} (r\,\mathrm{tg}\,z_*)^{2q}\cos^{2q+2}\tau - \\ - \tilde{a}_4 r^2 \sum_{q=0}^{\infty} (r\,\mathrm{tg}\,z_*)^{2q+1}\cos^{2q+3}\tau - \\ - \tilde{a}_5 r^2\sin\tau \sum_{q=0}^{\infty} (r\,\mathrm{tg}\,z_*)^{2q}\cos^{2q+1}\tau + \\ + \tilde{a}_5 r\sin\tau \sum_{q=0}^{\infty} (r\,\mathrm{tg}\,z_*)^{2q+1}\cos^{2q+2}\tau, \tag{I.62}$$

where

$$\left.\begin{aligned} \tilde{a}_1 &= \frac{A}{\cos z_*}(1+\cos^2\varphi_*), \\ \tilde{a}_2 &= \frac{A}{\cos z_*}[\sin 2\varphi_*\cos\varepsilon + \mathrm{tg}\,z_*(1+\cos^2\varphi_*)], \\ \tilde{a}_3 &= \frac{A}{\cos z_*}\sin 2\varphi_*\sin\varepsilon, \\ \tilde{a}_4 &= \frac{A}{\cos z_*}[(1+\cos^2\varphi_*)\,\mathrm{tg}^2 z_* + \sin^2\varphi_*\cos 2\varepsilon + \\ &\qquad + \sin 2\varphi_*\cos\varepsilon\,\mathrm{tg}\,z_*], \\ \tilde{a}_5 &= \frac{A}{\cos z_*}(\sin^2\varphi_*\sin 2\varepsilon + \sin 2\varphi_*\sin\varepsilon\,\mathrm{tg}\,z_*). \end{aligned}\right\} \tag{I.63}$$

To determine the Fourier coefficients of $B(r, \tau)$, we thus have to take the integrals

$$\left.\begin{aligned} &\int_0^{2\pi}\cos^{\mu}\tau\sin k\tau\,d\tau, \qquad \int_0^{2\pi}\cos^{\mu}\tau\cos k\tau\,d\tau, \\ &\int_0^{2\pi}\cos^{\mu}\tau\sin\tau\sin k\tau\,d\tau, \qquad \int_0^{2\pi}\cos^{\mu}\tau\sin\tau\cos k\tau\,d\tau. \end{aligned}\right\} \tag{I.64}$$

Since $\cos^{\mu}\tau$ is an even function, and $\cos^{\mu}\tau\sin\tau$ is odd, we have

$$\int_0^{2\pi}\cos^{\mu}\tau\sin k\tau\,d\tau = 0, \qquad \int_0^{2\pi}\cos^{\mu}\tau\sin\tau\cos k\tau\,d\tau = 0. \tag{I.65}$$

The remaining two integrals are evaluated without difficulty. As an example, let us calculate

$$\int_0^{2\pi}\cos^{\mu}\tau\cos k\tau\,d\tau = 2\int_0^{\pi}\cos^{\mu}\tau\cos k\tau\,d\tau. \tag{I.66}$$

If $\mu = 2m+1$, for even $k = 2n$ we have

$$\int_0^{2\pi}\cos^{2m+1}\tau\cos 2n\tau\,d\tau = 0, \tag{I.67}$$

and for odd $k = 2n+1$, seeing that

$$\cos^{2\mu+1}(\tau+\pi) = -\cos^{2\mu+1}\tau, \tag{I.68}$$

we have

$$\int_0^{2\pi} \cos^{2m+1}\tau \cos(2n+1)\tau\, d\tau = 4\int_0^{\frac{\pi}{2}} \cos^{2m+1}\tau \cos(2n+1)\tau\, d\tau. \tag{I.69}$$

We can now use the standard relation*

$$\frac{1}{F(x, y)} = \frac{2^{x+y-1}(x+y-1)}{\pi}\int_0^{\frac{\pi}{2}} \cos(x-y)\tau \cos^{x+y-2}\tau\, d\tau, \tag{I.70}$$

where $F(x, y)$ is the Euler integral of the first kind (beta function)

$$F(x, y) = \int_0^1 u^{x-1}(1-u)^{y-1}\, du. \tag{I.71}$$

From (I.70), taking

$$x = m+n+2, \quad y = m-n+1, \tag{I.72}$$

we obtain

$$4\int_0^{\frac{\pi}{2}} \cos^{2m+1}\cos(2n+1)\tau\, d\tau = \frac{4\pi}{2^{2m+2}(2m+2)F(m+n+2, m-n+1)}. \tag{I.73}$$

Seeing that for positive integers a and b

$$\frac{1}{F(a, b)} = \frac{a(a+b-1)!}{(b-1)!\,a!}, \tag{I.74}$$

we finally obtain

$$\left.\begin{aligned}\int_0^{2\pi} \cos^{2m+1}\tau \cos(2n+1)\tau\, d\tau &= \frac{\pi}{2^{2m}}\frac{(2m+1)!}{(m-n)!\,(m+n+1)!}\\ &(m \geqslant n,\ n \geqslant 0).\end{aligned}\right\} \tag{I.75}$$

Similarly

$$\int_0^{2\pi} \cos^{2m}\tau \cos(2n+1)\tau\, d\tau = 0, \tag{I.76}$$

$$\int_0^{2\pi} \cos^{2m}\tau \cos 2n\tau\, d\tau = 2\int_0^{\pi/2} \cos^{2m-1}\tau \cos(2n-1)\tau\, d\tau + 2\int_0^{\pi/2} \cos^{2m-1}\tau \cos(2n+1)\tau\, d\tau, \tag{I.77}$$

* See Whittaker, E. and G. Watson. A Course of Modern Analysis.—Cambridge. 1927.

so that

$$\left.\begin{aligned}\int_0^{2\pi} \cos^{2m}\tau \cos 2n\tau\, d\tau &= \frac{\pi}{2^{2n-1}} \frac{(2m)!}{(m-n)!\,(m+n)!}\\ &(m \geqslant n,\ n \geqslant 0).\end{aligned}\right\} \tag{I.78}$$

Seeing that

$$\int_0^{2\pi} \cos^{\mu}\tau \sin\tau \sin k\tau\, d\tau = \int_0^{2\pi} \cos^{\mu}\tau\,[\cos(k-1)\tau - \cos(k+1)\tau]\, d\tau, \tag{I.79}$$

we also find

$$\left.\begin{aligned}\int_0^{2\pi} \cos^{2m}\tau \sin\tau \sin 2n\tau\, d\tau &= 0,\\ \int_0^{2\pi} \cos^{2m+1}\tau \sin\tau \sin(2n+1)\tau\, d\tau &= 0;\end{aligned}\right\} \tag{I.80}$$

$$\left.\begin{aligned}\int_0^{2\pi} \cos^{2m-1}\tau \sin\tau \sin 2n\tau\, d\tau &=\\ &= \frac{\pi}{2^{2m+1}} \frac{(2m+1)!\,2n}{(m-n+1)!\,(m+n+1)!}\\ &(m \geqslant n-1,\ n \geqslant 1);\end{aligned}\right\} \tag{I.81}$$

$$\left.\begin{aligned}\int_0^{2\pi} \cos^{2m}\tau \sin\tau \sin(2n+1)\tau\, d\tau &= \frac{\pi}{2^{2m}} \frac{(2m)!\,(2n+1)}{(m-n)!\,(m+n+1)!}\\ &(m \geqslant n,\ n \geqslant 0).\end{aligned}\right\} \tag{I.82}$$

Using (I.62), we find the Fourier coefficients of $B(r, \tau)$:

$$\left.\begin{aligned}a_0 &= 2\tilde{a}_1,\\ a_1 &= -\tilde{a}_2 r,\\ b_1 &= \tilde{a}_3 r.\end{aligned}\right\} \tag{I.83}$$

For $n \geqslant 1$, Eqs. (I.75), (I.78), (I.81), (I.82) give

$$\left.\begin{aligned}a_{2n} &= \tilde{a}_4 r^2 \sum_{q=n-1}^{\infty} (r \operatorname{tg} z_*)^{2q} \frac{1}{2^{2q+1}} \frac{(2q+2)!}{(q-n+1)!\,(q+n+1)!},\\ b_{2n} &= -\tilde{a}_5 r^2 \sum_{q=n-1}^{\infty} (r \operatorname{tg} z_*)^{2q} \frac{1}{2^{2q+1}} \frac{(2q+1)!\,2n}{(q-n+1)!\,(q+n+1)!},\\ a_{2n+1} &=\\ &= -\tilde{a}_4 r^2 \sum_{q=n-1}^{\infty} (r \operatorname{tg} z_*)^{2q+1} \frac{1}{2^{2q+2}} \frac{(2q+3)!}{(q-n+1)!\,(q+n+2)!},\\ b_{2n+1} &=\\ &= \tilde{a}_5 r^2 \sum_{q=n-1}^{\infty} (r \operatorname{tg} z_*)^{2q+1} \frac{1}{2^{2q+2}} \frac{(2q+2)!\,(2n+1)}{(q-n+1)!\,(q+n+2)!}.\end{aligned}\right\} \tag{I.84}$$

Eqs. (I.84) can be written in a more convenient form by changing over to a new index $q = l + n - 1$. Then

$$\left.\begin{aligned}
a_{2n} &= \frac{\tilde{a}_4 r^2}{2}\left(\frac{r\,\mathrm{tg}\,z_*}{2}\right)^{2(n-1)}\sum_{l=0}^{\infty}\left(\frac{r\,\mathrm{tg}\,z_*}{2}\right)^{2l}\frac{(2n+2l)!}{l!\,(2n+l)!},\\
b_{2n} &= -\frac{\tilde{a}_5 r^2}{2}\left(\frac{r\,\mathrm{tg}\,z_*}{2}\right)^{2(n-1)}\sum_{l=0}^{\infty}\left(\frac{r\,\mathrm{tg}\,z_*}{2}\right)^{2l}\frac{(2n+2l-1)!\,2n}{l!\,(2n+l)!},\\
a_{2n+1} &= -\frac{\tilde{a}_4 r^2}{2}\left(\frac{r\,\mathrm{tg}\,z_*}{2}\right)^{2n-1}\sum_{l=0}^{\infty}\left(\frac{r\,\mathrm{tg}\,z_*}{2}\right)^{2l}\frac{(2n+2l+1)!}{l!\,(2n+l+1)!},\\
b_{2n+1} &= \frac{\tilde{a}_5 r^2}{2}\left(\frac{r\,\mathrm{tg}\,z_*}{2}\right)^{2n-1}\sum_{l=0}^{\infty}\left(\frac{r\,\mathrm{tg}\,z_*}{2}\right)^{2l}\frac{(2n+2l)!\,(2n+1)}{l!\,(2n+l+1)!}.
\end{aligned}\right\} \tag{I.85}$$

The series in (I.85) converge very rapidly for small r, so that excellent accuracy is ensured by retaining only the first terms.

Fesenkov's scattering function*

$$y_2 = 1 + a\cos\varphi + b\cos^2\varphi + c\cos^3\varphi \tag{I.86}$$

provides a better approximation to atmospheric scattering than Rayleigh's function (I.60). Using this B we obtain

$$\begin{aligned}
B = \tilde{a}_1 - \tilde{a}_2 r\cos\tau + \tilde{a}_3 r\sin\tau + \tilde{a}_4 r^2\cos^2\tau - \tilde{a}_5 r^2\cos\tau\sin\tau + \\
+ \tilde{a}_6 r^3\sum_{q=0}^{\infty}(r\,\mathrm{tg}\,z_*)^{2q+1}\cos^{2q+4}\tau - \tilde{a}_6 r^3\sum_{q=0}^{\infty}(r\,\mathrm{tg}\,z_*)^{2q}\cos^{2q+3}\tau + \\
+ \tilde{a}_7 r^3\sin\tau\sum_{q=0}^{\infty}(r\,\mathrm{tg}\,z_*)^{2q}\cos^{2q+2}\tau - \\
- \tilde{a}_7 r^3\sin\tau\sum_{q=0}^{\infty}(r\,\mathrm{tg}\,z_*)^{2q+1}\cos^{2q+3}\tau.
\end{aligned} \tag{I.87}$$

Here

$$\left.\begin{aligned}
\tilde{a}_1 &= \frac{A}{\cos z_*}(1 + a\cos\varphi_* + b\cos^2\varphi_* + c\cos^3\varphi_*),\\
\tilde{a}_2 &= \frac{A}{\cos z_*}[(1 + a\cos\varphi_* + b\cos^2\varphi_* + c\cos^3\varphi_*)\,\mathrm{tg}\,z_* + \\
&\quad + \sin\varphi_*\cos\varepsilon\,(a + 2b\cos\varphi_* + 3c\cos^2\varphi_*)],\\
\tilde{a}_3 &= \frac{A}{\cos z_*}\sin\varphi_*\sin\varepsilon\,(a + 2b\cos\varphi_* + 3c\cos^2\varphi_*),\\
\tilde{a}_4 &= \frac{A}{\cos z_*}[(1 + a\cos\varphi_* + b\cos^2\varphi_* + c\cos^3\varphi_*)\,\mathrm{tg}^2 z_* + \\
&\quad + \sin\varphi_*\cos\varepsilon\,\mathrm{tg}\,z_*\,(a + 2b\cos\varphi_* + 3c\cos^2\varphi_*) + \\
&\quad + \sin^2\varphi_*\cos 2\varepsilon\,(b + 3c\cos\varphi_*)],\\
\tilde{a}_5 &= \frac{A}{\cos z_*}[\sin\varphi_*\sin\varepsilon\,\mathrm{tg}\,z_*\,(a + 2b\cos\varphi_* + 3c\cos^2\varphi_*) + \\
&\quad + \sin^2\varphi_*\sin 2\varepsilon\,(b + 3c\cos\varphi_*)],\\
\tilde{a}_6 &= \frac{A}{\mathrm{tg}\,z_*}[(1 + a\cos\varphi_* + b\cos^2\varphi_* + c\cos^3\varphi_*)\,\mathrm{tg}^3 z_* + \\
&\quad + \sin\varphi_*\cos\varepsilon\,\mathrm{tg}^2 z_*\,(a + 2b\cos\varphi_* + 3c\cos^2\varphi_*) + \\
&\quad + \sin^2\varphi_*\cos 2\varepsilon\,\mathrm{tg}\,z_*\,(b + 3c\cos\varphi_*) + \\
&\quad + \sin^3\varphi_*\cos\varepsilon\,(c\cos^2\varepsilon - 3c\sin^2\varepsilon)],\\
\tilde{a}_7 &= \frac{A}{\mathrm{tg}\,z_*}[\sin\varphi_*\sin\varepsilon\,\mathrm{tg}^2 z_*\,(a + 2b\cos\varphi_* + 3c\cos^2\varphi_*) + \\
&\quad + \sin^2\varphi_*\sin 2\varepsilon\,\mathrm{tg}\,z_*\,(b + 3c\cos\varphi_*) + \\
&\quad + \sin^3\varphi_*\sin\varepsilon\,(3c\cos^2\varepsilon - c\sin^2\varepsilon)].
\end{aligned}\right\} \tag{I.88}$$

* See the book mentioned in footnote to p. 376.

For the Fourier coefficients we now have

$$\left.\begin{aligned} a_0 &= 2\tilde{a}_1, \\ a_1 &= -\tilde{a}_2 r, \quad b_1 = \tilde{a}_3 r, \\ a_2 &= \frac{\tilde{a}_4 r^2}{2}, \quad b_2 = -\frac{\tilde{a}_5 r^2}{2}, \end{aligned}\right\} \tag{I.89}$$

and for $n \geqslant 1$

$$\left.\begin{aligned} a_{2n+1} &= -\frac{\tilde{a}_6 r^3}{4}\left(\frac{r\,\mathrm{tg}\,z_*}{2}\right)^{2n-2}\sum_{i=0}^{\infty}\left(\frac{r\,\mathrm{tg}\,z_*}{2}\right)^{2i}\frac{(2n+2i+1)!}{i!\,(2n+i+1)!}, \\ b_{2n+1} &= \frac{\tilde{a}_7 r^3}{4}\left(\frac{r\,\mathrm{tg}\,z_*}{2}\right)^{2n-2}\sum_{i=0}^{\infty}\left(\frac{r\,\mathrm{tg}\,z_*}{2}\right)^{2i}\frac{(2n+2i)!\,(2n+1)}{i!\,(2n+i+1)!}, \\ a_{2n} &= \frac{\tilde{a}_6 r^3}{4}\left(\frac{r\,\mathrm{tg}\,z_*}{2}\right)^{2n-3}\sum_{i=0}^{\infty}\left(\frac{r\,\mathrm{tg}\,z_*}{2}\right)^{2i}\frac{(2n+2i)!}{i!\,(2n+i)!}, \\ b_{2n} &= -\frac{\tilde{a}_7 r^3}{4}\left(\frac{r\,\mathrm{tg}\,z_*}{2}\right)^{2n-3}\sum_{i=0}^{\infty}\left(\frac{r\,\mathrm{tg}\,z_*}{2}\right)^{2i}\frac{(2n+2i-1)!\,2n}{i!\,(2n+i)!}. \end{aligned}\right\} \tag{I.90}$$

If only the first terms are retained in each series, the amplitudes A_1, A_2 of the background modulation harmonics are written, using (I.42), in the form

$$A_1 = \frac{r_0^3}{3}\sqrt{\tilde{a}_2^2 + \tilde{a}_3^2}, \quad A_2 = \frac{r_0^4}{8}\sqrt{\tilde{a}_4^2 + \tilde{a}_5^2}, \tag{I.91}$$

and for $k \geqslant 3$ we have

$$A_k = \frac{r_0^{k+2}}{4(k+2)}\left(\frac{\mathrm{tg}\,z_*}{2}\right)^{k-3}\sqrt{\tilde{a}_6^2 + \tilde{a}_7^2}. \tag{I.92}$$

Eqs. (I.41), (I.88), (I.91), (I.92) lead to a number of qualitative and quantitative conclusions concerning the choice of the parameters of the chopping reticle in systems where stars are detected against the background of scattered solar radiation in the atmosphere. These conclusions even permit making some comparative estimates.

If m_m^A is the stellar magnitude of the sighted star, E_0 is the illumination (in luxes) produced by a zero magnitude star, and the coefficient A in (I.51a) gives the brightness in stilbs, we have from (I.41)

$$\lambda_k = \frac{E_0 \cdot 10^{-\frac{m_m^A}{2.5}}}{10^4 \pi A_k}. \tag{I.93}$$

Taking some λ^0 for the ratio of the star signal to the background modulation amplitude at the working frequency, we find

$$A_k \leqslant \frac{E_0 \cdot 10^{-\frac{m_m^A}{2.5}}}{10^4 \pi \lambda^0}. \tag{I.94}$$

This inequality, together with Eqs. (I.91), (I.92), (I.88), establishes a relation

among the following quantities:

A — the background brightness in the direction of the telescope axis,

φ_*, z_* — the working range of angles relative to the directions to the Sun and the zenith,

m_m^A — the minimum stellar magnitude at which the system ensures the specified signal-to-background ratio,

λ^0 — the signal-to-background ratio,

r_0 — the radius of the field of view,

k — the working harmonic index (i.e., the index of the first harmonic which ensures the required signal-to-background ratio); in the final analysis it determines the number of reticle cycles.

If, say, A, φ_*, z_*, m_m^A, λ^0 and r_0 are given, we can find the minimum number of cycles.

The choice of the minimum number of wedges is a highly important problem, since it determines the diameter of the central dead region in the chopper field (remember that the star image in the focal plane of the optical system is a disk of finite size) and thus largely affects the static and the dynamic errors of the system.

It should be further remembered that normally star indication systems operate at a high noise level. Therefore narrow-band filters must be used. On the other hand, the star image gradually moves across the chopper, so that the working frequency is affected (the chopper Doppler effect). If v is the image drift velocity at right angles to the chopper radius, and r_* is the distance of the star image from the chopper center, the working frequency will change by $\Delta(n\omega) = nv/r_*$. This frequency drift, proportional to the number of reticle cycles n, inevitably sets a limit to the minimum filter passband that can be used.

We see from Eqs. (I.88), (I.91), (I.92) that the maximum values of A_k are attained for minimum φ_*, maximum z_*, and $\varepsilon = 0$.

The last condition, as we see from (I.55), (I.56), signifies that the directions to the star, the Sun, and the zenith are coplanar.

For $\varepsilon = 0$

$$\tilde{a}_3 = \tilde{a}_5 = \tilde{a}_7 = 0 \tag{I.95}$$

so that

$$\left.\begin{aligned} A_1 &= \frac{r_0^3 \tilde{a}_2}{3}, \quad A_2 = \frac{r_0^4 \tilde{a}_4}{8}, \\ A_k &= \frac{r_0^{k+2}}{4(k+2)} \left(\frac{\operatorname{tg} z_*}{2}\right)^{k-3} \tilde{a}_6 \qquad (k \geqslant 3). \end{aligned}\right\} \tag{I.96}$$

Consider some numerical estimates. Let $\lambda^0 = 10$, $m_m^A = 3$, $E_0 = 2.3 \cdot 10^{-6}$, $r_0 = 0.01$, $z_* \leqslant 5\pi/12$, $\varphi_* > \pi/9$, $A = 0.05$, $a = 0.05$, $b = 1.1$, $c = 0.7$. Then $\tilde{a}_2 = 2.1$, $\tilde{a}_4 = 8$, $\tilde{a}_6 = 30$, $\operatorname{tg} z_* = 3.7$. Inserting in (I.96), we get

$$A_1 = 7 \cdot 10^{-5}, \quad A_2 = 10^{-8}. \tag{I.97}$$

For $k \geqslant 3$

$$A_k = 0.33 \frac{(0{,}019)^{k+2}}{k+2}. \tag{I.98}$$

Figure A.4 plots $-\log A_k$ as a function of k.

The right-hand side of inequality (I.94) for the above numerical values of the parameters gives

$$\frac{E_\lambda \cdot 10^{-\frac{m_m^A}{2.5}}}{10^4 \pi \lambda^0} = 4.6 \cdot 10^{-13}. \tag{I.99}$$

Minus the logarithm of this number is also marked in Figure A.4.

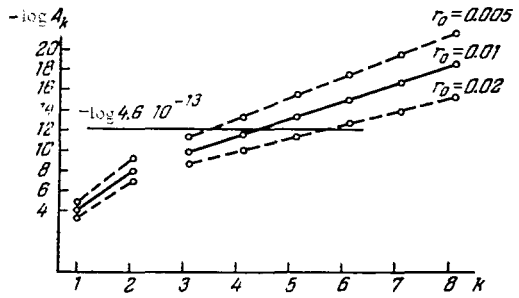

FIGURE A 4.

We see from the diagram that fifth and higher harmonics can be used as the working frequencies in this case.

Harmonics 1 through 4 should either be filtered out by the chopper or suppressed in the photocell output channel.

As another example, let us consider the dependence of the lowest working frequency on the angle of view of the system.

From (I.96) we see that for

$$r_0' = \mu r_0 \tag{I.100}$$

A_k' is related to A_k and μ for all k by the equality

$$A_k' = \mu^{k-2} A_k. \tag{I.101}$$

The plot of $-\log A_k'(k)$ is therefore displaced by $(k+2)\log \mu$ in the vertical direction in relation to the plot of $-\log A_k(k)$.

The dashed line in Figure A.4 gives $-\log A_k'$ for $\mu = 2$ and $\mu = 1/2$, i.e., for $r_0 = 0.02$ and $r_0 = 0.005$. We see from the diagram that changing r_0 by a factor of 2 in either direction involves a simultaneous change of unity in the working frequency.

Note that as λ^0 is changed by a factor of 10, the line

$$\log \frac{E_0 \cdot 10^{-\frac{m_m^A}{2,5}}}{10^4 \pi \lambda^0} = \text{const} \tag{I.102}$$

is displaced by a unit interval along the axis $-\log A_k$.

Let us derive some energy estimates.

The total energy T of all the modulation harmonics is given by Parseval's theorem* in the form

$$T=\sum_{k=1}^{\infty}\left[(a_k^0)^2+(b_k^0)^2\right]=\frac{\omega}{\pi}\int_0^{2\pi/\omega}|P_0(t)|^2-\frac{(a_0^0)^2}{2}. \tag{I.103}$$

A similar relation holds for the background harmonics.

For any reticle

$$T=2p^2\alpha(1-\alpha), \tag{I.104}$$

where α is the ratio of the transparent to the filled area, i.e.,

$$\alpha=\frac{1}{2\pi}\sum_{j=0}^{n-1}(\tau_{j+1}-\tau_j). \tag{I.105}$$

T is maximum for $\alpha=1/2$.

A uniform reticle, a chopping semidisk, and a dual-cycle reticle have $\alpha=1/2$, and the total energy of the modulated signal reaches its maximum value

$$T=0.5p^2. \tag{I.106}$$

For a semidisk reticle $\alpha=0.25$ and thus

$$T=0.375p^2. \tag{I.107}$$

For a single-cycle chopper $\alpha=\tau_0/2\pi$ and

$$T=\frac{p^2\tau_0}{\pi}\left(1-\frac{\tau_0}{2\pi}\right). \tag{I.108}$$

If the working frequencies comprise l harmonics starting with the m-th harmonic, we have

$$T_1=\sum_{k=m}^{m+l}\left[(a_k^0)^2+(b_k^0)^2\right]. \tag{I.109}$$

For example, if three harmonics are used starting with the fifth, i.e., the frequencies 5ω, 6ω, 7ω, we have for a reticle with $\tau_0=\pi/6$

$$T_1=0.034p^2. \tag{I.110}$$

The total energy of this slit by (I.108) is $\approx 0.15p^2$.

For a semidisk (with working frequencies 5ω and 6ω) we find

$$T_1=0.024p^2. \tag{I.111}$$

For a six-cycle semidisk reticle ($n=6$) we have at the working frequencies 5ω, 6ω, 7ω

$$T_1=\frac{p^2}{\pi^2}\left[1+\frac{2(n^2+1)}{n^2-1}\frac{1}{\sin^2\frac{\pi}{2n}}\right]\approx 0.19p^2. \tag{I.112}$$

* See Tolstov, G.P. Ryady Fur'e (Fourier Series). — Gostekhizdat. 1951.

For a dual-cycle reticle with $n = 6$, $m = 8$, i.e., for working frequencies 5ω, 6ω, 7ω, 8ω, 9ω, we get

$$T_1 = \frac{p^2}{\pi^2}\left[2 + \frac{1}{(n-1)^2}\left(\operatorname{ctg}\frac{\pi}{2n} - \operatorname{ctg}\frac{3\pi}{2m}\right)^2 + \right.$$
$$+ \frac{1}{(n+1)(m-1)}\left(\operatorname{ctg}\frac{\pi}{2n} + \operatorname{ctg}\frac{\pi}{2m}\right)^2 +$$
$$\left. + \frac{1}{(m+1)^2}\left(\operatorname{ctg}\frac{\pi}{2m} - \operatorname{ctg}\frac{3\pi}{2n}\right)^2\right] \approx 0.39p^2. \qquad \text{(I.113)}$$

Finally, for a six-cycle uniform reticle ($n = 6$) and a single harmonic 6ω we have

$$T_1 = \frac{4p^2}{\pi^2} \approx 0.4p^2. \qquad \text{(I.114)}$$

Note that estimate (I.114) is independent of n, and estimates (I.112) and (I.113) for $n \geqslant 5$ are fairly insensitive to n and m, since for large n we have the asymptotic approximations

$$T_1 = \frac{p^2}{\pi^2}\left(1 + \frac{8}{\pi^2}\right) \approx 0.18p^2 \qquad \text{(I.115)}$$

in place of (I.112) and

$$T_1 = \frac{p^2}{\pi^2}\left(2 + \frac{176}{9\pi^2}\right) \approx 0.395p^2 \qquad \text{(I.116)}$$

in place of (I.113).

Therefore, in practice, for $n \geqslant 5$, estimates (I.112), (I.113) or (I.115), (I.116) are independent of n and m.

The above numerical estimates show that the single-cycle reticle and the semidisk at frequencies $k \geqslant 5$ are highly disadvantageous from the point of view of energy.

A uniform reticle, on the other hand, ensures a nearly perfect utilization of the energy of the modulated signal, even if only a single harmonic is used ($0.4p^2$, compared to the theoretical maximum of $0.5p^2$).

The semidisk reticle, as we see from (I.112), (I.115), passes at the frequencies $(n-1)\omega$, $n\omega$, $(n+1)\omega$ less than half the energy passed by the uniform reticle at the single frequency $n\omega$ and 0.75 of the energy passed by the semidisk.

Comparison of estimates (I.113), (I.116) with (I.114) shows that at the frequencies $(n-1)\omega$, $n\omega$, $(m-1)\omega = (n+1)\omega$, $m\omega$, $(m+1)\omega$ the dual-cycle reticle passes virtually the same energy as the uniform reticle at the single frequency $n\omega$.

Note that the feasibility of the energy relations above is determined by the entire signal reduction channel. The final energy estimates of various choppers should therefore take into consideration the signal reduction circuits and the high level of extraneous noise in the system.

In conclusion, let us estimate the error associated with the truncation of the series (I.90) after the first term.

Consider the sum entering the expression (I.90) for a_{2n-1}:

$$\sum_{i=0}^{\infty}\left(\frac{r\operatorname{tg} z_*}{2}\right)^{2i}\frac{(2n+2i+1)!}{i!(2n+i+1)!} = 1 + \sum_{i=1}^{\infty}\left(\frac{r\operatorname{tg} z_*}{2}\right)^{2i}\frac{(2n+2i+1)!}{i!(2n+i+1)!}. \qquad \text{(I.117)}$$

Writing y for the sum on the left, we see that

$$y - \left(\frac{r \operatorname{tg} z_*}{2}\right) My \leqslant 1, \tag{I.118}$$

where

$$M = \max \frac{(2n+2j+2)(2n+2j+3)}{(j+1)(2n+j+2)} = 2n+3. \tag{I.119}$$

From (I.118), (I.119) we have

$$y \leqslant \frac{1}{1-(2n+3)\left(\frac{r \operatorname{tg} z_*}{2}\right)^2}. \tag{I.120}$$

Similarly, for the sums entering the expressions for b_{2n+1}, a_{2n}, b_{2n} we find

$$\left.\begin{aligned}
&\sum_{i=0}^{\infty}\left(\frac{r \operatorname{tg} z_*}{2}\right)^{2i}\frac{(2n+2i)!\,(2n+1)}{i!\,(2n+i+1)!} \leqslant \frac{1}{1-(2n+1)\left(\frac{r \operatorname{tg} z_*}{2}\right)^2},\\
&\sum_{i=0}^{\infty}\left(\frac{r \operatorname{tg} z_*}{2}\right)^{2i}\frac{(2n+2i)!}{i!\,(2n+i)!} \leqslant \frac{1}{1-(2n+2)\left(\frac{r \operatorname{tg} z_*}{2}\right)^2},\\
&\sum_{i=0}^{\infty}\left(\frac{r \operatorname{tg} z_*}{2}\right)^{2i}\frac{(2n+2i-1)!\,2n}{i!\,(2n+1)!} \leqslant \frac{1}{1-2n\left(\frac{r \operatorname{tg} z_*}{2}\right)^2}.
\end{aligned}\right\} \tag{I.121}$$

For $n = 12$, $r = 0.02$, $\operatorname{tg} z_* = 4$ we have

$$\frac{1}{1-(2n+3)\left(\frac{r \operatorname{tg} z_*}{2}\right)^2} \approx 1 + 0.04. \tag{I.122}$$

Error estimates for partial sums containing two and more terms are obtained along the same lines.

Appendix II

*NAVIGATION USING A DIRECTIONAL GYROSCOPE AND A DOPPLER VELOCITY METER**

Consider the navigation of an object along a trajectory using a directional gyro with latitude compensation at the starting point. To ensure that the object actually reaches the end point of the programmed trajectory, the directional gyro is initially deflected by a certain angle from the orthodromy passing through the starting and the terminal point of the trajectory. The arrival time is fixed by comparing the integrated readings of the Doppler velocity meter with the calculated range.

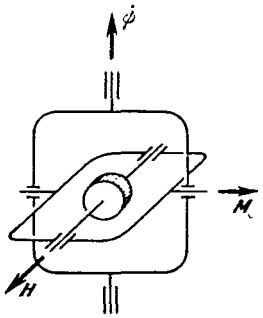

FIGURE A.5.

We will determine the initial parameters of motion and estimate some of the errors of this navigation technique associated with inaccuracies in the initial setting of the parameters.

Let the object move at a constant distance R_0 from the Earth's center; its velocity $v(t)$ is a known function of time.

The equations of motion are written for the schematic diagram of the directional gyro in a three-degrees-of-freedom gimbal. The axis of the outer gimbal is vertical: it fixes the course angle. The angular momentum vector H of the gyroscope lies in the plane of the horizon. The mutually perpendicular orientation of the two gimbals is ensured by a special frame correction arrangement (not shown in the figure). The latitude error of the gyro is compensated by a correcting torque M_x, applied to the inner gimbal axis.

We introduce a right orthogonal system $O\xi\eta\zeta$ fixed to the Earth. The origin of this system is at the Earth's center. The axis $O\zeta$ points along the Earth's spin vector ω, and the axis $O\xi$ is the intersection line of the equator with the plane of the Greenwich meridian.

The position of an object on the surface of a concentric sphere s of radius R_0 spinning together with the Earth is described by spherical coordinates: the longitude λ reckoned in the plane of the equator from the axis $O\xi$ and the latitude φ reckoned from the equatorial plane.

The coordinates of the starting and the terminal point are λ_1, φ_1 and λ_2, φ_2, respectively.

A Darboux trihedron O_1xyz with its axes directed to the points of the compass is associated with the current position of the moving object. Its origin is at the point O_1 currently occupied by the object on the sphere s.

* Andreev, V.D. – Izvestiya AN SSSR, tech. cyber., No. 6, 1963.

The axis O_1y is directed up along the geocentric vertical, and the axis O_1x points north along the meridian.

The mutual orientation of the axes $O\xi\eta\zeta$ and O_1xyz is described by the following matrix of direction cosines:

$$\begin{array}{c|ccc} & x & y & z \\ \xi & -\sin\varphi\cos\lambda & \cos\varphi\cos\lambda & -\sin\lambda \\ \eta & -\sin\varphi\sin\lambda & \cos\varphi\sin\lambda & \cos\lambda \\ \zeta & \cos\varphi & \sin\varphi & 0. \end{array} \tag{II.1}$$

We introduce the co-moving axes $O_1x_1y_1z_1$ with the origin at the point O_1. The axis O_1y_1 is directed up along the vertical, and the axis O_1x_1 points along the velocity vector v of the object. The orientation of the axes $O_1x_1y_1z_1$ relative to O_1xyz is described by the following matrix:

$$\begin{array}{c|ccc} & x_1 & y_1 & z_1 \\ x & \cos\psi & 0 & -\sin\psi \\ y & 0 & 1 & 0 \\ z & \sin\psi & 0 & \cos\psi. \end{array} \tag{II.2}$$

We see from (II.2) that the positive angles ψ are reckoned from north to east.

If the outer gimbal axis is directed along the axis O_1y_1 (O_1y), the two gimbal axes meet at the point O_1. Seeing that the projection of the absolute angular velocity of the axes $O_1x_1y_1z_1$ on the axis O_1y_1 is

$$\omega_{y_1} = (\omega + \dot{\lambda})\sin\varphi - \dot{\psi}, \tag{II.3}$$

we write the equations of motion of the object and the directional gyro in the axes $O\xi\eta\zeta$ in the form

$$\dot{\varphi} = \frac{v}{R_0}\cos\psi, \quad \dot{\lambda} = \frac{v\sin\psi}{R_0\cos\varphi}, \quad \dot{\psi} = (\omega + \dot{\lambda})\sin\varphi - \frac{M_c}{H}. \tag{II.4}$$

The initial conditions for Eqs. (II.4) are

$$\varphi(0) = \varphi_1, \quad \lambda(0) = \lambda_1, \quad \psi(0) = \psi_0. \tag{II.5}$$

In case of continuous latitudinal compensation of the directional gyro, we have

$$\frac{M_c}{H} = \omega\sin\varphi, \tag{II.6}$$

and the equations of motion along the orthodromy take the form

$$\dot{\varphi} = \frac{v}{R_0}\cos\psi, \quad \dot{\lambda} = \frac{v\sin\psi}{R_0\cos\varphi}, \quad \dot{\psi} = \dot{\lambda}\sin\varphi. \tag{II.7}$$

The initial value $\psi(0)$ in this case is the angle ψ_0 that the orthodromy through the starting (φ_1, λ_1) and the terminal (φ_2, λ_2) points of the trajectory makes with the axis O_1x at the point $(\varphi_1\lambda_1)$, so that

$$\left.\begin{aligned} \sin\psi_0 &= \frac{\cos\varphi_2\sin(\lambda_2 - \lambda_1)}{\sin S_0}, \\ \cos\psi_0 &= \frac{\sin\varphi_2 - \sin\varphi_1\cos S_0}{\cos\varphi_1\sin S_0}, \end{aligned}\right\} \tag{II.8}$$

where S_0 is the angle between the directions to the points (φ_1, λ_1) and (φ_2, λ_2) in the orthodromy plane, given by

$$\cos S_0 = \cos\varphi_1 \cos\varphi_2 \cos(\lambda_2 - \lambda_1) + \sin\varphi_1 \sin\varphi_2. \tag{II.9}$$

In the case under discussion, the latitudinal correction is applied only at the starting point, i.e.,

$$\frac{M}{H} = \omega \sin\varphi_1. \tag{II.10}$$

The equations of motion (II.4) thus take the form

$$\left.\begin{aligned} \dot{\varphi} &= \frac{v}{R_0}\cos\psi, \quad \dot{\lambda} = \frac{v\sin\psi}{R_0\cos\varphi}, \\ \dot{\psi} &= (\omega + \dot{\lambda})\sin\varphi - \omega\sin\varphi_1. \end{aligned}\right\} \tag{II.11}$$

The initial conditions for Eqs. (II.11) are

$$\varphi(0) = \varphi_1, \quad \lambda(0) = \lambda_1, \quad \psi(0) = \psi_0 + \Delta\psi_0. \tag{II.12}$$

In (II.12), ψ_0 is defined by (II.8) and $\Delta\psi_0$ should be determined from the condition that the trajectory passes through the terminal point, e.g., from the equality

$$\lambda(\varphi_2) = \lambda_2. \tag{II.13}$$

If t_1 is the time when condition (II.13) is satisfied, the angular range S_1 between the starting and the terminal point along the trajectory is given by the integral

$$S_1 = \int_0^{t_1} \sqrt{\dot{\varphi}^2 + \dot{\lambda}^2\cos^2\varphi}\, dt = \int_0^{t_1} \frac{v}{R_0}\, dt. \tag{II.14}$$

The angular range S_1 naturally may differ from the range S_0 specified by Eq. (II.9).

Before proceeding with the solution of Eqs. (II.11) and the determination of the starting parameters of motion, let us consider in some detail the simplifying assumptions introduced in the derivation of Eqs. (II.11).

The object was assumed to move over the surface of a sphere s of radius R_0 concentric with the Earth and sharing its axial rotation. The outer gimbal axis was further assumed to be directed along the axis O_1y_1 (O_1y), i.e., along the normal to the surface of the sphere s, whereas in systems where the outer gimbal axis is caged to a gyrovertical with pendulum correction, it slightly deviates from the direction O_1y_1.

In the derivation of Eqs. (II.11) the velocity vector v was taken to coincide with the longitudinal axis; the corresponding direction in the system O_1xyz was defined by the gyro output axis ψ. In fact, however, the longitudinal axis O_1x_1' may drift from the direction O_1x_1 through an angle

$$\beta = \frac{v_{z_1}'}{v_{x_1}'}. \tag{II.15}$$

This drift may be caused by transverse wind, Coriolis forces, imperfect

geometry of the object, and other factors. The drift angle β is measured with a Doppler velocity meter and is compensated by the guidance system, so that in fact we have in Eqs. (II.11)

$$v(t) = v_c(t)\cos\beta. \tag{II.16}$$

It is readily seen that, under certain restrictions on the angular range S_0, the above simplifications do not lead to noticeable errors in the determination of ψ_0 and $\Delta\psi_0$.

In the calculation of ψ_0 and $\Delta\psi_0$, $\cos\beta$ can be taken equal to unity and R_0 can be set equal, say, to the maximum radius of curvature of the geoid. When the range S_1 is measured, however, the Earth's flattening should be taken into consideration, since the Doppler velocity meter gives the projection of the object's velocity relative to the Earth onto a plane tangent to the geoid.*

For fairly large β, Eq. (II.16) should be taken into consideration when integrating the longitudinal Doppler readings. Let us return to Eqs. (II.11). For constant $v = v_0$ this system can be integrated, at least in quadratures.

Indeed, inserting $\dot\lambda$ from the second equation in (II.11) in the third and dividing through the third equation by the first, we obtain the following relation between ψ and φ:

$$\frac{d(\sin\psi)}{d\varphi} - \sin\psi\operatorname{tg}\varphi = \frac{R_0\omega}{v_0}(\sin\varphi - \sin\varphi_1). \tag{II.17}$$

This is a linear differential equation in $\sin\psi$. Its solution is

$$\sin\psi = -\frac{\omega R_0}{v_0\cos\varphi}\left(\frac{\cos 2\varphi}{4} + \sin\varphi\sin\varphi_1\right) + C_1. \tag{II.18}$$

$\varphi(t)$ is obtained from the equation

$$\frac{v_0}{R_0}t + C_2 = \int_{\varphi_1}^{\varphi}\frac{d\varphi}{\sqrt{1-\sin^2\psi}}, \tag{II.19}$$

where $\sin\psi$ is defined by Eq. (II.18).

Given $\varphi(t)$, we find $\psi(t)$ from (II.18), and from the second equation in (II.11) we get

$$\lambda(t) = \frac{v_0}{R_0}\int_0^t\frac{\sin\psi}{\cos\varphi}dt + \lambda_1. \tag{II.20}$$

The application of the quadratures (II.19), (II.20) to the determination of the initial parameters of motion is not easy, however: first, the integrands are fairly complex and second, no immediate technique is available for their generalization to the case of variable v.

We will therefore derive an approximate solution of Eqs. (II.11), remembering that the solutions of Eqs. (II.11) are close to the solutions of Eqs. (II.7). We will thus consider Eqs. (II.11) in a coordinate system fixed to the orthodromy through the points (φ_1, λ_1), (φ_2, λ_2), which is a solution of Eqs. (II.7).

The orthodromic coordinates $O\xi_1\eta_1\zeta_1$ are defined as follows. The axis $O\xi_1$ is directed from the Earth's center to the starting point (φ_1, λ_1), the axis $O\eta_1$

* See §5.2 (page 193).

is normal to the orthodromy plane, and the axis $O\xi_1$ lies in the orthodromy plane, pointing in the direction of the terminal point. The mutual orientation of the axes $O\xi\eta\zeta$ and $O\xi_1\eta_1\zeta_1$ is described by the following matrix of direction cosines:

$$\begin{array}{c|ccc} & \xi_1 & \eta_1 & \zeta_1 \\ \xi & \beta_{11} & \beta_{12} & \beta_{13} \\ \eta & \beta_{21} & \beta_{22} & \beta_{23} \\ \zeta & \beta_{31} & \beta_{32} & \beta_{33}. \end{array} \tag{II.21}$$

Clearly,

$$\beta_{13} = \cos\varphi_1 \cos\lambda_1, \quad \beta_{23} = \cos\varphi_1 \sin\lambda_1, \quad \beta_{33} = \sin\varphi_1. \tag{II.22}$$

The other β_{ik} are determined from the equalities

$$\xi_1 = \frac{\rho_2 - \rho_1 \cos S_0}{\sin S_0}, \quad \eta_1 = \frac{\rho_1 \times \rho_2}{\sin S_0}, \tag{II.23}$$

where ξ_1, η_1 are the unit vectors of the axes $O\xi_1$, $O\eta_1$, and ρ_1, ρ_2 are the unit vectors in the directions from the point O to the starting and the terminal point, respectively.

From (II.23) we have

$$\left.\begin{aligned}
\beta_{11} &= \frac{1}{\sin S_0}(\cos\varphi_2 \cos\lambda_2 - \cos\varphi_1 \cos\lambda_1 \cos S_0) = \\
&= -\sin\varphi_1 \cos\lambda_1 \cos\psi_0 - \sin\lambda_1 \sin\psi_0, \\
\beta_{21} &= \frac{1}{\sin S_0}(\cos\varphi_2 \sin\lambda_2 - \cos\varphi_1 \sin\lambda_1 \cos S_0) = \\
&= -\sin\varphi_1 \sin\lambda_1 \cos\psi_0 + \cos\lambda_1 \sin\psi_0, \\
\beta_{31} &= \frac{1}{\sin S_0}(\sin\varphi_2 - \sin\varphi_1 \cos S_0) = \cos\varphi_1 \cos\psi_0, \\
\beta_{12} &= \frac{1}{\sin S_0}(\cos\varphi_1 \sin\lambda_1 \sin\varphi_2 - \cos\varphi_2 \sin\lambda_2 \sin\varphi_1) = \\
&= -\sin\varphi_1 \cos\lambda_1 \sin\psi_0 + \cos\psi_0 \sin\lambda_1, \\
\beta_{22} &= \frac{1}{\sin S_0}(\sin\varphi_1 \cos\varphi_2 \cos\lambda_2 - \cos\varphi_1 \sin\varphi_2 \cos\lambda_1) = \\
&= -\sin\varphi_1 \sin\lambda_1 \sin\psi_0 - \cos\lambda_1 \cos\psi_0, \\
\beta_{32} &= \frac{1}{\sin S_0}\cos\varphi_1 \cos\varphi_2 \sin(\lambda_2 - \lambda_1) = \cos\varphi_1 \sin\psi_0.
\end{aligned}\right\} \tag{II.24}$$

The position of the object in the axes $O\xi_1\eta_1\zeta_1$ is determined by its angular range S along the orthodromy and the distance μ from the orthodromy; the range S is reckoned from the starting point of the trajectory; positive μ are taken in the direction from the orthodromy to the axis $O\eta_1$.

If ρ is the unit vector in the direction to the current position of the object on the sphere s, its direction cosines

$$\cos\mu \sin S, \quad \sin\mu, \quad \cos\mu \cos S \tag{II.25}$$

in the axes $O\xi_1\eta_1\zeta_1$ and the direction cosines

$$\cos\varphi \cos\lambda, \quad \cos\varphi \sin\lambda, \quad \sin\varphi \tag{II.26}$$

in the axes $O\xi\eta\zeta$ are related, in virtue of matrix (II.21), by the equalities

$$\left.\begin{aligned}\cos\varphi\cos\lambda &= \beta_{11}\cos\mu\sin S+\beta_{12}\sin\mu+\beta_{13}\cos\mu\cos S,\\ \cos\varphi\sin\lambda &= \beta_{21}\cos\mu\sin S+\beta_{22}\sin\mu+\beta_{23}\cos\mu\cos S,\\ \sin\varphi &= \beta_{31}\cos\mu\sin S+\beta_{32}\sin\mu+\beta_{33}\cos\mu\cos S,\end{aligned}\right\} \tag{II.27}$$

which express the coordinates φ and λ in terms of S and μ.

The equations of motion (II.11) in the axes $O\xi_1\eta_1\zeta_1$ take the form

$$\left.\begin{aligned}\dot{\mu} &= a(t)\sin\theta,\\ \dot{S} &= \frac{a(t)\cos\theta}{\cos\mu},\\ \dot{\theta} &= -\omega(\sin\varphi-\sin\varphi_1),\end{aligned}\right\} \tag{II.28}$$

where

$$\dot{\theta} = -\dot{\psi}+\dot{\lambda}\sin\varphi,\quad a(t)=\frac{v(t)}{R_0}. \tag{II.29}$$

Inserting $\sin\varphi$ from (II.27), (II.24), (II.22), we obtain

$$\left.\begin{aligned}\dot{S} &= \frac{a(t)\cos\theta}{\cos\mu},\\ \dot{\mu} &= a(t)\sin\theta,\\ \dot{\theta} &= -\omega(\cos\varphi_1\cos\psi_0\cos\mu\sin S+\\ &\quad+\cos\varphi_1\sin\psi_0\sin\mu+\sin\varphi_1\cos\mu\cos S-\sin\varphi_1).\end{aligned}\right\} \tag{II.30}$$

The initial conditions for Eqs. (II.30) are

$$\left.\begin{aligned}&S(0)=0,\quad \mu(0)=0,\\ &\theta(0)=\theta_0=\psi_0-\psi_{(0)}=-\Delta\psi_0.\end{aligned}\right\} \tag{II.31}$$

The trajectory will pass through the terminal point $(\varphi_2,\ \lambda_2)$ if the error $\theta_0=-\Delta\psi_0$ is determined from the condition

$$\mu(S_0)=0. \tag{II.32}$$

The angles μ and θ being small, we can replace Eqs. (II.30) by

$$\left.\begin{aligned}\dot{S} &= a(t),\\ \dot{\theta} &= -\omega(\cos\varphi_1\cos\psi_0\sin S+\sin\varphi_1\cos S-\sin\varphi_1),\\ \dot{\mu} &= a(t)\,\theta.\end{aligned}\right\} \tag{II.33}$$

Eqs. (II.33), as is readily seen, can be integrated successively, one after the other, giving

$$\left.\begin{aligned}S &= \int_0^t a(t)\,dt,\\ \theta &= -\omega\int_0^t(\cos\varphi_1\cos\psi_0\sin S+\sin\varphi_1\cos S-\sin\varphi_1)\,dt+\theta_0,\\ \mu &= \int_0^t a(t)\,\theta\,dt.\end{aligned}\right\} \tag{II.34}$$

If we take

$$\left.\begin{aligned} a(t) &= \frac{v(t)}{R_0} \quad \text{for } t \leqslant t_*, \\ a(t) &= \frac{v_0}{R_0} = a_0 = \text{const for } t > t_*. \end{aligned}\right\} \tag{II.35}$$

insertion in the first equation in (II.34) gives

$$\left.\begin{aligned} S &= \int_0^t a(t)\,dt \qquad (t \leqslant t_*), \\ S &= \int_0^{t_*} a(t)\,dt + a_0(t - t_*) \qquad (t > t_*). \end{aligned}\right\} \tag{II.36}$$

From the second equality in (II.34) we have for $t \leqslant t_*$

$$\begin{aligned} \theta = &-\omega \sin\varphi_1 \int_0^t \left(\cos \int_0^t a(t)\,dt\right) dt - \\ &-\omega \cos\varphi_1 \cos\psi_0 \int_0^t \left(\sin \int_0^t a(t)\,dt\right) dt + \omega t \sin\varphi_1 + \theta_0. \end{aligned} \tag{II.37}$$

For $t > t_*$ we get

$$\begin{aligned} \theta = &\omega \sin\varphi_1 \left(\frac{\sin S_*}{a_0} - q_1^*\right) - \omega \cos\varphi_1 \cos\psi_0 \left(\frac{\cos S_*}{a_0} + q_2^*\right) - \\ &-\frac{\omega}{a_0} \sin\varphi_1 \sin[S_* + a(t - t_*)] + \\ &+\frac{\omega}{a_0} \cos\varphi_1 \cos\psi_0 \cos[S_* + a(t - t_0)] + \omega t \sin\varphi_1 + \theta_0, \end{aligned} \tag{II.38}$$

where

$$\left.\begin{aligned} S_* &= \int_0^{t_*} a(t)\,dt, \quad q_1^* = \int_0^{t_*} \left(\cos \int_0^t a(t)\,dt\right) dt, \\ q_2^* &= \int_0^{t_*} \left(\sin \int_0^t a(t)\,dt\right) dt. \end{aligned}\right\} \tag{II.39}$$

Finally, from the third equality in (II.34), we get for $t \leqslant t_*$

$$\begin{aligned} \mu = &-\omega \sin\varphi_1 \int_0^t \left[a(t) \int_0^t \left(\cos \int_0^t a(t)\,dt\right) dt\right] dt - \\ &-\omega \cos\varphi_1 \cos\psi_0 \int_0^t \left[a(t) \int_0^t \left(\sin \int_0^t a(t)\,dt\right) dt\right] dt + \\ &+\omega \sin\varphi_1 \int_0^t t a(t)\,dt + \theta_0 \int_0^t a(t)\,dt, \end{aligned} \tag{II.40}$$

and for $t > t_*$

$$\begin{aligned} \mu = &\omega \sin\varphi_1 \left(p_3^* - p_1^* - \frac{\cos S_*}{a_0} - \frac{a_0 t_*^2}{2}\right) - \\ &-\omega \cos\varphi_1 \cos\psi_0 \left(p_2^* + \frac{\sin S_*}{a_0}\right) + a_0 \left[\omega \sin\varphi_1 \left(\frac{\sin S_*}{a_0} - q_1^*\right) - \right. \\ &\left. -\omega \cos\varphi_1 \cos\psi_0 \left(\frac{\cos S_*}{a_0} + q_2^*\right)\right](t - t_*) + \frac{\omega}{a_0} \sin\varphi_1 \cos\left[S_* + a_0(t - t_*)\right] + \\ &+\frac{\omega}{a_0} \cos\varphi_1 \cos\psi_0 \sin[S_* + a_0(t - t_*)] + a_0 \omega \frac{t^2}{2} \sin\varphi_1 + \theta_0 [S_* + a_0(t - t_*)]. \end{aligned} \tag{II.41}$$

Here

$$\left.\begin{aligned} p_1^* &= \int_0^{t_*}\left[a(t)\int_0^t\left(\cos\int_0^t a(t)\,dt\right)dt\right]dt, \\ p_2^* &= \int_0^{t_*}\left[a(t)\int_0^t\left(\sin\int_0^t a(t)\,dt\right)dt\right]dt, \quad p_3^* = \int_0^{t_*} t a(t)\,dt. \end{aligned}\right\} \tag{II.42}$$

We can now find θ_0. Suppose that at the time of arrival t_1 at the terminal point, when we should have

$$\mu(t_1) = 0, \tag{II.43}$$

the following inequality is satisfied:

$$t > t_*. \tag{II.44}$$

Since

$$S(t_1) = S_0, \tag{II.45}$$

we have

$$t_1 - t_* = \frac{S_0 - S_*}{a_0}, \quad t_1 = \frac{S_0 - S_*}{a_0} + t_*. \tag{II.46}$$

Inserting $t = t_1$ in (II.41), we obtain after some simplifications

$$\begin{aligned} \theta_0 = \frac{\omega\cos\varphi_1\cos\psi_0}{S_0}\Big[&\frac{\sin S_* - \sin S_0}{a_0} + \\ &+\Big(q_2^* + \frac{\cos S_*}{a_0}\Big)(S_0 - S_*) + p_2^*\Big] + \\ + \frac{\omega\sin\varphi_1}{S_0}\Big[&\frac{\cos S_* - \cos S_0}{a_0} + \Big(q_1^* - \frac{\sin S_*}{a_0}\Big)(S_0 - S_*) + \\ &+ t_*(S_* - S_0) - \frac{1}{2a_0}(S_0 - S_*)^2 - p_3^* + p_1^*\Big]. \end{aligned} \tag{II.47}$$

Let us estimate the errors associated with the substitution of Eqs. (II.33) for Eqs. (II.30).

Subtracting Eqs. (II.33) from Eqs. (II.30), we obtain for the differences $\Delta\mu$, ΔS, $\Delta\theta$ between the two sets of solutions

$$\left.\begin{aligned} \Delta\dot{S} &= a_0\Big(1 - \frac{\cos\theta}{\cos\mu}\Big), \\ \Delta\dot{\theta} &= \omega\,\Delta S(\cos\varphi_1\cos\psi_0\cos S + \sin\varphi_1\sin S) - \\ &- \omega(1 - \cos\mu)(\cos\varphi_1\cos\psi_0\sin S - \sin\varphi_1\cos S) - \\ &- \omega\sin\mu\cos\varphi_1\sin\psi_0, \\ \Delta\dot{\mu} &= a(\theta - \sin\theta + \Delta\theta). \end{aligned}\right\} \tag{II.48}$$

Seeing that

$$\left.\begin{aligned} |\cos\varphi_1\cos\psi_0\cos S + \sin\varphi_1\sin S| &\leqslant 1, \\ |\cos\varphi_1\cos\psi_0\sin S - \sin\varphi_1\cos S| &\leqslant 1, \end{aligned}\right\} \tag{II.49}$$

and retaining only the leading terms on the right in (II.48), we obtain

$$\left.\begin{aligned} &|\Delta S| \leqslant S_0 \max\frac{\theta^2}{2},\\ &|\Delta\theta| < \frac{\omega S_0}{a_0}\left(S_0 \max\frac{\theta^2}{2} + \max|\mu|\right),\\ &|\Delta\mu| < S_0\left(\max\frac{|\theta^3|}{6} + |\Delta\theta|\right). \end{aligned}\right\} \tag{II.50}$$

To obtain estimates for the right-hand sides of (II.50), we take S, θ and μ from Eqs. (II.34) for constant $a = a_0$:

$$\left.\begin{aligned} &S = a_0 t,\\ &\theta = \theta_0 - \omega\left(\sin\varphi_1\frac{\sin a_0 t}{a_0} - \cos\varphi_1\cos\psi_0\frac{\cos a_0 t - 1}{a_0} - t\sin\varphi_1\right),\\ &\mu = a_0\theta_0 t + \omega\left(\sin\varphi_1\frac{\cos a_0 t - 1}{a_0} + \right.\\ &\qquad\left. + \cos\varphi_1\cos\psi_0\frac{\sin a_0 t}{a_0} + \frac{a_0 t^2}{2}\sin\varphi_1 - t\cos\varphi_1\cos\psi_0\right). \end{aligned}\right\} \tag{II.51}$$

where

$$\theta_0 = \frac{\omega\cos\varphi_1\cos\psi_0}{a_0}\left(1 - \frac{\sin S_0}{S_0}\right) - \frac{\omega}{a_0}\sin\varphi_1\left(\frac{S_0}{2} - \frac{\cos S_0 - 1}{S_0}\right). \tag{II.52}$$

From (II.51) and (II.52) we have

$$\left.\begin{aligned} &\max|\theta| \approx |\theta(S_0)| \approx \frac{\omega S_0^2}{3a_0},\\ &\max|\mu| \approx \left|\mu\left(\frac{S_0}{\sqrt{3}}\right)\right| \approx \frac{\omega S_0^3}{9a_0\sqrt{3}}. \end{aligned}\right\} \tag{II.53}$$

Hence

$$\left.\begin{aligned} &|\Delta S| \leqslant \frac{\omega^2 S_0^5}{18a_0^2},\\ &|\Delta\theta| \leqslant \frac{\omega^2 S_0^4}{9a_0^2}\left(\frac{1}{\sqrt{3}} + \frac{\omega S_0^2}{2a_0}\right),\\ &|\Delta\mu| \leqslant \frac{\omega^2 S_0^5}{9a_0^2}\left(\frac{1}{\sqrt{3}} + \frac{\omega S_0^2}{2a_0}\right). \end{aligned}\right\} \tag{II.54}$$

Eq. (II.47) for θ_0 can be simplified. Series expanding $\sin S_0$, $\cos S_0$ in powers of S_0 and retaining only terms to third order in S_0, we find

$$\theta_0 = \frac{\omega\cos\varphi_1\cos\psi_0}{6a_0}\left(S_0^2 + \varepsilon_1 + \frac{\varepsilon_2}{S_0}\right) - \frac{\omega\sin\varphi_1}{24a_0}\left(S_0^3 + \varepsilon_3 + \frac{\varepsilon_4}{S_0}\right). \tag{II.55}$$

Here ε_1, ε_2, ε_3, ε_4 are constants characterizing the initial part of the trajectory. They are expressed in terms of S_*, t_*, q_1^*, q_2^*, p_1^*, p_2^*, p_3^* by the following equalities

$$\left.\begin{aligned} &\varepsilon_1 = 6(a_0 q_2^* + \cos S_* - 1),\\ &\varepsilon_2 = 6(\sin S_* - S_*\cos S_* - a_0 q_2^* S_* + p_2^*),\\ &\varepsilon_3 = 24(a_0 t_* - a_0 q_1^* + \sin S_* - S_*),\\ &\varepsilon_4 = 24\left(1 - \cos S_* - S_*\sin S_* + a_0 q_1^* S_* - t_* S_* + \frac{S_*^2}{2} + p_3^* - p_1^*\right). \end{aligned}\right\} \tag{II.56}$$

The error in θ_0 and the associated error $\Delta\mu(S_0)$ following the substitution of (II.55) for (II.47) are estimated by the equalities

$$|\Delta\theta_0| \leqslant \frac{\omega S_0^4}{5!\, a_0}, \quad |\Delta\mu(S_0)| \leqslant \frac{\omega S_0^5}{5!\, a_0}. \tag{II.57}$$

The error $\Delta\mu(S_0)$ associated with the difference between the calculated and the actual velocity $v(t)$ is estimated by

$$|\Delta\mu(S_0)| \leqslant \frac{|\Delta a|\,\omega S_0^3}{3a_0^2}, \tag{II.58}$$

where $|\Delta a|$ is the deviation of the true $a(t)$ from the calculated value.

The assumption of constant R_0 is also equivalent to some deviation of the calculated velocity from the true value.

Eqs. (II.8), (II.9), (II.55), (II.39), (II.42), and the last equality in (II.31) enable us to calculate ψ_0, $\Delta\psi_0$ and to estimate the resulting error $\Delta\mu(S_0)$ at the time of arrival.

Let us now determine the programmed range. The first equation in (II.54) shows that the error ΔS due to the deviation of the trajectory from the orthodromy plane is small. The determination of S_1 therefore reduces to the determination, with adequate precision, of the arc of the ellipse traced by the intersection of the geoid with the orthodromy plane from the starting point (φ_1, λ_1) to the terminal point (φ_2, λ_2).

Using the matrix (II.21), we see that the geoid meets the orthodromy plane along an ellipse whose semimajor axis lies in the Earth's equatorial plane and is equal to the radius of the Earth's equatorial section, i.e., the Earth's semimajor axis a; the semiminor axis of the ellipse is given by

$$b' = a\sqrt{1 - e^2(1 - \cos^2\varphi_1 \sin^2\psi_0)}, \tag{II.59}$$

where

$$e^2 = \frac{a^2 - b^2}{a^2} \tag{II.60}$$

is the eccentricity of the geoid.

The square of the eccentricity e' of the ellipse is

$$(e')^2 = e^2(1 - \cos^2\varphi_1 \sin^2\psi_0). \tag{II.61}$$

The length of the arc of the ellipse in the orthodromy plane is clearly given by

$$l = a\int_{\alpha}^{S_0+\alpha} \sqrt{1 - e^2(1 - \cos^2\varphi_1 \sin^2\psi_0)\cos^2 S}\, dS, \tag{II.62}$$

where α is the angle in the orthodromy plane between the vector ρ_1 and the intersection line of the orthodromy plane with the equator.

Series expanding the integrand in (II.62) in powers of e^2 and integrating, we get

$$l = aS_0\left[1 - \frac{e^2}{4}(1 - \cos^2\varphi_1 \sin^2\psi_0)\right]. \tag{II.63}$$

The truncation error in this case is estimated as

$$|\Delta l| \leq \frac{ae^2}{4} S_0. \tag{II.64}$$

The determination of the initial parameters ψ_0, $\Delta\psi_0$, and l from known geographical coordinates of the starting (φ_1', λ) and the terminal (φ_2', λ_2) point of the trajectory thus involves the following computational stages.

Determination of the geocentric latitudes φ_1, φ_2 from the geographical latitudes φ_1', φ_2':

$$\varphi_{1,2} = \varphi_{1,2}' - \frac{e^2}{2} \sin 2\varphi_{1,2}'. \tag{II.65}$$

Determination of ψ_0 and S_0 from the equalities

$$\left.\begin{aligned} \cos S_0 &= \cos\varphi_1 \cos\varphi_2 \cos(\lambda_2 - \lambda_1) + \sin\varphi_1 \sin\varphi_2, \\ \sin\psi_0 &= \frac{\cos\varphi_2 \sin(\lambda_2 - \lambda_1)}{\sin S_0}, \\ \cos\psi_0 &= \frac{\sin\varphi_2 - \sin\varphi_1 \cos S_0}{\cos\varphi_1 \sin S_0} \end{aligned}\right\} \tag{II.66}$$

Calculation of l:

$$l = aS_0\left[1 - \frac{e^2}{4}(1 - \cos^2\varphi_1 \sin^2\psi_0)\right]. \tag{II.67}$$

Finally, calculation of $\Delta\psi_0$ from the equation

$$\Delta\psi_0 = -\frac{\omega \cos\varphi_1 \cos\psi_0}{6a_0}\left(S_0^2 + \varepsilon_1 + \frac{\varepsilon_2}{S_0}\right) + \frac{\omega \sin\varphi_1}{24a_0}\left(S_0^3 + \varepsilon_3 + \frac{\varepsilon_4}{S_0}\right). \tag{II.68}$$

In (II.65)—(II.68), ϵ^2, a, ω, a_0, ε_1, ε_2, ε_3, ε_4 are known constants.

The overall maximum errors in navigating the object to the terminal point of the trajectory with inaccurate initial parameters are estimated by the following inequalities:

$$|\Delta l| \leq \frac{ae^2 S_0}{4} + \frac{\omega^2 S_0^5}{18a_0^2},$$

$$|\Delta\mu| \leq \frac{\omega_0^2 S_0^5}{9a_0^2}\left(\frac{1}{\sqrt{3}} + \frac{\omega S_0^2}{2a_0} + \frac{3a_0}{40\omega}\right) + \frac{|\Delta a|\,\omega S_0^3}{3a_0^2}.$$

Appendix III

*ACCELEROMETER GUIDANCE**

We will consider a guidance system directly utilizing the accelerometer readings, without determination of the object coordinates. The equations of motion of the object near a programmed trajectory will be derived and analyzed.

Let the programmed trajectory be given in the plane through the Earth's center O_1 maintaining an invariable orientation relative to the Earth. We introduce a right orthogonal system $O_1\xi_0\eta_0\zeta_0$, whose plane $O_1\xi_0\eta_0$ coincides with the plane of the programmed trajectory. The actual position of the center of mass O of the object in the axes $O_1\xi_0\eta_0\zeta_0$ is determined by its distance R from the Earth's center O_1 and the two angles Δz and S. Δz is the angle between the direction O_1O and the plane $O_1\xi_0\eta_0$. Positive angles correspond to deviations of the direction O_1O toward the axis ζ_0. S is the angle between the axis $O_1\eta_0$ and the projection of the direction O_1O onto the plane $O_1\xi_0\eta_0$.

Suppose that S and R are supplied by some sources of information on board the object. The navigation of the object can thus be considered as a guidance problem in which the angle Δz is invariably zero with known accuracy.

The problem evidently can be solved if we first measure the exact value of Δz (e.g., using inertial systems) and then apply this value to guidance purposes. However, knowledge of the exact value of Δz is not absolutely essential. In the following we consider a guidance system which uses the readings of a suitably aligned accelerometer; these readings constitute a certain linear combination of Δz and its derivatives.

Suppose that gyroscopes on board the moving object fix the direction ζ_0 normal to the plane of the programmed trajectory. An accelerometer is mounted in this direction, its readings being a_{ζ_0}. The accelerometer reading corresponding to $\Delta z \equiv 0$ is designated $a^0_{\zeta_0}$. a_{ζ_0} is a function of S, Δz, R, their derivatives, and time. $a^0_{\zeta_0}$, on the other hand, depends only on S, R, and time. For known S and R, $a^0_{\zeta_0}$ can be formed at any given time. The difference

$$\delta a_{\zeta_0} = a_{\zeta_0} - a^0_{\zeta_0} \qquad \text{(III.1)}$$

can be adopted as a measure of the deviation of the actual trajectory from the programmed path and thus provides a guidance function.

Let us derive the expressions for a_{ζ_0}, $a^0_{\zeta_0}$, δa_{ζ_0}. The following errors are taken into consideration: accelerometer errors Δa_{ζ_0}, errors Δx, ΔR in the information on S and R, and orientation errors of the accelerometer sensitive axis.

* Andreev, V.D. and I.V. Novozhilov.– Inzhenernyi Zhurnal, Mechanics of Rigid Bodies, No. 2. 1966.

We introduce right orthogonal axes $O\xi\eta\zeta$, $O\xi'\eta'\zeta'$, $O\xi'\eta'\zeta'$. Their common origin is at the center of mass of the object. The axis η is directed along the line O_1O, the axis ξ is parallel to the plane of the programmed trajectory. The axes ζ', ζ' point along the axis ζ_*, the axis η' makes an angle $S' = S - \Delta x$ with the axis η^0, and the axis η' makes an angle S with the axis η_*.

The axes $O\xi^0\eta^0\zeta^0$ are obtained by turning the axes $O\xi'\eta'\zeta'$ through a small angle Δx around the axis ζ', and the axes $O\xi\eta\zeta$ are generated by turning the axes $O\xi^0\eta^0\zeta^0$ through an angle Δz about the axis ξ^0. The direction cosines between the axes ξ, η, ζ and ξ', η', ζ' are given by the matrix

$$\begin{array}{c|ccc} & \xi' & \eta' & \zeta' \\ \hline \xi & 1 & \Delta x & 0 \\ \eta & -\Delta x & 1 & \Delta z \\ \zeta & 0 & -\Delta z & 1 \end{array} \qquad \text{(III.2)}$$

To find the direction cosines between the axes ξ^0, η^0, ζ^0 and ξ', η', ζ', we should clearly take $\Delta z = 0$ in (III.2).

The instrumental error in the orientation of the accelerometer sensitive axis ζ_*, i.e., its deviation from the direction $\zeta^0(\zeta')$, is described by two small angles α_* and β_*, so that the direction cosines of the axis ζ_* relative to the axes ξ', η', ζ' are β_*, $-\alpha_*$, and 1. Let the accelerometer sensitive mass be a unit point mass at the point O. For the accelerometer readings we then have

$$a_{\zeta_*} = -W_{\zeta_*} + g_{\zeta_*} + \Delta a_{\zeta_*} \qquad \text{(III.3)}$$

where W_{ζ_*} is the ζ_* projection of the absolute acceleration of the center of mass of the object, g_{ζ_*} is the projection of the Earth's gravitation on the accelerometer sensitive mass, and Δa_{ζ_*} is the accelerometer error.

The projections W_ξ, W_η, W_ζ of the absolute acceleration of the point O on the axes $O\xi\eta\zeta$ are given by

$$\left.\begin{aligned} W_\xi &= -R(\dot{\omega}_\zeta - \omega_\xi\omega_\eta) - 2\dot{R}\omega_\zeta, \\ W_\eta &= \ddot{R} - R(\omega_\xi^2 + \omega_\zeta^2), \\ W_\zeta &= R(\dot{\omega}_\xi + \omega_\eta\omega_\zeta) - 2\dot{R}\omega_\xi, \end{aligned}\right\} \qquad \text{(III.4)}$$

where dots denote time differentiation; ω_ξ, ω_η, ω_ζ are the projections of the absolute angular velocity of the axes $O\xi\eta\zeta$ on the corresponding unit vectors.

If the Earth's gravitational field is central, we have $g_\xi = g_\zeta = 0$, $g_\eta = -g$. From matrix (III.2), Eqs. (III.3), (III.4), and the definition of the angles α_*, β_* we thus find, dropping terms of second order of smallness,

$$a_{\zeta_*} = -W_\xi\beta_* - (W_\eta + g)(\Delta z - \alpha_*) - W_\zeta + \Delta a_{\zeta_*} \qquad \text{(III.5)}$$

To find $a^0_{\zeta_0}$ $(= a_{\zeta_0})$, we should further take $\alpha_* = 0$, $\beta_* = 0$, $\Delta z = 0$. Then $a^0_{\zeta_0} = -W_{\zeta^0}$.

To form W_{ζ^0} from (III.4), we require R, $\dot{R}$, and the absolute velocity components ω_{ξ^0}, ω_{η^0}, ω_{ζ^0} of the axes $\xi^0\eta^0\zeta^0$. These components are functions of S, $\dot{S}$, and t. However, only $\omega_{\xi'}$, $\omega_{\eta'}$, $\omega_{\zeta'}$ are available, which are functions of S', $\dot{S}'$, and t; instead of R and $\dot{R}$ we only have the distances $R + \Delta R$, $\dot{R} + \Delta\dot{R}$. $a^0_{\zeta_0}$ is therefore formed as follows:

$$a^0_{\zeta_0} = -W_{\zeta_0} = -(R + \Delta R)(\dot{\omega}_{\xi'} + \omega_{\eta'}\omega_{\zeta'}) - 2\omega_{\xi'}(\dot{R} + \Delta\dot{R}). \qquad \text{(III.6)}$$

From Eqs. (III.5), (III.6) and an expression analogous to (III.1) we now find

$$\delta a_{\zeta_*} = -W_\xi \beta_* - (W_\eta + g)(\Delta z - \alpha_*) - W_\zeta + \Delta a_\zeta + W_{\zeta_0}. \tag{III.7}$$

In the right-hand side of Eq. (III.7) we insert for $\omega_{\xi'}$, $\omega_{\eta'}$, $\omega_{\zeta'}$ their expressions in terms of ω_ξ, ω_η, ω_ζ, Δz, Δx, $\Delta \dot{z}$, $\Delta \dot{x}$. By matrix (III.2) we have

$$\left.\begin{aligned} \omega_{\xi'} = \omega_\xi - \omega_\eta \Delta x - \Delta \dot{z}, \quad \omega_{\eta'} = \omega_\xi \Delta x + \omega_\eta - \omega_\zeta \Delta z, \\ \omega_{\zeta'} = \omega_\eta \Delta z + \omega_\zeta - \Delta \dot{x}. \end{aligned}\right\} \tag{III.8}$$

Inserting (III.8) in (III.7), substituting for W_ξ, W_η, W_ζ from (III.4), and putting $x = R\Delta x$, $z = R\Delta z$, we find

$$\begin{aligned} \delta a_{\zeta_*} = \ddot{z} - (g/R - \omega_\xi^2 - \omega_\eta^2) z - x(\dot{\omega}_\eta - \omega_\xi \omega_\zeta) - 2\omega_\eta \dot{x} + \\ + \Delta R(\dot{\omega}_\xi + \omega_\eta \omega_\zeta) + 2\omega_\xi \Delta \dot{R} + \alpha_* [g + \ddot{R} - R(\omega_\xi^2 + \omega_\zeta^2)] + \\ + \beta_* [R(\dot{\omega}_\xi - \omega_\xi \omega_\eta) + 2\dot{R}\omega_\eta] + \Delta a_\zeta. \end{aligned} \tag{III.9}$$

The accelerometer signal δa_{ζ_*} is used for guidance. If the guidance process is stable and the signal δa_{ζ_*} is delivered with a sufficiently high gain, we may take the trajectory to be described by the equation $\delta a_{\zeta_*} = 0$, or

$$\ddot{z} + (g/R - \omega_\xi^2 - \omega_\eta^2) z = f_\zeta(\Delta R, \Delta \dot{R}, x, \dot{x}, \alpha_*, \beta_*, \Delta a_\zeta), \tag{III.10}$$

where f_ζ is a homogeneous function of first order of its arguments.

Let for the time being ΔR, $\Delta \dot{R}$, x, $\dot{x}$, α_*, β_*, Δa_ζ and thus also the function f_ζ be negligibly small. Then from (III.3)

$$\ddot{z} + (g/R - \omega_\xi^2 - \omega_\eta^2) z = 0. \tag{III.11}$$

For $\omega_\xi^2, \omega_\eta^2 \ll \omega_0^2 = g/R$, when the effect of ω_ξ, ω_η is ignorable, the z component of motion amounts to periodic oscillations with Schuler frequency and amplitude determined by the initial value of the coordinate z of the object and its initial rate of change.

So far we have neglected guidance in the longitudinal coordinate S. The value of S is known (with an error Δx). We can thus devise a system using accelerometer information on the coordinate S for guidance purposes (the accelerometer signal will control the engine thrust, say).

If S is the programmed value of S', the accelerometer should be mounted along the axis ξ'. For δa_{ξ_*} we obtain, as in (III.2),

$$\begin{aligned} \delta a_{\xi_*} = \ddot{x} + (g/R - \omega_\eta^2 - \omega_\zeta^2) x - z(\dot{\omega}_\eta + \omega_\xi \omega_\zeta) - \\ - 2\omega_\eta \dot{z} - 2\omega_\xi \Delta \dot{R} - \Delta R(\dot{\omega}_\zeta - \omega_\xi \omega_\eta) - \\ - \gamma_* [g + \ddot{R} - R(\omega_\xi^2 + \omega_\zeta^2)] + \delta_* [R(\dot{\omega}_\xi - \omega_\eta \omega_\zeta) + 2\dot{R}\omega_\xi] + \Delta a_\xi, \end{aligned}$$

where x/R is the deviation of the actual coordinate S of the object from its programmed value S', and the angles γ_* and δ_* are the instrumental orientation errors of the accelerometer axis a_ξ, analogous to the angles α_*, β_*. In this case, given stable and sufficiently effective guidance, the S component of motion will be described by the equation $\delta a_{\xi_*} = 0$. Together with the equation

$\delta a_{\xi_*} = 0$ this gives

$$\left.\begin{aligned}
&\ddot{x} + (\omega_\eta^2 - \omega_\xi^2 - \omega_\eta^2)\,x - z\,(\dot{\omega}_\eta + \omega_\xi\omega_\zeta) - 2\omega_\eta\dot{z} = \\
&\quad = 2\omega_{\zeta_*}\Delta\dot{R} + \Delta R\,(\dot{\omega}_\zeta - \omega_\xi\omega_\eta) + \\
&\quad + \gamma_*\,[g + \ddot{R} - R\,(\omega_\xi^2 + \omega_\zeta^2)] - \\
&\quad - \delta_*\,[R\,(\dot{\omega}_\xi + \omega_\eta\omega_\zeta) + 2\dot{R}\omega_\xi] + \Delta a_\xi, \\
&\ddot{z} + (\omega_\zeta^2 - \omega_\xi^2 - \omega_\eta^2)\,z + x\,(\dot{\omega}_\eta - \omega_\xi\omega_\zeta) + 2\omega_\eta\dot{x} = \\
&\quad = 2\omega_{\xi_*}\Delta\dot{R} + \Delta R\,(\dot{\omega}_\xi + \omega_\eta\omega_\zeta) + \\
&\quad + \alpha_*\,[g + \ddot{R} - R\,(\omega_\xi^2 + \omega_\zeta^2)] + \\
&\quad + \beta_*\,[R\,(\dot{\omega}_\zeta - \omega_\xi\omega_\eta) + 2\dot{R}\omega_\eta] + \Delta a_\zeta.
\end{aligned}\right\} \qquad \text{(III.12)}$$

This system describes the deviation of the motion of the center of mass from the programmed trajectory. If the accelerometers a_ξ, a_ζ are mounted on the same platform, we have $\beta_* = \delta_*$ and Eqs. (III.12) fully coincide with position error equations of a two-accelerometer inertial system.

Note that a third accelerometer can be mounted along the η' axis for guidance in the coordinate R. Then the equations $\delta a_{\xi_*} = 0$, $\delta a_{\zeta_*} = 0$, $\delta a_{\eta_*} = 0$ again fully coincide with the equations of a three-accelerometer inertial system.

In certain cases accelerometer signals are not used for guidance in one of the two coordinates (Δz and S), and the corresponding coordinates are measured with an inertial system. The combined system of guidance and inertial error equations then have the form (III.12). The situation is entirely similar for the three coordinates Δz, S, R.

Eqs. (III.12) are invariant under rotations of the system $O\xi\eta\zeta$ about the axis η. It thus follows that our original assumption of a programmed trajectory lying in a plane of fixed orientation relative to the Earth is not essential. However, to simplify further treatment, we will retain this assumption.

The above results were obtained under conditions of stable and sufficiently effective (rigid) accelerometer guidance. The feasibility of these conditions is by no means certain. To check this point, we should consider in some detail the motion of the guided object, which is done here for an aircraft.

We start with the equations of motion. The projection of forces on the axis $O\zeta^0$ gives

$$m\,(W_{\zeta^0} - g_{\zeta^0}) = F^1_{\zeta^0}, \qquad \text{(III.13)}$$

where $m = \text{const}$ is the mass of the object, W_{ζ^0} is the projection of the absolute acceleration of the center of mass of the object, mg_{ζ^0} is the projection of the gravitational attraction, $F^1_{\zeta^0}$ is the projection of aerodynamic forces and engine thrust.

The expression for the difference $W_{\zeta^0} - g_{\zeta^0}$ can be found from Eqs. (III.5), (III.4), if in (III.5) we change the sign on the right and take $\alpha_* = \beta_* = \Delta a_\zeta = 0$.

To find $F^1_{\zeta^0}$, let V be the velocity of the object relative to the earth, V^* the air velocity of the object, v the wind velocity at the point O. Then

$$V^* = V - v, \quad V_{\zeta^0} = R\,\Delta\dot{z}. \qquad \text{(III.14)}$$

The axes $Ox^*y^*z^*$ are obtained from $O\xi\eta\zeta$ by two successive rotations: through an angle ψ_0 around the axis η and through an angle θ_0 around the axis z^* (bringing the axis x^* to coincide with the vector V^*). If V^*_ζ, $V^*_\eta \ll V^*$, we have

$$\psi_0 = -V^*_\zeta/V^*, \quad \theta_0 = V^*_\eta/V^*. \tag{III.15}$$

We introduce a right orthogonal system of coordinates $Oxyz$ fixed to the object; the axis x is parallel to the wing chord and points in the direction of flight, the axis y lies in the plane of symmetry of the object, and the z axis is directed to starboard. Assuming small deviations of the axes $Oxyz$ from $Ox^*y^*z^*$, we define their relative orientation by three small rotations: through the angles γ_0, β around the axes x^*, y^* (this gives what is generally called a semifixed system of coordinates) and through the angle α around the axis z^*. α and β are the angle of attack and the angle of side slip, respectively.

The aerodynamic forces are generally defined in the semifixed axes by the following projections:* the drag X, the lift Y, and the side force Z. The thrust P is assumed to lie in the plane of symmetry of the object, making a small angle φ to the axis x.

To terms of second order in ψ_0, θ_0, α, β, γ_0, φ, Δz, we now have

$$F^1_{\zeta^0} = (P - X)(-\psi_0 - \beta) + Y(\gamma_0 + \Delta z) + Z. \tag{III.16}$$

For Z we write

$$Z = m\left(C^\beta_z \beta + C^{\delta_x}_z \delta_x + C^{\delta_y}_z \delta_y\right) + m\,\Delta f_z. \tag{III.17}$$

Here δ_x, δ_y, δ_z are the small adjustment angles of the ailerons, the elevators, and the rudders, C^β_z, $C^{\delta_x}_z$, $C^{\delta_y}_z$ are the aerodynamic coefficients, m is the mass of the object, $m\Delta f_z$ is a perturbation force.

For midcourse flight, when the difference $P - X$ is small, the first term on the right in (III.16) is ignorable; we may further take $R = \text{const}$ and $Y = mg$. Eq. (III.13), using (III.16), (III.17), takes the form

$$W_{\zeta^0} - g_{\zeta^0} = g(\gamma_0 + \Delta z) + C^\beta_z \beta + C^{\delta_x}_z \delta_x + C^{\delta_y}_z \delta_y + \Delta f_z. \tag{III.18}$$

We will now derive the equations of the angular motion of the object about its center of mass, which are needed for the determination of the angle β in (III.18). The orientation of the body axes $Oxyz$ relative to the axes $O\xi\eta\zeta$ is described by small rotation angles around the axes ξ, η, ζ: the yaw angle ψ, the pitch angle ϑ, and the roll angle γ. Then, to terms of second order

$$\psi_0 - \psi = -\beta, \quad \gamma_0 = \gamma. \tag{III.19}$$

We are mainly concerned with the motion of the center of mass of the object. Since the oscillation frequencies corresponding to this motion are substantially less than the oscillation frequencies characterizing the angular stabilization of the object, the equations of motion about the center of mass can therefore be written in the form $M_x = 0$, $M_y = 0$, $M_z = 0$, where M_x, M_y, M_z are the projections of the external torque on the axes $Oxyz$.

* See, e.g., Lebedev, A.A. and L.S.Chernobrovkin. Dinamika poleta (Flight Dynamics). — Oborongiz. 1962.

The aerodynamic symmetry of the object suggests that for small α, β, δ_x, δ_y, δ_z the torques M_x, M_y are independent of α, δ_z. Therefore, to terms of second order of smallness, we have from $M_x = 0$, $M_y = 0$*

$$\left.\begin{aligned} M_x^{\beta}\beta + M_x^{\delta_x}\delta_x + M_x^{\delta_y}\delta_y + \Delta M_x = 0, \\ M_y^{\beta}\beta + M_y^{\delta_x}\delta_x + M_y^{\delta_y}\delta_y + \Delta M_y = 0, \end{aligned}\right\} \tag{III.20}$$

where M_x^{β}, M_y^{β}, $M_x^{\delta_x}$, $M_x^{\delta_y}$, $M_y^{\delta_x}$, $M_y^{\delta_y}$ are the aerodynamic torque coefficients, ΔM_x, ΔM_y are the perturbing torques.

To complete the set of the equations of motion, we require expressions for the control adjustments, which can be written in the form

$$\left.\begin{aligned} \delta_x = n_0(\gamma + \Delta z - \alpha_*), \quad \delta_y = m_0(\psi - \beta_*) + k_0 u, \\ L(u) = N(\delta a_{z_*}). \end{aligned}\right\} \tag{III.21}$$

Here n_0, m_0, k_0 are the guidance system inductances. The factors $\psi - \beta_*$, $\gamma + \Delta z - \alpha_*$ are the angular stabilization signals from the gimbals of the gyroscopic platform generating the axes $O\xi'\eta'\zeta'$.

The last equation in (III.21) describes the operation of the accelerometer output processor, u is the output signal of this device, i.e., the processed accelerometer output. L and N are at this stage arbitrary operators satisfying the condition

$$L(0) = N(0), \qquad \text{or} \qquad u(\delta a_{z_*} = 0) = 0. \tag{III.22}$$

Inserting for $W_{z_*} - g_{z_*}$ in Eq. (III.18) its expression from (III.5), (III.4), and considering the resulting equation in conjunction with Eqs. (III.14), (III.15), (III.19)—(III.21), (III.9), we obtain a set of equations describing the guided motion of the object:

$$\left.\begin{aligned} &[\ddot{R} - R(\omega_\xi^2 + \omega_\zeta^2) + g]\Delta z + R(\dot{\omega}_\xi + \omega_\eta\omega_\zeta) + 2\dot{R}\omega_\xi = \\ &\quad = g(\gamma + \Delta z) + C_z^{\beta}\beta + C_z^{\delta_x}\delta_x + C_z^{\delta_y}\delta_y + \Delta f_z, \\ &\delta_x = n_0(\gamma + \Delta z - \alpha_*), \quad \delta_y = m_0(\psi - \beta_*) + k_0 u, \\ &L(u) = N(\delta a_{z_*}), \\ &M_x^{\beta}\beta + M_x^{\delta_x}\delta_x + M_x^{\delta_y}\delta_y + \Delta M_x = 0, \\ &M_y^{\beta}\beta + M_y^{\delta_x}\delta_x + M_y^{\delta_y}\delta_y + \Delta M_y = 0, \\ &\delta a_{z_*} = -\ddot{z} - (g/R - \omega_\xi^2 - \omega_\eta^2)z - x(\dot{\omega}_\eta - \omega_\xi\omega_\zeta) - \\ &\quad - 2\omega_\eta\dot{x} + \Delta R(\dot{\omega}_\xi + \omega_\eta\omega_\zeta) + 2\Delta\dot{R}\omega_\xi + \\ &\quad + \alpha_*[g + \ddot{R} - R(\omega_\xi^2 + \omega_\zeta^2)] + \\ &\quad + \beta_*[R(\dot{\omega}_\zeta - \omega_\xi\omega_\eta) + 2\dot{R}\omega_\eta] + \Delta a_\zeta, \\ &\psi_0 = -(R\Delta\dot{z} - v_\zeta)/V^*, \quad \psi_0 - \psi = -\beta, \quad z = R\Delta z. \end{aligned}\right\} \tag{III.23}$$

The variational form of these equations describes the motion of the object about the unperturbed, programmed state, defined by the conditions

$$u = \Delta z = z \equiv 0, \quad \Delta R = x = 0, \quad \beta_* = \alpha_* = \Delta a_\zeta = 0,$$
$$\Delta f_z = \Delta M_x = \Delta M_y = v_\zeta = 0.$$

* See, e.g., the book mentioned in footnote to previous page.

For the unperturbed motion

$$\left.\begin{aligned} &R(\dot{\omega}_\xi+\omega_\eta\omega_\zeta)+2\dot{R}\omega_\xi=g\gamma+C_z^\beta\beta+C_z^{\delta_x}\delta_x+C_z^{\delta_y}\delta_y,\\ &\delta_x=n_0\gamma,\quad \delta_y=m_0\psi,\quad \psi=\beta,\\ &M_x^\beta\beta+M_x^{\delta_x}\delta_x+M_x^{\delta_y}\delta_y=0,\\ &M_y^\beta\beta+M_y^{\delta_x}\delta_x+M_y^{\delta_y}\delta_y=0. \end{aligned}\right\} \qquad \text{(III.24)}$$

For the sake of simplicity, we do not introduce any additional symbol to denote the variations of the variables γ, δ_x, δ_y, β, ψ relative to the values specified by Eqs. (III.24). Thus, using the symbol of the variable for its variation, we obtain Eqs. (III.23), (III.24) in variational form:

$$\left.\begin{aligned} &\varepsilon_\zeta-C_z^\beta\beta-C_z^{\delta_x}\delta_x-C_z^{\delta_y}\delta_y-g(\gamma+\Delta z)=\Delta f_z,\\ &M_x^\beta\beta+M_x^{\delta_x}\delta_x+M_x^{\delta_y}\delta_y=-\Delta M_x,\\ &M_y^\beta\beta+M_y^{\delta_x}\delta_x+M_y^{\delta_y}\delta_y=-\Delta M_y,\\ &\delta_x-n_0(\gamma+\Delta z)=-n_0\alpha_*,\quad \delta_y-m_0\psi-k_0u=-m_0\beta_*,\\ &L(u)-N(\delta a_{\zeta_*})=0,\quad \delta a_{\zeta_*}+\varepsilon_\zeta=f_\zeta,\\ &\varepsilon_\zeta-\ddot{z}-(\omega_0^2-\omega_\xi^2-\omega_\eta^2)z=0\qquad (\omega_0^2=g/R),\\ &z=R\Delta z,\\ &\psi-\beta+\frac{R\Delta\dot{z}}{V}=\frac{v_\zeta}{V}\qquad (V=V^*=\text{const}). \end{aligned}\right\} \qquad \text{(III.25)}$$

Here, as in Eq. (III.10),

$$\begin{aligned} f_\zeta=&-x(\dot{\omega}_\eta-\omega_\xi\omega_\zeta)-2\omega_\eta\dot{x}+\Delta R(\dot{\omega}_\xi+\omega_\eta\omega_\zeta)+\\ &+2\Delta\dot{R}\omega_\xi+\alpha_*[g+\ddot{R}-R(\omega_\xi^2+\omega_\zeta^2)]+\\ &+\beta_*[R(\dot{\omega}_\zeta-\omega_\xi\omega_\eta)+2\dot{R}\omega_\eta]. \end{aligned}$$

Suppose now that the object is statically stable in its course. We can thus take $m_0 = 0$. The first seven equations in (III.25) are then solved independently of the last three equations.

Setting zero on the right in the first seven equations in (III.25) (this implies that wind is the only perturbing factor in the system), we obtain the particular solution $\varepsilon_\zeta=\beta=\delta_x=\delta_y=u=\delta a_{\zeta_*}=\gamma+\Delta z=0$. The last three equations in (III.25) give

$$\left.\begin{aligned} &\ddot{z}+(\omega_0^2-\omega_\xi^2-\omega_\eta^2)z=0,\\ &z=R\Delta z,\\ &\psi=(-R\Delta\dot{z}+v_\zeta)/V. \end{aligned}\right\} \qquad \text{(III.26)}$$

The first equation in (III.26) coincides with the previous equation (III.11).

Note that Eq. (III.11) was obtained assuming a sufficiently effective guidance. Now we see that the same result is obtained for any guidance (provided that there are no perturbations). The z component of motion in this case is not affected by wind perturbation.

Further note that these results are valid for quite general operators L, N, nonlinear operators included. The only constraint imposed on these operators are the equalities (III.22).

Eliminating between Eqs. (III.25) the variables $\gamma + \Delta z$, β, δ_x, δ_y, ε_z, δa_{z_0}, we write

$$\left.\begin{aligned} &\ddot{z} + (\omega_0^2 - \omega_\xi^2 - \omega_\eta^2)\, z = -\mu k_0 u, \\ &L(u) = N\left[-\ddot{z} - (\omega_0^2 - \omega_\xi^2 - \omega_\eta^2)\, z\right]. \end{aligned}\right\} \tag{III.27}$$

In the derivation of Eqs. (III.27), as before, wind was the only perturbation in the system. The coefficient μ in the first equation in (III.27) depends only on the parameters of the object. Its explicit expression is not needed in what follows.

If L and N are polynomials of the differentiation operator p, and $\omega_\xi^2 \ll \omega_0^2$, $\omega_\eta^2 \ll \omega_0^2$, $\mu k_0 = \text{const}$, we can investigate the dependence of the general solution of Eqs. (III.27) on the form of the operators L, N. If $z(p)$ is the Carson–Heaviside transform of the solution $z(t)$, we write

$$z(p) = \frac{L(p)(p^2 z_0 + p\dot{z}_0) - \mu k_0 Q(p, z_0, u_0, \dot{z}_0, \dot{u}_0, \ldots)}{(p^2 + \omega_0^2)[L(p) - \mu k_0 N(p)]}, \tag{III.28}$$

where Q is a linear homogeneous function of the initial conditions.

The factor $(p^2 + \omega_0^2)$ in the denominator on the right in (III.28) is responsible for the undamped Schuler-frequency oscillations in the solution. One of the main problems in choosing the operators $L(p)$ and $N(p)$ is how to eliminate these undamped oscillations. The simplest approach is by taking

$$L(p) = p^2 + \omega_0^2, \quad N(p) = 1, \tag{III.29}$$

when for $\mu k_0 = \mu k_{01} \gg \omega_0^2$ we have

$$z(p) = \frac{p^2 z_0 + p\dot{z}_0}{p^2 + \mu k_{01}} + \frac{\mu k_{01} p(z_0 - u_0) + p(\dot{z}_0 - \dot{u}_0)}{(p^2 + \omega_0^2)(p^2 + \mu k_{01})}. \tag{III.30}$$

If

$$u_0 = z_0, \quad \dot{u}_0 = \dot{z}_0, \tag{III.31}$$

then

$$z(p) = \frac{p^2 z_0 + p\dot{z}_0}{p^2 + \mu k_{01}}. \tag{III.32}$$

Eqs. (III.29), (III.31) signify that the accelerometer readings and the initial conditions give the exact z, i.e., the situation is reduced to guidance with an ordinary inertial system. Note that Eqs. (III.29) are readily generalized to conditions of the form

$$L(p) = (p^2 + \omega_0^2)\Lambda(p), \quad N(p) = -\Lambda(p),$$

where $\Lambda(p)$ is a general operator. The analogs of Eqs. (III.31), however, are more complicated in this case.

Another extreme case of (III.29), (III.31) is

$$L(p) = 1, \quad N(p) = -1, \tag{III.33}$$

which leads to direct accelerometer guidance. Then, irrespective of the value of $k_{01} = k_{02}$, we have from (III.28)

$$z(p) = \frac{p^2 z_0 + p\dot{z}_0}{p^2 + \omega_0^2}, \tag{III.34}$$

i.e., $z(t)$ has the same form as before, assuming an infinitely large k_0.

Note that the choice (III.33) ignores the problem of filtering of the high-frequency accelerometer components. This filtering can always be accomplished using filters with small time constants, so that the long-period components are not affected.

Let us assess the contribution of the right-hand sides of Eqs. (III.25). From the first seven equations in (III.25), taking $L = 1$, $N = -1$, we obtain $(1 + \mu k_{02})\,\delta a_{z_0} = \Phi$, where Φ is a function of Δf_x, ΔM_x, ΔM_y, $n_0 \mu_*$, f_{z_0}, i.e., a function of the right-hand sides of Eqs. (III.25). For $\mu k_{02} \gg 1$ the effect of this function is ignorable.

The preceding analysis led to the existence of undamped oscillations (this emerges from Eqs. (III.30), (III.34)). Damping can be introduced in the equations of motion if we use guidance information on $\dot{z}$. This is achieved for $m_0 \neq 0$, say, since in virtue of the last equation in (III.25) we have $\psi = (-\dot{z} + v_z)/V$. From Eqs. (III.25) for $m_0 \neq 0$, $L = p^2 + \omega_0^2$, $N = -1$, $\omega_\xi^2 \ll \omega_0^2$, $\omega_\eta^2 \ll \omega_0^2$, $\mu k_{01} = \text{const}$ (and $\mu k_{01} \gg \omega_0^2$), $\Phi = 0$, we have instead of (III.30)

$$z(p) = \frac{p^2 z_0 + p\dot{z}_0 + p\lambda_1 z_0 + \lambda_1 v_z}{p^2 + \lambda_1 p + \mu k_{01}} + \frac{\mu k_{01}\,[p(z_0 - u_0) + p(\dot{z}_0 - \dot{u}_0)]}{(p^2 + \omega_0^2)(p^2 + \lambda_1 p + \mu k_{01})}. \tag{III.35}$$

Here for $\mu k_{01} \gg \omega_0^2$, $\lambda_1 = m_0 \mu / V$.

The first term on the right in Eq. (III.35) corresponds to a damped transient. The second term for $z_0 \neq u_0$, $\dot{z}_0 \neq \dot{u}_0$ gives an undamped component. For $z_0 = u_0$, $\dot{z}_0 = \dot{u}_0$ this component drops out. If $v_z = \text{const}$, we have a static deviation $z_1 = \lambda_1 v_z / \mu k_{01}$.

For $L = 1$ and $N = -1$, taking again $m_0 \neq 0$, $\Phi = 0$; ω_ξ^2, $\omega_\eta^2 \ll \omega_0^2$; $\mu k_{02} = \text{const}$ (and $\mu k_{02} \gg 1$), we replace (III.30) by

$$z(p) = \frac{p^2 z_0 + p\dot{z}_0 + p\lambda_2 z_0 + \lambda_2 v_z}{p^2 + \lambda_2 p + \omega_0^2}.$$

Here for $\mu k_{02} \gg 1$, $\lambda_2 = m_0 k_{02} V$. In this case the free oscillations in the system are eventually damped to zero and we are left with the static deviation $z_2 = \lambda_2 v_z / \omega_0^2$. It is readily seen that for $k_{01} = \omega_0^2 k_0^2$ the static deviations z_1, z_2 are equal to each other, and $\lambda_1 \gg \lambda_2$.

REFERENCES

1. ADAMS, D. E. Introduction to Inertial Navigation. — J. Institute of Navigation (London), 9, No. 3. 1956.
2. ANDREEV, V. D. Integration of the equations of errors of an inertial navigation system for Keplerian motion of the object. — Prikladnaya Matematika i Mekhanika, 29, No. 2. 1965.
3. ANDREEV, V. D. Determination of optimal parameters of light indication systems. — Izvestiya AN SSSR, Technical Cybernetics, No. 4. 1963.
4. ANDREEV, V. D. A solution of the Stokes problem for a geoid level surface. — Prikladnaya Matematika i Mekhanika, 30, No. 2. 1966.
5. ANDREEV, V. D. Theory of inertial systems for autonomous measurement of the coordinates of a moving object. — Prikladnaya Matematika i Mekhanika, 28, No. 1. 1964.
6. ANDREEV, V. D. Theory of Schuler-tuned systems. — Prikladnaya Matematika i Mekhanika, 29, No. 6. 1965.
7. ANDREEV, V. D. General equations of inertial navigation. — Prikladnaya Matematika i Mekhanika, 28, No. 2. 1964.
8. ANDREEV, V. D. A case of small oscillations of a physical pendulum with a moving suspension point. — Prikladnaya Matematika i Mekhanika, 22, No. 6. 1958.
9. ANDREEV, V. D. A navigation technique using a directional gyroscope and a Doppler velocity meter. — Izvestiya AN SSSR, Technical Cybernetics, No. 6. 1963.
10. ANDREEV, V. D. Errors in inertial navigation systems. — Izvestiya AN SSSR, Technical Cybernetics, No. 2. 1964.
11. ANDREEV, V. D. Unperturbed operation equations of an inertial system measuring curvilinear coordinates. — Prikladnaya Matematika i Mekhanika, 29, No. 5. 1965.
12. ANDREEV, V. D. Equations of errors of an inertial system measuring general curvilinear coordinates of a moving object. — Izvestiya AN SSSR, Mechanics, No. 4. 1965.
13. ANDREEV, V. D. The second group of errors of inertial navigation systems. — Izvestiya AN SSSR, Technical Cybernetics, No. 3. 1964.
14. ANDREEV, V. D. Transmission of the useful signal and the background signal through the reticles of light indication systems. — Izvestiya AN SSSR, Technical Cybernetics, No. 3. 1963.

15. ANDREEV, V.D. Teoriya inertsial'noi navigatsii (avtonomnye sistemy) (Theory of Inertial Navigation: Autonomous Systems). — Nauka. 1966.
16. ANDREEV, V.D., E.A. DEVYANIN, and A.P. DEM'YANOVSKII. The theory of inertial systems without gyroscopic sensitive elements. — Inzhenernyi Zhurnal, Mechanics of Rigid Bodies, No.1. 1966.
17. ANDREEV, V.D. and I.V. NOVOZHILOV. Accelerometer guidance. — Inzhenernyi Zhurnal, Mechanics of Rigid Bodies, No.2. 1966.
18. APPELL, P. Traité de Mécanique Rationelle. — Paris, Gauthiers—Villars. 1953.
19. AROYAN, G.F. Technical Spatial Filtering. — Proceedings of the IRE, No.9. 1959.
20. ASTAF'EV, G.P., V.S. SHEBSHAEVICH, and Yu.A. YURKOV. Radionavigatsionnye ustroistva i sistemy (Radio Navigation Systems). — Sovetskoe Radio. 1958.
21. BELETSKII, V.V. Motion of artificial Earth satellites about the center of mass. — In: "Iskusstvennye sputniki Zemli," No.1. ANSSSR. 1958.
22. BLAZHKO, S.N. Kurs sfericheskoi astronomii (A Course in Spherical Astronomy). — Gostekhizdat. 1954.
23. BLYUMIN, G.D. and M.V. CHICHINADZE. Unperturbability conditions for an aided single-rotor gyrocompass. — Izvestiya ANSSSR, Mechanics and Machine Building, No.5. 1964.
24. BLYUMIN, G.D. and M.V. CHICHINADZE. Unperturbability conditions for an aided single-rotor gyrocompass. — Izvestiya ANSSSR, Mechanics and Machine Building, No.3. 1964.
25. BODNER, V.A., V.E. OVCHAROV, and V.P. SELEZNEV. Synthesis of invariant damped systems with arbitrary period. — Doklady ANSSSR, 125, No.3. 1959.
26. BOL'SHAKOV, I.A. Transmission of regular and random signals through a commutation-type phase detector. — Vestnik Moscow University, No.6. 1958.
27. BUELL, H. Doppler, Inertial and Doppler Inertial Techniques. — Journal of the Institute of Navigation (Washington), Vol.II, 3. 1964.
28. BULGAKOV, B.V. Kolebaniya (Oscillations). — Gostekhizdat. 1954.
29. BULGAKOV, B.V. Prikladnaya teoriya giroskopov (Applied Theory of Gyroscopes). — Gostekhizdat. 1955.
30. BULGAKOV, B.V. and Ya.N. ROITENBERG. The theory of gyro horizons. — Izvestiya ANSSSR, tech. sci., No.3. 1948.
31. BUTTWEILER, B. Modulated Infrared Systems. — Journal of the Optical Society of America, 9, No.9:51. 1961.
32. VINNITSKII, A.S. Ocherk osnov radiolokatsii pri nepreryvnom izluchenii radiovoln (Continuous Radar Techniques). — Sovetskoe Radio. 1961.
33. GECKELER, I.W. Kreiselkompas und Schiffsmanöver. — Ingeneur Archiv, 4, Nos.1—2. Berlin. 1933.
34. GECKELER, I.W. Kreiselmechanik des Anschütz-Raumkompasses. — Ingeneur Archiv, 6, No.4. Berlin. 1935.
35. GORENSHTEIN, I.A., I.A. SHUL'MAN, and A.S. SAFARYAN. Inertsial'naya navigatsiya (Inertial Navigation). — Sovetskoe Radio. 1962.

36. GRAMMEL, R. Der Kreisel, Vol. 2. — Berlin, Springer-Verlag. 1950.
37. GRAUR, A.V. Matematicheskaya kartografiya (Mathematical Cartography). — Leningrad, Izd. LGU. 1956.
38. GRUSHINSKII, N.P. Teoriya figury Zemli (Theory of the Earth's Figure). — Fizmatgiz. 1963.
39. GOURSAT, E. Cours d'Analyse. — Paris, Gauthier—Villars. 1905.
40. DANILIN, V.P. Giroskopicheskie pribory (Gyroscopic Instruments). — Izd. Vysshaya Shkola. 1965.
41. DEVYANIN, E.A. On the equations of automatic trackers. — Izvestiya ANSSSR, Technical Cybernetics, No. 1. 1965.
42. DEVYANIN, E.A. and A.P. DEM'YANOVSKII. Inertial determination of the orientation and the angular velocities of a satellite. — Inzhenernyi Zhurnal, Mechanics of Rigid Bodies, No. 2. 1966; DEVYANIN, E.A. Supplements to the above, ibid., No. 5. 1966.
43. DRAPER, C.S., W. WRIGLEY, and L.R. GROHE. The Floating Integrating Gyro and its Application to Geometrical Stabilization Problems in Moving Bases. — Aeronautical Engineering Review, **15**, No. 6: 46. June 1956.
44. DRAPER, C.S., W. WRIGLEY, and J. HOVORKA. Inertial Guidance. — New York, Pergamon Press. 1960.
45. DUBOSHIN, G.N. Nebesnaya mekhanika. Osnovnye zadachi i metody (Celestial Mechanics: Problems and Methods). — Fizmatgiz. 1963.
46. DUBOSHIN, G.N. Teoriya prityazheniya (Theory of Gravitational Attraction). — Fizmatgiz. 1961.
47. ZHBANOV, Yu.K. A study of free oscillations in an autonomous system which measures the position of a moving object. — Prikladnaya Matematika i Mekhanika, **24**, No. 6. 1960.
48. ZAKHARIN, M.I. On the construction of an inertial navigation system with a platform which is stationary in the inertial space. — Izvestiya ANSSSR, Dept. Tech. Sci., power engineering and automation, No. 1. 1961.
49. Stellar Inertial Navigation in Aviation. — Radioelektronika za rubezhom, No. 35. 1963.
50. PITTMAN, D., ed. Inertial Guidance Systems. [Russian translation. 1964.]
51. ISHLINSKII, A.Yu. The theory of the gyrocompass horizon. — Prikladnaya Matematika i Mekhanika, **20**, No. 4. 1956.
52. ISHLINSKII, A.Yu. The theory of the gyropendulum. — Prikladnaya Matematika i Mekhanika, **21**, No. 1. 1957.
53. ISHLINSKII, A.Yu. The theory of complex gyrostabilizing systems. — Prikladnaya Matematika i Mekhanika, **22**, No. 3. 1958.
54. ISHLINSKII, A.Yu. Mekhanika giroskopicheskikh sistem (Mechanics of Gyroscopic Systems). — Izdatel'stvo ANSSSR. 1963.
55. ISHLINSKII, A.Yu. Autonomous determination of the position of a moving object using a three-degrees-of-freedom gyrocompass, a course gyroscope, and an integrating device. — Prikladnaya Matematika i Mekhanika, **23**, No. 1. 1959.
56. ISHLINSKII, A.Yu. Relative equilibrium of a physical pendulum with a moving pivot. — Prikladnaya Matematika i Mekhanika, **20**, No. 3. 1956.

57. ISHLINSKII, A.Yu. The equations of the problem of determination of the position of a moving object using gyroscopes and accelerometers. — Prikladnaya Matematika i Mekhanika, 21, No.6. 1957.
58. ISHLINSKII, A.Yu. The equations of the precession theory of gyroscopes in the form of equations of motion of the representing point in the image plane. — Prikladnaya Matematika i Mekhanika, 23, No.5. 1959.
59. ISHLINSKII, A.Yu. Theory of the two-gyro vertical. — Prikladnaya Matematika i Mekhanika, 21, No.2. 1957.
60. KLIMOV, D.M. On the unperturbability of a gyroscopic frame. — Prikladnaya Matematika i Mekhanika, 28, No.3. 1964.
61. KONDRAT'EV, K.Ya. Luchistaya energiya solntsa (The Sun's Radiant Energy). — Gidrometeoizdat. 1951.
62. KOCHIN, N.E. Vektornoe ischislenie i nachala tenzornogo ischisleniya (Vector Analysis and Elements of Tensor Calculus). — Izd. ANSSSR. 1951.
63. KOSHLYAKOV, V.N. The theory of the gyrocompass. — Prikladnaya Matematika i Mekhanika, 23, No.5. 1959.
64. KOSHLYAKOV, V.N. The stability of the gyrocompass horizon in the presence of dissipative forces. — Prikladnaya Matematika i Mekhanika, 26, No.3 1962.
65. KOSHLYAKOV, V.N. Reducibility of the equations of motion of a gyrocompass horizon. — Prikladnaya Matematika i Mekhanika, 25, No.5. 1961.
66. KRASOVSKII, A.A. and G.S. POSPELOV. Osnovy avtomatiki i tekhnicheskoi kibernetiki (Elements of Automation and Engineering Cybernetics). — Gosenergoizdat. 1962.
67. KRYLOV, A.N. and Yu.N. KRUTKOV. Obshchaya teoriya giroskopov i nekotorykh tekhnicheskikh ikh primenenii (General Theory of Gyroscopes and of Some of Their Technical Applications). — Izd. ANSSSR. 1932.
68. KUDREVICH, B.I. Theory of gyroscopic instruments. — Collected Works, Vol.1. Sudpromgiz. 1963.
69. KUZNETSOV, D.S. Spetsial'nye funktsii (Special Functions). — Izd. Vysshaya Shkola. 1962.
70. KULIKOV, K.A. Izmenyaemost' shirot and dolgot (Variations of Latitude and Longitude). — Fizmatgiz. 1962.
71. LEDEBEV, A.A. and L.S. CHERNOBROVKIN. Dinamika poleta (Flight Dynamics). — Oborongiz. 1962.
72. Litton LN-3 Navigation System. — Interavia Air Letter, No.4787, 24/VII. 1961.
73. LUR'E, A.I. Analiticheskaya mekhanika (Analytical Mechanics). — Fizmatgiz. 1961.
74. LUR'E, A.I. Free fall of a point mass on board a satellite. — Prikladnaya Matematika i Mekhanika, No.1. 1963.
75. LYAPUNOV, A.M. Obshchaya zadacha ob ustoichivosti dvizheniya (General Problem of the Stability of Motion). — ONTI. 1935.
76. McCLURE, C.L. Theory of Inertial Guidance. — Prentice Hall. 1960.
77. McCONNEL, A.J. Application of Tensor Analysis. — Dover. 1957.
78. MAL'TSEV, A.I. Osnovy lineinoi algebry (Elements of Linear Algebra). — Gostekhizdat. 1948.

79. MERKIN, D.R. Giroskopicheskie sistemy (Gyroscopic Systems). — Gostekhizdat. 1953.
80. MERKIN, D.R. Stability of motion of a gyroframe. — Prikladnaya Matematika i Mekhanika, 25, No.6. 1961.
81. METELITSYN, I.I. Gyroscopic stabilization. — Doklady AN SSSR, 36, No.1. 1952.
82. MIKHAILOV, A.A. Kurs gravimetrii i teorii figury Zemli (A Course in Gravimetry and the Theory of the Earth's Figure). — Redbyuro GUGK, SNK SSSR. 1939.
83. NIKOLAI, E.L. Giroskop v kardanovom podvese (A Gimbaled Gyroscope). — Nauka. 1964.
84. O'DONNELL, C.F. Inertial Navigation. — J.Franklin Inst., 266, No.4. October 1958; 266, No.5. November 1958.
85. PARUSNIKOV, N.A. and L.I. KLIGER. Equations of small oscillations of the Schuler vertical. — Inzhenernyi Zhurnal, Mechanics of Rigid Bodies, No.5. 1966.
86. POGORELOV, D.A. Teoriya keplerovykh dvizhenii letatel'nykh apparatov (Theory of Kepler Motion of Flying Vehicles). — Fizmatgiz. 1961.
87. PRIVALOV, I.I. Analiticheskaya geometriya (Analytical Geometry). — Gostekhizdat. 1952.
88. Problemy inertsial'noi navigatsii (Problems of Inertial Navigation), N.I. Borisov, ed. — IL. 1961. [Russian translations.]
89. PUGACHEV, V.S. Teoriya sluchainykh funktsii i ee primenenie k zadacham avtomaticheskogo upravleniya (Theory of Random Functions and Its Application to Automatic Control). — Gostekhizdat. 1957.
90. RASHEVSKII, P.K. Kurs differentsial'noi geometrii (A Course in Differential Geometry). — Gostekhizdat. 1956.
91. RIVKIN, S.S. Teoriya giroskopicheskikh ustroistv (Theory of Gyroscopic Devices), Part 1. — Sudpromgiz. 1962; Part 2. — Izd. Sudostroenie. 1964.
92. WRIGLEY, W., R.B. WOODBURY, and J. HOVORKA. Inertial Guidance. — New York, Inst. Aeronaut. Sciences. 1957.
93. ROITENBERG, Ya.N. Sustained oscillations of gyro stabilizers. — Prikladnaya Matematika i Mekhanika, 11, No.2. 1947.
94. ROITENBERG, Ya.N. Giroskopy (Gyroscopes). — Nauka. 1966.
95. ROITENBERG, Ya.N. A many-gyro vertical. — Prikladnaya Matematika i Mekhanika, 10, No.1. 1946.
96. ROITENBERG, Ya.N. Fast meridional positioning of the gyrocompass. — Prikladnaya Matematika i Mekhanika, No.5. 1959.
97. ROITENBERG, Ya.N. Meridional setting of a gyrocompass during the starting of the gyro rotors. — Prikladnaya Matematika i Mekhanika, 24, No.1. 1960.
98. RUMYANTSEV, V.V. Stability of motion of a gimbaled gyroscope. — Prikladnaya Matematika i Mekhanika, 22, Nos.3,4. 1958.
99. RYZHIK, I.M. and I.S. GRADSHTEIN. Tablitsy integralov, summ, ryadov i proizvedenii (Tables of Integrals, Sums, Series, and Products). — Gostekhizdat. 1951.

100. SAVANT, C.I., R.C. HOWARD, C.B. SOLLOWAY, and C.A. SAVANT. Principles of Inertial Navigation. — New York, McGraw-Hill. 1961.
101. SLATER, I.M. and D.B. DUNCAN. Inertial Navigation. — Aeronautical Engineering Review, 15, No.1. January 1956.
102. SLOMYANSKII, G.A. and Yu.N. PRYADILOV. Poplavkovye giroskopy i ikh primenenie (Floating Gyroscopes and Their Application). — Oborongiz. 1958.
103. SMIRNOV, V.I. Kurs vysshei matematiki (A Course in Higher Mathematics), Vol.3, Part 1. — Gostekhizdat. 1953.
104. SOLODOVNIKOV, V.V. Vvedenie v statisticheskuyu dinamiku sistem avtomaticheskogo upravleniya (Introduction to Statistical Dynamics of Automatic Control Systems). — Gostekhizdat. 1952.
105. STEPANOV, V.V. Kurs differentsial'nykh uravnenii (A Course in Differential Equations). — Gostekhizdat. 1953.
106. SUBBOTIN, M.F. Kurs nebesnoi mekhaniki (A Course in Celestial Mechanics), Vol.2. 1937.
107. SUSLOV, T.K. Teoreticheskaya mekhanika (Theoretical Mechanics). — Gostekhizdat. 1944.
108. TOLSTOV, G.P. Ryady Fur'e (Fourier Series). — Gostekhizdat. 1951.
109. WHITTAKER, E. and G. WATSON. A Course of Modern Analysis. — Cambridge. 1927.
110. FEODOS'EV, V.I. and G.B. SINYAREV. Vvedenie v raketnuyu tekhniku (Introduction to Rocket Technology). — Oborongiz. 1961.
111. FRIDLENDER, G.O. Inertsial'nye sistemy navigatsii (Inertial Navigation Systems). — Fizmatgiz. 1961.
112. HOCH, H. Das physikalische Pendel im radialsymmetrischen Erdgewichtskraftfeld. — Zeitschrift für ang. Mathematik und Physik, 24, No.5/6. 1944.
113. CHETAEV, N.G. Ustoichivost' dvizheniya (Stability of Motion). — Gostekhizdat. 1955.
114. SCHULER, M. Die Störung von Pendel und Kreiselapparaten durch die Beschleunigung des Fahrzeuges. — Physikalische Zeitschrift, 24, No.16. Leipzig. 1923.

National Aeronautics and Space Administration
WASHINGTON, D. C. 20546

OFFICIAL BUSINESS

POSTAGE AND FEES PAID
NATIONAL AERONAUTICS AND
SPACE ADMINISTRATION

05U 001 46 51 3DS 69304 00903
AIR FORCE WEAPONS LABORATORY/WLIL/
KIRTLAND AIR FORCE BASE, NEW MEXICO 87117

ATT E. LOU BOWMAN, CHIEF,TECH. LIBRARY

NASA TT F-564
TT 69-55027

Printed in Jerusalem, Israel

5444